

A study memo on applied mathematics

Contents

1	Integer Programming	2
1.1	MILP and Branch-and-Bound Method	2
1.2	Meyer's Fundamental Theorem	2
1.2.1	Main result	2
1.2.2	Fourier elimination and Farkas Lemma	4
1.2.3	Polyhedron and Minkowski Weyl Theorem	6
1.2.4	Perfect formulation and Meyer's Fundamental theorem	9
1.2.5	Sharp MILP Formulation	11
1.2.6	Review	12
1.3	MILP formulation	13
1.3.1	Minimal formulation	13
1.3.2	Locally ideal formulation	13
2	Event graph analysis	20
2.1	Max-plus algebra	20
2.2	Petri net and Event graph	21

This is a study memo of [1], [3].

1 Integer Programming

1.1 MILP and Branch-and-Bound Method

Definition 1.1 (MILP: Mixed integer linear programming). *Let*

$$(S1) \ A \in M(m, n, \mathbb{Q}), \ G \in M(m, p, \mathbb{Q}), \ b \in \mathbb{Q}^m, \ c \in \mathbb{R}^n, \ h \in \mathbb{R}^p.$$

$$(S2) \ S := \{(x, y) \in (\mathbb{Z}_+)^n \times (\mathbb{R}_+)^p \mid g(x, y) := Ax + Gy \leq b\}$$

We call the following problem a MILP.

$$\begin{aligned} \max \quad & f(x, y) := c^t x + h^t y \\ \text{subject to} \quad & (x, y) \in S \end{aligned}$$

We succeed notations in Definition 1.1. And we set

$$S^0 := \{(x, y) \in (\mathbb{R}_+)^n \times (\mathbb{R}_+)^p \mid Ax + Gy \leq b\}$$

Let us assume the MILP has a optimal solution (x^*, y^*) and the optimal optimal value z^* . So $S^0 \neq \phi$. Let us fix $(x, y) \in S^0$.

Algorithm Branch-and-Bound Method

Input: $S^0 \neq \phi$

Step 1: Take a $(x^0, y^0) \in S^0$ and $(\underline{x}, \underline{y}, \underline{z}) \leftarrow (x_0, y_0, f(x^0, y^0))$ and $\mathcal{S} \leftarrow S_0$

Step 2: Take $j \in \{1, 2, \dots, n\}$. $S_{00} := \{(x, y) \in S \mid x_j \leq \lfloor x_j^0 \rfloor\}$ and $S_{01} := \{(x, y) \in S \mid x_j \geq \lceil x_j^0 \rceil\}$ and
MILP₀₀ : $\max f(S_{00})$ and MILP₀₁ : $\max f(S_{01})$.

Delete S_0 from \mathcal{S} and add S_{00} and S_{01} to \mathcal{S} .

Step 3: **for** $S_\alpha \in \mathcal{S}$ **do**

Solve $LP_\alpha : \max f(S_\alpha)$.

if LP_α is not feasible **then**

Delete S_α from \mathcal{S} .

else

We set (x^α, y^α) which is a optimal solution and z^α which is its optimal value.

Delete S_α from \mathcal{S} .

if $x^\alpha \in \mathbb{Z}_+^n$ **then**

if $z^\alpha > \underline{z}$ **then**

$$(\underline{x}, \underline{y}, \underline{z}) \leftarrow (x^\alpha, y^\alpha, f(x^\alpha, y^\alpha)).$$

end if

else $z^\alpha > \underline{z}$

Take $j \in \{1, 2, \dots, n\}$. $S_{\alpha 0} := \{(x, y) \in S_\alpha \mid x_j \leq \lfloor x_j^\alpha \rfloor\}$ and $S_{\alpha 1} := \{(x, y) \in S_\alpha \mid x_j \geq \lceil x_j^\alpha \rceil\}$.

Add $S_{\alpha 0}$ and $S_{\alpha 1}$ to \mathcal{S} .

end if

end if

end for

Output: $(\underline{x}, \underline{y}, \underline{z})$.

1.2 Meyer's Fundamental Theorem

1.2.1 Main result

The propositions shown in this subsection will not be presented with proofs in this subsection, but will be presented with proofs in the subsections that follow.

Definition 1.2 (Polyhedron). *Let* $A \in M(m, n, \mathbb{R}), b \in \mathbb{R}^m$. *We call*

$$P := \{x \in \mathbb{R}^n \mid Ax \leq b\}$$

a Polyhedron in \mathbb{R}^n or a \mathcal{H} -polyhedron. We call the right side \mathcal{H} -representation. If $A \in M(m, n, \mathbb{Q}), b \in \mathbb{Q}^m$ then P is a rational polyhedron.

Definition 1.3 (Recession cone). *Let P be a nonempty polyhedron. We call*

$$\text{rec}(P) := \{r \in \mathbb{R}^n \mid x + \lambda r \in P, \lambda \in \mathbb{R}_+\}$$

the recession cone of P .

Notation 1.1. *Let*

$$(S1) \ A \in M(m, n, \mathbb{Q}), \ G \in M(m, p, \mathbb{Q}), \ b \in \mathbb{Q}^m, \ c \in \mathbb{R}^n, \ h \in \mathbb{R}^p.$$

We set

$$P(A, G, b) := \{(x, y) \in (\mathbb{R}_+)^n \times (\mathbb{R}_+)^p \mid g(x, y) := Ax + Gy \leq b\}$$

Definition 1.4 (Convex, Convex combination). *Let $A \subset \mathbb{R}^n$. We say A is convex if $\sum_{i=1}^n \lambda_i a_i \in A$ for $a_1, \dots, a_n \in A$ and $\lambda_1, \dots, \lambda_n \in [0, 1]$ such that $\sum_{i=1}^n \lambda_i = 1$. We call the sum*

$$\sum_{i=1}^n \lambda_i a_i$$

convex combination of a_1, \dots, a_n .

Proposition 1.1. *Let*

$$(S1) \ A \in M(m, n, \mathbb{Q}), \ G \in M(m, p, \mathbb{Q}), \ b \in \mathbb{Q}^m, \ c \in \mathbb{R}^n, \ h \in \mathbb{R}^p.$$

$$(S2) \ S := \{(x, y) \in (\mathbb{Z}_+)^n \times (\mathbb{R}_+)^p \mid g(x, y) := Ax + Gy \leq b\}$$

Then

(i)

$$\sup\{c^t x + h^t y \mid (x, y) \in S\} = \sup\{c^t x + h^t y \mid (x, y) \in \text{conv}(S)\}$$

Furthermore, there is $(x, y) \in S$ such that $c^t x + h^t y = \sup\{c^t x + h^t y \mid (x, y) \in S\} \iff$ there is $(x, y) \in \text{conv}(S)$ such that $c^t x + h^t y = \sup\{c^t x + h^t y \mid (x, y) \in S\}$

(ii) $\text{ex}(\text{conv}(S)) \subset S$

Theorem 1.1 (Meyer(1974)[2] Fundamental Theorem). *Here are the settings and assumptions.*

$$(S1) \ A \in M(m, n, \mathbb{Q}), \ G \in M(m, p, \mathbb{Q}), \ b \in \mathbb{Q}^m.$$

$$(S2) \ S := \{(x, y) \in P(A, G, b) \mid x \in (\mathbb{Z}_+)^n\}.$$

Then there are $A' \in M(m, n, \mathbb{Q}), G' \in M(m, p, \mathbb{Q}), b' \in \mathbb{Q}^m$ such that

$$\text{conv}(S) = P(A', G', b')$$

By Proposition 1.1 and Theorem 1.1, MILP

$$\begin{aligned} \max \ f(x, y) &:= c^t x + h^t y \\ \text{subject to } (x, y) &\in S \end{aligned}$$

is equal to a pure LP

$$\begin{aligned} \max \ f(x, y) &:= c^t x + h^t y \\ \text{subject to } (x, y) &\in P(A', G', b') \end{aligned}$$

We set

$$\tilde{A} := \begin{pmatrix} A \\ A' \end{pmatrix}, \tilde{G} := \begin{pmatrix} G \\ G' \end{pmatrix}, \tilde{b} := \begin{pmatrix} b \\ b' \end{pmatrix},$$

Then clearly

$$S = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p \mid (x, y) \in P(\tilde{A}, \tilde{G}, \tilde{b}), x \in \mathbb{Z}^n\}$$

and MILP

$$\begin{aligned} \max \ f(x, y) &:= c^t x + h^t y \\ \text{subject to } (x, y) &\in S \end{aligned}$$

has a continuous relaxation

$$\begin{aligned} \max f(x, y) &:= c^t x + h^t y \\ \text{subject to } (x, y) &\in P(\tilde{A}, \tilde{G}, \tilde{b}) \end{aligned}$$

whose optimal value is equal to the one of the original MILP. And we can effectively find an optimal solution of this continuous relaxation which is contained in S .

From the above discussion, the following can be shown.

Proposition 1.2. *Here are the settings and assumptions.*

$$(S1) \ A \in M(m, n, \mathbb{Q}), \ G \in M(m, p, \mathbb{Q}), \ b \in \mathbb{Q}^m, \ c \in \mathbb{R}^n, \ h \in \mathbb{R}^p.$$

$$(S2) \ S := \{(x, y) \in P(A, G, b) \mid x \in (\mathbb{Z}_+)^n\}.$$

Then there is $M \in \mathbb{N}$ and are $\tilde{A} \in M(M, n, \mathbb{Q}), \tilde{G} \in M(M, p, \mathbb{Q}), \tilde{b} \in \mathbb{Q}^M$ such that

$$S = P(\tilde{A}, \tilde{G}, \tilde{b}) \cap \mathbb{Z}_+^n \times \mathbb{R}_+^p$$

and

$$\text{conv}(S) = P(\tilde{A}, \tilde{G}, \tilde{b})$$

1.2.2 Fourier elimination and Farkas Lemma

Definition 1.5 (Conic combination). *Let $v_1, \dots, v_m \in \mathbb{R}^n$. For every $\lambda_1, \dots, \lambda_m \geq 0$, we call $\sum_{i=1}^m \lambda_i v_i$ a conic combination of v_1, \dots, v_m .*

Theorem 1.2 (Fourier Elimination). *Let*

$$(S1) \ A \in M(m, n, \mathbb{R}), \ b \in \mathbb{R}^m.$$

$$(S2) \ I^+ := \{i \mid a_{i,n} > 0\}, \ I^- := \{i \mid a_{i,n} < 0\}, \ I^0 := \{i \mid a_{i,n} = 0\}.$$

$$(S3) \ a'_{i,k} := \frac{a_{i,k}}{|a_{i,n}|} \ (i \in I^+ \cup I^-, k \in \{1, 2, \dots, n-1\}), \ b'_i := \frac{b_i}{|a_{i,n}|} \ (i \in I^+ \cup I^-).$$

$$(S4) \ \tilde{A} := (A, b) \in M(m, n+1, \mathbb{R}).$$

$$(S5) \ \text{We set } \tilde{A}_{n-1} \in M(\#I^+ * \#I^- + \#I^0, n, \mathbb{R}) \text{ and } b' \in \mathbb{R}(\#I^+ * \#I^- + \#I^0) \text{ by}$$

$$(kq\text{-th row of } \tilde{A}_{n-1}) = \frac{1}{|a_{k,n}|} (k\text{-th row of } \tilde{A}) + \frac{1}{|a_{q,n}|} (q\text{-th row of } \tilde{A}) \ (\forall k \in I^+, \forall q \in I^-)$$

and

$$((\#I^+ * \#I^- + j)\text{-th row of } \tilde{A}') = (j\text{-th row of } \tilde{A}) \ (j = 1, 2, \dots, \#I^0)$$

$$(S6) \ x^i := (x_1, \dots, x_i) \ (x \in \mathbb{R}^n)$$

Then

(i) $Ax \leq b, x \in \mathbb{R}^n$ is feasible if and only if

$$\begin{aligned} \sum_{i=1}^{n-1} (a'_{k,i} + a'_{q,i}) x_i &\leq b'_k + b'_q \ (\forall k \in I^+, \forall q \in I^-), \\ \sum_{i=1}^{n-1} a_{p,i} x_i &\leq b_p \ (\forall p \in I^0) \end{aligned}$$

(ii) If $A \in M(m, n, \mathbb{Q})$ and $b \in \mathbb{Q}^m$, then $a'_{k,i}, a'_{q,i}, b'_k, b'_q \in \mathbb{Q} \ (\forall k \in I^+, \forall i \in \{1, 2, \dots, n-1\}, \forall q \in I^-)$.

(iii) $\{x \in \mathbb{R}^n \mid Ax \leq b\} \neq \emptyset \iff \{x \in \mathbb{R}^{n+1} \mid \tilde{A}(x^t, -1)^t \leq 0\} \neq \emptyset \iff \{x \in \mathbb{R}^n \mid \tilde{A}_{n-1}((x^{n-1})^t, -1)^t \leq 0\} \neq \emptyset$.

(iv) For each $i \in \{0, 1, \dots, n-1\}$, there is $m_i \in \mathbb{N}$ and $\tilde{A}_i \in M(m_i, i+1, \mathbb{R})$ such that every row of \tilde{A}_i is a conic combination of rows of \tilde{A} and

$$\{x \in \mathbb{R}^n \mid Ax \leq b\} \neq \emptyset \iff \{x \in \mathbb{R}^i \mid \tilde{A}_i((x^i)^t, -1)^t \leq 0\}$$

(v) If $\tilde{A} \in M(m, n+1, \mathbb{Q})$ then $\tilde{A}_i \in M(m_i, i+1, \mathbb{Q}) \ i \in \{0, 1, \dots, n-1\}$.

$$(vi) \{x \in \mathbb{R}^n | Ax \leq b\} \neq \emptyset \iff \tilde{A}_0 \leq 0.$$

Proof of the 'only if' part in (i). Let us assume $x \in \mathbb{R}^n$ such that $Ax \leq b$. Then

$$\sum_{i=1}^{n-1} a'_{k,i} x_i + x_n \leq b'_k \quad (\forall k \in I^+)$$

and

$$\sum_{i=1}^{n-1} a'_{q,i} x_i - x_n \leq b'_q \quad (\forall q \in I^-)$$

So, by adding the left and right sides of these two inequalities, respectively, the following holds.

$$\begin{aligned} \sum_{i=1}^{n-1} (a'_{k,i} + a'_{q,i}) x_i &\leq b'_k + b'_q \quad (\forall k \in I^+, \forall q \in I^-), \\ \sum_{i=1}^{n-1} a_{p,i} x_i &\leq b_p \quad (\forall p \in I^0) \end{aligned}$$

□

Proof of the 'if' part in (i). Let us assume

$$\begin{aligned} \sum_{i=1}^{n-1} (a'_{k,i} + a'_{q,i}) x_i &\leq b'_k + b'_q \quad (\forall k \in I^+, \forall q \in I^-), \\ \sum_{i=1}^{n-1} a_{p,i} x_i &\leq b_p \quad (\forall p \in I^0) \end{aligned}$$

Then

$$\sum_{i=1}^{n-1} a'_{k,i} x_i - b'_k \leq -\left(\sum_{i=1}^{n-1} a'_{q,i} - b'_q\right) \quad (\forall k \in I^+, \forall q \in I^-)$$

We set

$$x_n := \min\left\{-\left(\sum_{i=1}^{n-1} a'_{k,i} - b'_k\right) \mid k \in I^+\right\}$$

Then

$$x_n \geq \max\left\{\left(\sum_{i=1}^{n-1} a'_{q,i} - b'_q\right) \mid q \in I^-\right\}$$

So, $Ax \leq b$. □

Proof of (ii)-(iv). These are followed by (i). □

Theorem 1.3 (Farkas Lemma I). *Let*

$$(S1) \quad A \in M(m, n, \mathbb{R}), \quad b \in \mathbb{R}^m.$$

Then

$$\{x \in \mathbb{R}^n | Ax \leq b\} = \emptyset \iff \{v \in \mathbb{R}^m | A^t v = 0, b^t v < 0, v \geq 0\} \neq \emptyset$$

Proof of 'only if' part. By Fourier elimination method (iv), there are $m_0 \in \mathbb{N}$ and $U \in M(m_0, n, \mathbb{R})$ such that $U \geq 0$ and $U\tilde{A} = (O_{m_0, n-1}, b^0)$ and $b^0 \not\geq 0$. Then there is $u \in \mathbb{R}^{m_0}$ such that $u^t b^0 < 0$. We set

$$v := (u^t U)^t$$

Then $v \geq 0$ and $Av = 0$ and $v^t b < 0$. □

Proof of 'if' part. Let us assume $\exists v \in \mathbb{R}^m$ such that $v^t A = 0$ and $v^t b < 0$ and $v \geq 0$. For any $x \in \mathbb{R}^n$, $v^t Ax = 0$. So, $Ax \not\leq b$. \square

Theorem 1.4 (Farkas Lemma II). *Let*

$$(S1) \ A \in M(m, n, \mathbb{R}), \ b \in \mathbb{R}^m.$$

Then

$$\{x \in \mathbb{R}^n | Ax = b, x \geq 0\} \neq \emptyset \iff \{u \in \mathbb{R}^m | A^t u \leq 0\} \subset \{u \in \mathbb{R}^m | u^t b \leq 0\}$$

Proof of ' \implies '. Let us fix $x \in \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$. Let us fix any $u \in \{u \in \mathbb{R}^m | A^t u \leq 0\}$. So, $b^t u \leq 0$. \square

Proof of ' \impliedby '. Let us assume

$$\{x \in \mathbb{R}^n | Ax = b, x \geq 0\} = \emptyset$$

Then

$$\{x \in \mathbb{R}^n | Ax \leq b, -Ax \leq -b, x \geq 0\} = \{x \in \mathbb{R}^n | Bx \leq c\} = \emptyset$$

Here,

$$B := \begin{pmatrix} A \\ -A \\ -I_n \end{pmatrix}, \ c := \begin{pmatrix} b \\ -b \\ O_{n,1} \end{pmatrix}$$

and I_n is the n -th unit matrix. By Farkas Lemma I, there are $v \in \mathbb{R}_+^n$ and $v' \in \mathbb{R}_+^m$ and $w \in \mathbb{R}_+^n$ such that

$$B^t \begin{pmatrix} v \\ v' \\ w \end{pmatrix} = 0, \ \begin{pmatrix} v \\ v' \\ w \end{pmatrix}^t c < 0$$

This implies

$$A(-(v - v')) = -w, \ -(v - v')^t b > 0$$

We set $u := -(v - v')$. Then

$$u \in \{u \in \mathbb{R}^m | A^t u \leq 0\} \setminus \{u \in \mathbb{R}^m | u^t b \leq 0\}$$

\square

1.2.3 Polyhedron and Minkowski Weyl Theorem

Definition 1.6 (Polytope). *We say $A \subset \mathbb{R}^n$ is a polytope if there are finite vectors $v_1, \dots, v_m \in \mathbb{R}^n$ such that $A = \text{conv}(v_1, \dots, v_m)$. We call v_1, \dots, v_m vertices of A . If $v_1, \dots, v_m \in \mathbb{Q}^n$, we call A is a rational polytope.*

Definition 1.7 (Cone). *We say $C \subset \mathbb{R}^n$ is a cone if $0 \in C$ and for every $x \in C$ and $\lambda \in \mathbb{R}_+$ $\lambda x \in C$.*

By the definition of cone, the following holds.

Proposition 1.3. *Any cone containing nonzero vector is not bounded.*

Definition 1.8 (Convex Cone). *We say $C \subset \mathbb{R}^n$ is a convex cone if C is cone and every conic combination of finite vectors of C is contained in C .*

Because every intersection of convex cones is also convex cone, the following holds.

Proposition 1.4 (Convex Cone generated by a set). *Let us assume A is any subset of \mathbb{R}^n . Then there is the minimum convex cone containing A . We denote this convex cone by $\text{cone}(A)$.*

Definition 1.9 (Polyhedral cone). *Let*

$$(S1) \ A \in M(m, n, \mathbb{Q}).$$

We call

$$P := \{x \in \mathbb{R}^n | Ax \leq 0\}$$

a Polyhedral cone.

Theorem 1.5 (Minkowski Weyl Theorem for cones). *Let*

$$(S1) \ C \subset \mathbb{R}^n.$$

Then C is a Polyhedral cone if and only if C is finite generated cone.

STEP1. Proof of 'if' part. Let us assume C is finite generated cone. Then there is $r_1, \dots, r_k \in \mathbb{R}$ such that $C = \text{cone}(r_1, \dots, r_k)$. We set $R = (r_1, \dots, r_k)$.

By applying Fourier elimination method k times to the the following inequality

$$-\mu \leq 0, R\mu \leq x, -R\mu \leq -x$$

and Fourier elimination method (vi), there is $A \in M(m, n, \mathbb{R})$ such that the above inequality is equivalent to

$$Ax \leq 0$$

So, $C = \{x \in \mathbb{R}^n | Ax \leq 0\}$. □

STEP2. Proof of 'only if' part. Let us assume C is a Polyhedral cone. So, there is $A \in M(m, n, \mathbb{R})$ such that $C = \{x \in \mathbb{R}^n | Ax \leq 0\}$. We set $C^* := \{y \in \mathbb{R}^n | \exists \nu \in \mathbb{R}_+^m \text{ such that } A^t \nu = y\}$. Then

$$C^* = \text{cone}(a^1, \dots, a^m)$$

Here, $a^i \in \mathbb{R}^n$ is the i -th row vector of A ($i = 1, 2, \dots, m$). By STEP1, there is $R \in M(n, k, \mathbb{R})$ such that

$$C^* = \{y \in \mathbb{R}^n | R^t y \leq 0\}$$

We denote the i -th column vector of R by r^i ($i = 1, 2, \dots, k$). We will show

$$C = \text{cone}(r_1, \dots, r_k)$$

Let us fix any $x \in \text{cone}(r_1, \dots, r_k)$. Then there are $\nu_1, \dots, \nu_k \in \mathbb{R}_+$ such that $x = R\nu$. Because $a_i = A^t e_i$ ($i = 1, 2, \dots, m$), $a_i \in C^*$ ($i = 1, 2, \dots, m$). So, $AR \leq 0$. This implies $Ax = AR\nu \leq 0$. This means $x \in C$. We have shown $\text{cone}(r_1, \dots, r_k) \subset C$.

Let us fix any $\bar{x} \in \text{cone}(r_1, \dots, r_k)^c$. So, $\{\nu \in \mathbb{R}^k | R\nu = \bar{x}, \nu \geq 0\} = \emptyset$. By Farkas Lemma II, there is $y \in \mathbb{R}^n$ such that $R^t y \leq 0$ and $y^t \bar{x} > 0$. So, $y \in C^*$. Then there are $\nu \in \mathbb{R}_+^m$ such that $y = A^t \nu$. So, $\nu^t A \bar{x} > 0$. Because $\nu \in \mathbb{R}_+^m$, this implies $A \bar{x} \not\leq 0$. This means $\bar{x} \in C^c$. Consequently $C \subset \text{cone}(r_1, \dots, r_k)$. □

Definition 1.10 (Minkowski sum). Let $A, B \subset \mathbb{R}^n$. We call

$$A + B$$

the Minkowski sum of A and B .

Proposition 1.5. Let

- (i) Minkowski sum of any two convex set is convex.
- (ii) For any two subset $A, B \subset \mathbb{R}^n$,

$$\text{conv}(A + B) = \text{conv}(A) + \text{conv}(B)$$

Proof of (i). Let $A, B \subset \mathbb{R}^n$ be convex. For any $a_1, \dots, a_m \in A$ and $b_1, \dots, b_m \in B$ and $\lambda_1, \dots, \lambda_m \in [0, 1]$ such that $\sum_{i=1}^m \lambda_i = 1$,

$$\sum_{i=1}^m \lambda_i (a_i + b_i) = \sum_{i=1}^m \lambda_i a_i + \sum_{i=1}^m \lambda_i b_i \in A + B$$

So, $A + B$ is convex. □

Proof of (ii). By (i), $\text{conv}(A) + \text{conv}(B)$ is convex. And $A + B \subset \text{conv}(A) + \text{conv}(B)$. So, $\text{conv}(A + B) \subset \text{conv}(A) + \text{conv}(B)$. Let us fix any $a_1, \dots, a_k \in A$ and $b_1, \dots, b_l \in B$ and $\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_l \in [0, 1]$ such that $\sum_{i=1}^k \lambda_i = 1$ and $\sum_{i=1}^l \mu_i = 1$. Then

$$\sum_{i=1}^k \lambda_i a_i + \sum_{j=1}^l \mu_j b_j = \sum_{j=1}^l \mu_j \left(\sum_{i=1}^k \lambda_i a_i + b_j \right) = \sum_{j=1}^l \mu_j \left(\sum_{i=1}^k \lambda_i (a_i + b_j) \right) = \sum_{i,j} \lambda_i \mu_j (a_i + b_j) \in \text{conv}(A + B)$$

□

Theorem 1.6 (Minkowski-Weyl Theorem). *A subset $P \subset \mathbb{R}^n$ is a Polyhedron if and only if there is a polytope Q a finite generated cone C such that*

$$P = Q + C$$

We call the right side \mathcal{V} -representation and call P a \mathcal{V} -polyhedron.

Proof of ‘only if’ part. Let us fix $A \in M(m, n, \mathbb{R})$ and $b \in \mathbb{R}^m$ such that $P = \{x \in \mathbb{R}^n | Ax \leq b\}$. We set

$$C_P := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} | Ax - yb \leq 0, y \leq 0\}$$

Then clearly

$$P = \{x \in \mathbb{R}^n | (x, 1) \in C_P\}$$

By Minkowski Weyl Theorem for cones, there are $r^1, r^2, \dots, r^K \in \mathbb{R}^{n+1}$ such that

$$C_P := \text{cone}(r^1, r^2, \dots, r^K)$$

Because C_P is a cone, we can assume $r_{n+1}^i = 0$ or 1 ($\forall i$). So, there are $u_1, \dots, u_k \in \mathbb{R}^n$ and $v_1, \dots, v_l \in \mathbb{R}^n$ such that

$$C_P = \text{cone}\left(\begin{pmatrix} u_1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} u_k \\ 1 \end{pmatrix}, \begin{pmatrix} v_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} v_l \\ 0 \end{pmatrix}\right)$$

So,

$$P = \text{conv}(u^1, \dots, u^k) + \text{cone}(v^1, \dots, v^l)$$

□

Proof of ‘if’ part. We assume we can get

$$P = \text{conv}(u^1, \dots, u^k) + \text{cone}(v^1, \dots, v^l)$$

Then

$$P = \text{cone}\left(\begin{pmatrix} u^1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} u^k \\ 1 \end{pmatrix}, \begin{pmatrix} v^1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} v^l \\ 0 \end{pmatrix}\right) \cap \mathbb{R}^n \times \{1\}$$

Because $\text{cone}\left(\begin{pmatrix} u^1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} u^k \\ 1 \end{pmatrix}, \begin{pmatrix} v^1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} v^l \\ 0 \end{pmatrix}\right)$ is a Polyhedral cone, P is a Polyhedron. □

Proof of the last part. □

Proposition 1.6. *Let*

(i) *Bounded Polyhedron is polytone.*

(ii) *If $A \in M(m, n, \mathbb{Q})$ and $b \in \mathbb{Q}^m$, then there are $v_1, \dots, v_k \in \mathbb{Q}^n$ and $r_1, \dots, r_l \in \mathbb{Z}^n$ such that*

$$P := \{x \in \mathbb{R}^n | Ax \leq b\} = \text{conv}(v_1, \dots, v_k) + \text{cone}(r_1, \dots, r_l)$$

If P is bounded, P is a rational polytope.

(iii) *$P \subset \mathbb{R}^n$ is a rational polyhedron if and only if P is a minkowski sum of a rational polytope and a convex cone generated by finite rational vectors.*

Proof of (i). By Proposition 1.3, (i) holds. □

Proof of (ii). By the proof of Theorem 1.5, (ii) holds. □

Proof of (iii). By the proof of Theorem 1.5, (iii) holds. □

1.2.4 Perfect formulation and Meyer's Fundamental theorem

Proposition 1.7. *Here are the settings and assumptions.*

$$(S1) \ A \in M(m, n, \mathbb{Q}), \ G \in M(m, p, \mathbb{Q}), \ b \in \mathbb{Q}^m, \ c \in \mathbb{R}^n, \ h \in \mathbb{R}^p.$$

$$(S2) \ S := \{(x, y) \in (\mathbb{Z}_+)^n \times (\mathbb{R}_+)^p \mid g(x, y) := Ax + Gy \leq b\}$$

Then

(i)

$$\sup\{c^t x + h^t y \mid (x, y) \in S\} = \sup\{c^t x + h^t y \mid (x, y) \in \text{conv}(S)\}$$

Furthermore, there is $(x, y) \in S$ such that $c^t x + h^t y = \sup\{c^t x + h^t y \mid (x, y) \in S\} \iff$ there is $(x, y) \in \text{conv}(S)$ such that $c^t x + h^t y = \sup\{c^t x + h^t y \mid (x, y) \in S\}$

(ii) $\text{ex}(\text{conv}(S)) \subset S$

Proof of the first part of (i). Because $S \subset \text{conv}(S)$,

$$\sup\{c^t x + h^t y \mid (x, y) \in S\} \leq \sup\{c^t x + h^t y \mid (x, y) \in \text{conv}(S)\}$$

We can assume $z^* = \sup\{c^t x + h^t y \mid (x, y) \in S\} < \infty$. Let us set $H := \{(x, y) \in \mathbb{R}^{n+p} \mid c^t x + h^t y \leq z^*\}$. Because H is convex and $S \subset H$, $\text{conv}(S) \subset H$. So,

$$\sup\{c^t x + h^t y \mid (x, y) \in S\} \geq \sup\{c^t x + h^t y \mid (x, y) \in \text{conv}(S)\}$$

□

Proof of the last part of (i). The part of \implies is clear. We set $d := (c, h)$. Let us assume there is $\bar{z} = (\bar{x}, \bar{y})$ such that $d^t \bar{z} = \sup\{c^t x + h^t y \mid (x, y) \in \text{conv}(S)\}$. Then there are $\lambda_1, \dots, \lambda_k > 0$ and $z_1, \dots, z_k \in S$ such that $\bar{z} = \sum_{i=1}^k \lambda_i z_i$. Clearly $d^t z_i \leq d^t \bar{z} (\forall i)$. Because $d^t \bar{z} = \sum_{i=1}^k d^t \lambda_i z_i$, there is i such that $d^t z_i \geq d^t \bar{z}$. So, $d^t z_i = \sup\{c^t x + h^t y \mid (x, y) \in \text{conv}(S)\}$. □

Proof of (ii). Let us fix any $v \in \text{ex}(\text{conv}(S))$. Because $\text{ex}(\text{conv}(S)) \subset \text{conv}(S)$, there are $\lambda_1, \dots, \lambda_m \in (0, 1]$ and $v_1, \dots, v_m \in S$ such that $v = \sum_{i=1}^m \lambda_i v_i$. We can assume $m > 1$. We set $v' := \sum_{i=2}^m \frac{\lambda_i}{1 - \lambda_1} v_i$. Then $v' \in \text{conv}(S)$. Because $v = \lambda_1 v_1 + (1 - \lambda_1)v'$ and $v \in \text{ex}(\text{conv}(S))$, $v = v_1 \in S$. □

Proposition 1.8. *Let $r^1, \dots, r^K \in \mathbb{R}^n$. Then*

$$\text{conv}\left(\sum_{i=1}^K \mathbb{Z}_+ r^i\right) = \text{cone}(r_1, \dots, r^K)$$

Proof. We will show this by Mathematical induction. If $K = 1$, then this proposition holds. Let us fix any $k \in \mathbb{N}$ and assume this proposition holds for every $K \leq k$.

We set $C := \text{conv}(\sum_{i=1}^{k+1} \mathbb{Z}_+ r^i)$. Clearly $C \subset \text{cone}(r^1, \dots, r^{k+1})$. Let us fix $x \in \text{cone}(r^1, \dots, r^{k+1})$. Then there are $\mu_1, \dots, \mu_{k+1} \in \mathbb{R}_+$ such that $x = \sum_{i=1}^{k+1} \mu_i r^i$. We can assume $\mu_{k+1} > 0$. We set $\lambda := \frac{2\mu_{k+1}}{\lceil 2\mu_{k+1} \rceil}$. Because $0 \in C$, $2\mu_{k+1} r^{k+1} = (1 - \lambda)0 + \lambda \lceil 2\mu_{k+1} \rceil r^{k+1} \in C$. By Mathematical induction assumption, $\sum_{i=1}^k 2\mu_i r^i \in C$. So,

$$\sum_{i=1}^{k+1} \mu_i r^i = \frac{1}{2}(2\mu_{k+1} r^{k+1} + \sum_{i=1}^k 2\mu_i r^i) \in C$$

So, $\text{cone}(r^1, \dots, r^{k+1}) \subset C$. □

Theorem 1.7 (Meyer(1974)[2] Fundamental Theorem). *Here are the settings and assumptions.*

$$(S1) \ A \in M(m, n, \mathbb{Q}), \ G \in M(m, p, \mathbb{Q}), \ b \in \mathbb{Q}^m, \ c \in \mathbb{R}^n, \ h \in \mathbb{R}^p.$$

$$(S2) \ S := \{(x, y) \in P(A, G, b) \mid x \in \mathbb{Z}^n\}.$$

Then there are $A' \in M(m, n, \mathbb{Q}), G' \in M(m, p, \mathbb{Q}), b' \in \mathbb{Q}^m, c \in \mathbb{R}^n, h \in \mathbb{R}^p$ such that

$$\text{conv}(S) = P(A', G', b')$$

STEP1. Decomposition of S . We can assume $S \neq \emptyset$. Then by Proposition1.6, there are $v^1, \dots, v^t \in \mathbb{Q}^{n+p}$ and $r^1, \dots, r^q \in \mathbb{Z}^{n+p}$ such that

$$P := P(A, G, b) = \text{conv}(v^1, \dots, v^t) + \text{cone}(r^1, \dots, r^q)$$

We set

$$T := \left\{ \sum_{i=1}^s \lambda_i v^i + \sum_{j=1}^q \mu_j r^j \mid 0 \leq \lambda_i, \mu_j \leq 1 \ (\forall i, j), \sum_{i=1}^s \lambda_i = 1 \right\} = \text{conv}(v^1, \dots, v^t) + \sum_{j=1}^q [0, 1] r_j$$

Then T is bounded. There is $M \in \mathbb{N}$ and $D \in M(M, n+p, \mathbb{Q})$ such that

$$T = \left\{ z \in \mathbb{R}^{n+p} \mid \exists \lambda \in \mathbb{R}_+^n, \exists \mu \in \mathbb{R}_+^p \text{ s.t. } D \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \leq z, \sum_{i=1}^s \lambda_i \leq 1, -\sum_{i=1}^s \lambda_i \leq -1, \mu \leq 1 \right\}$$

By Fourier elimination method, there are $C \in M(M, n, \mathbb{R})$ and $d \in \mathbb{Q}^n$ such that $T = \{x \in \mathbb{R}^n \mid Cx \leq d\}$. So, by Proposition1.6, T is a rational polytope.

Let

$$T_I := \{(x, y) \in \mathbb{Z}^n \times \mathbb{R}^p \mid (x, y) \in T\}, R_I := \left\{ \sum_{j=1}^q \mu_j r^j \mid \mu_j \in \mathbb{Z}_+ \ (\forall j) \right\}$$

We will show

$$S = T_I + R_I$$

Because $T_I + R_I \subset T$ and i -th component of $T_I + R_I$ is integer for every $i \in \{1, 2, \dots, s\}$, $T_I + R_I \subset S$.

Let us fix any $(x, y) \in \mathbb{Z}^n \times \mathbb{R}^p$ such that $(x, y) \in S$. Then there are $\lambda_1, \dots, \lambda_s, \mu_1, \dots, \mu_q \in [0, 1]$ such that $\sum_{i=1}^s \lambda_i = 1$ and

$$(x, y) = \sum_{i=1}^s \lambda_i v^i + \sum_{j=1}^q \mu_j r^j$$

We set

$$(x', y') := \sum_{i=1}^s \lambda_i v^i + \sum_{j=1}^q (\mu_j - \lfloor \mu_j \rfloor) r^j, r := \sum_{j=1}^q \lfloor \mu_j \rfloor r^j$$

Then $(x', y') \in T_I$ and $r \in R_I$. So, $(x, y) \in T_I + R_I$. Consequently, $S = T_I + R_I$. \square

STEP2. Proof that $\text{conv}(S)$ is a rational polyhedron. By Proposition1.5 and STEP1,

$$\text{conv}(S) = \text{conv}(T_I) + \text{conv}(R_I)$$

Because $\text{conv}(R_I) = \text{conv}(r^1, \dots, r^q)$, by Proposition1.8, $\text{conv}(R_I)$ is a rational polyhedral cone. So, it is enough to show

$$\text{conv}(T_I) \text{ is a rational polytope}$$

Since T is bounded, $X := \{x \in \mathbb{Z}^n \mid \exists y \in \mathbb{R}^p \text{ such that } (x, y) \in T_I\}$ is bounded and so is a finite set.

For each $x \in X$, we set $T_x := \{(x, y) \mid \exists y \in \mathbb{R}^p \text{ such that } (x, y) \in T_I\}$. For any $\bar{x} \in X$,

$$T_{\bar{x}} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p \mid x = \bar{x} \text{ and } (x, y) \in T\}$$

Because T is a rational polytope, $T_{\bar{x}}$ is a rational polytope. We denote the set of all vertices of $T_{\bar{x}}$ by $V_{\bar{x}}$ for any $\bar{x} \in X$. We set $V := \cup_{x \in X} V_x$. V is a finite set. We will show

$$\text{conv}(T_I) = \text{conv}(V)$$

Because $T_I = \cup_{x \in X} T_x = \cup_{x \in X} \text{conv}(V_x) \subset \text{conv}(V)$, $\text{conv}(T_I) \subset \text{conv}(V)$. Because $V = \cup_{x \in X} V_x \subset \cup_{x \in X} \text{conv}(V_x) = \cup_{x \in X} T_x = \text{conv}(T_I)$, $\text{conv}(V) \subset \text{conv}(T_I)$. So, $\text{conv}(T_I) = \text{conv}(V)$. Consequently, $\text{conv}(T_I)$ is a rational polytope. \square

By the proof of Theorem1.7, the following holds.

Theorem 1.8. *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $G \in M(m, p, \mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

(S2) $S := \{(x, y) \in P(A, G, b) \mid x \in \mathbb{Z}^n\}$.

Then there are

$$a_1, \dots, a_k \in P(A, G, b) \cap \mathbb{Z}^n \times \mathbb{Q}^p = S$$

and

$$r_1, \dots, r_l \in \mathbb{Z}^{n+p}$$

such that

$$\text{conv}(S) = \text{conv}(a_1, \dots, a_k) + \text{cone}(r_1, \dots, r_l)$$

1.2.5 Sharp MILP Formulation

Definition 1.11 (MILP Formulation). *Here are the settings and assumptions.*

$$(S1) \ A \in M(m, n, \mathbb{Q}), \ G \in M(m, p, \mathbb{Q}), \ B \in M(m, t, \mathbb{Q}), \ b \in \mathbb{Q}^m.$$

$$(S2) \ S \subset \mathbb{Q}^n.$$

$$(S3) \ T(A, G, B, b) := \{(x, y, z) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Z}^t \mid Ax + Gy + Bz \leq b\}.$$

We say (A, G, B, b) is a MILP formulation for S if and only if S is equal to the image of

$$p_n : T(A, G, B, b) \ni (x, y, z) \mapsto x \in \mathbb{Q}^n$$

Clearly the following holds.

Proposition 1.9. *Here are the settings and assumptions.*

$$(S1) \ A \in M(m, n, \mathbb{Q}), \ G \in M(m, p, \mathbb{Q}), \ b \in \mathbb{Q}^m, \ c \in \mathbb{R}^n, \ h \in \mathbb{R}^p.$$

$$(S2) \ S := \{(x, y) \in P(A, G, b) \mid x \in (\mathbb{Z}_+)^n\}.$$

(S3) We set

$$\tilde{A} := \begin{pmatrix} A \\ E_n \\ O_{p,n} \\ O_{n,n} \end{pmatrix}, \tilde{G} := \begin{pmatrix} G \\ O_{n,p} \\ -E_p \\ O_{n,p} \end{pmatrix}, \tilde{B} := \begin{pmatrix} B \\ -E_n \\ O_{p,n} \\ -E_n \end{pmatrix}, \tilde{b} := \begin{pmatrix} b \\ 0_n \\ 0_p \\ 0_n \end{pmatrix}$$

Then $(\tilde{A}, \tilde{G}, \tilde{B}, \tilde{b})$ is a MILP formulation for S .

Definition 1.12 (Sharp MILP Formulation). *Here are the settings and assumptions.*

$$(S1) \ A \in M(m, n, \mathbb{Q}), \ G \in M(m, p, \mathbb{Q}), \ B \in M(m, t, \mathbb{Q}), \ b \in \mathbb{Q}^m.$$

$$(S2) \ S \subset \mathbb{Q}^n.$$

(Aq) (A, G, B, b) is a MILP formulation for S .

We say (A, G, B, b) is sharp MILP formulation for S if and only if $\text{conv}(S)$ is equal to the image of

$$p_n : \tilde{T}(A, G, B, b) \ni (x, y, z) \mapsto x \in \mathbb{Q}^n$$

Here, $\tilde{T}(A, G, B, b)$ is a LP relaxation of $T(A, G, B, b)$.

Theorem 1.9. *Here are the settings and assumptions.*

$$(S1) \ S \subset \mathbb{Q}^n.$$

(A1) There are $A \in M(m, n, \mathbb{Q}), G \in M(m, p, \mathbb{Q}), B \in M(m, t, \mathbb{Q}), b \in \mathbb{Q}^m$ such that (A, G, B, b) is a MILP formulation for S .

Then there are $M \in \mathbb{N}$ and $\tilde{A} \in M(M, n, \mathbb{Q}), \tilde{G} \in M(M, p, \mathbb{Q}), \tilde{B} \in M(M, t, \mathbb{Q}), \tilde{b} \in \mathbb{Q}^M$ such that $(\tilde{A}, \tilde{G}, \tilde{B}, \tilde{b})$ is a sharp MILP formulation for S .

Proof. We set

$$T_I := \{(x, y, z) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Z}^t \mid Ax + Gy + Bz \leq b\}$$

and $p_1 : T_I \ni (x, y, z) \mapsto x \in \mathbb{Q}^n$. Because (A, G, B, b) is a MILP formulation for S ,

$$p_1(T_I) = S$$

By Theorem 1.2.4, there are $M \in \mathbb{N}$ and $\tilde{A} \in M(M, n, \mathbb{Q}), \tilde{G} \in M(M, p, \mathbb{Q}), \tilde{B} \in M(M, t, \mathbb{Q}), \tilde{b} \in \mathbb{Q}^M$ such that

$$T_I = \{(x, y, z) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Z}^t \mid \tilde{A}x + \tilde{G}y + \tilde{B}z \leq \tilde{b}\}$$

$$\text{conv}(T_I) = \{(x, y, z) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Q}^t \mid \tilde{A}x + \tilde{G}y + \tilde{B}z \leq \tilde{b}\}$$

Because $\text{conv}(S) = \text{conv}(p_1(T_I)) = p_1(\text{conv}(T_I))$,

$$\text{conv}(S) = p_1(\text{conv}(T_I))$$

So, $(\tilde{A}, \tilde{G}, \tilde{B}, \tilde{b})$ is a sharp MILP formulation for S . □

1.2.6 Review

Meyer theorem states that the convex hull of the feasible region of MILP is a rational polyhedron. So, the feasibility and the optimal value of MILP are equivalent to the feasibility and the optimal value of some LP, respectively. By methods such as simplex method, we can find this LP solution in extreme points of feasible region. By Proposition 1.7, this extreme point is a solution of original MILP problem.

I think there are the following three ideas that are important in the proof of Meyer theorem.

1. Fourier elimination method
2. Expressing the feasible region of MILP or LP in terms of the Minkowski sum of bounded and unbounded parts
3. Going back and forth between integer and continuous parts of a polyhedron

Fourier elimination method plays an important role throughout this section. Fourier elimination method is a method of solving linear inequalities

$$Ax \leq b \quad (1.2.1)$$

focusing on the sign of the coefficients of a certain variable and using only non-negative multipliers to eliminate the variable. (1.2.1) corresponds to another two linear inequalities. If there is a solution of (1.2.1), then there is $U \in M(m_0, n, \mathbb{R})$ such that $U \geq 0$ and $UA = 0$ and

$$0 \leq Ub \quad (1.2.2)$$

By focusing on row vectors of U , if there is no solutions of (1.2.1), then there is $u \in \mathbb{R}_+^n$ such that

$$A^t u = 0, u^t b < 0, u \geq 0 \quad (1.2.3)$$

Correspondance between (1.2.1) and (1.2.3) is stated by Farkas Lemma.

For idea 2 on LP feasible region P , we state this idea as Minkowski Weyl Theorem.

$$P = \text{conv}(v^1, \dots, v^s) + \text{cone}(r^1, \dots, r^q) \quad (1.2.4)$$

By increasing the dimension of the solution space of the simultaneous inequalities by one as follows, Minkowski Weyl Theorem is boiled down to the case in P is a polyhedral cone.

$$P = \tilde{P} \cap \mathbb{R}^n \times \{1\}, \tilde{P} := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid (A, -b) \begin{pmatrix} x \\ y \end{pmatrix} \leq 0\} \quad (1.2.5)$$

By Fourier elimination method and Farkas Lemma, any polyhedral cone is equivalent to finite generated convex cone.

Meyer theorem is the following.

Theorem 1.10. *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q}), G \in M(m, p, \mathbb{Q}), b \in \mathbb{Q}^m, c \in \mathbb{R}^n, h \in \mathbb{R}^p$.

(S2) $S := \{(x, y) \in P(A, G, b) \mid x \in \mathbb{Z}^n\}$.

Then $\text{conv}(S)$ is a rational polyhedron.

In the proof of Meyer theorem, we focus on Polyhedron $P := P(A, G, b)$ which is containing S . By Minkowski Weyl Theorem, we get

$$P = \text{conv}(v^1, \dots, v^s) + \text{cone}(r^1, \dots, r^q)$$

We focus a bounded part of P

$$T = \text{conv}(v^1, \dots, v^s) + \sum_{j=1}^q [0, 1] r_j$$

We denote an integer part of T by T_I and denote an integer part of $\text{cone}(r^1, \dots, r^q)$ by R_I . Then we get

$$S = T_I + R_I$$

So,

$$\text{conv}(S) = \text{conv}(T_I) + \text{conv}(R_I)$$

Because $\text{conv}(T_I)$ is a rational polytope and $\text{conv}(R_I)$ is a rational polyhedral cone, $\text{conv}(S)$ is a rational polyhedron.

1.3 MILP formulation

1.3.1 Minimal formulation

Definition 1.13 (Implied equations, redundant inequalities, and facet). *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $b \in \mathbb{Q}^m$, $P := \{x \in \mathbb{Q}^n | Ax \leq b\}$. a_i is the i -th row vector of A .

Then

- (i) We say $F \subset P$ is a face of P if and only if $F = \{x | a_i^T x = b_i (\forall i \in L)\}$ for some $L \subset \{1, 2, \dots, m\}$.
- (ii) We say $F \subset P$ is a proper face of P if and only if F is a face and $F \neq \phi$ and $F \neq P$.
- (iii) We say $F \subset P$ is a facet of P if and only if F is a proper face and maximum with respect to inclusion.
- (iv) We say $a_i x \leq b_i$ ($i \in L$) is implied equations of P if and only if $a_i x \leq b_i$ ($\forall i \in L$) for any $x \in P$.
- (v) We say $a_i x \leq b_i$ ($i \in L$) is facet defining inequalities of P if and only if $F := \{x | a_i x \leq b_i (\forall i \in L)\}$ is a facet of P .
- (vi) We say $a_i x \leq b_i$ ($i \in L$) is redundant inequalities of P if and only if there is a subset $I \subset \{1, 2, \dots, m\}$ such that $P = \{x | a_i x \leq b_i (\forall i \in I \setminus L)\}$.
- (vii) We say $L \subset \{1, 2, \dots, m\}$ is a minimal formulation of P if and only if $P = \{x | a_i x \leq b_i (\forall i \in L)\}$ and there is no $i \in L$ such that $a_i \leq b$ is a redundant inequality of P .

1.3.2 Locally ideal formulation

Proposition 1.10 (Standard equity form for LP). *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $b \in \mathbb{Q}^m$.

(S2) $S := \{x \in \mathbb{Q}^n | Ax \leq b\}$.

(S3) We set for $x \in S$,

$$\Phi(x) := (y^+, y^-, z)$$

Here,

$$y_i^+ := \max\{x_i, 0\} \quad (i = 1, 2, \dots, n)$$

$$y_i^- := \max\{-x_i, 0\} \quad (i = 1, 2, \dots, n)$$

$$z_j := (a_j, x) - b_j \quad (j = 1, 2, \dots, m)$$

(S4) $\tilde{S} := \{(y^+, y^-, z) \in \mathbb{Q}_+^n | A(y^+ - y^-) + z \leq b\}$.

Then Φ is a bijective from S to \tilde{S} . We call \tilde{S} the standard equity form of S . We call each z_j a slack variable.

Definition 1.14 (Basic feasible solution for LP). *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $b \in \mathbb{Q}^m$.

Then

- (i) For $x \in \mathbb{Q}^n$, we say \bar{x} is a basic solution of $Ax = b$ if and only if $\{a_i | a_i \text{ is the } i\text{-th column of } A \text{ and } \bar{x}_i > 0\}$ are linear independent.
- (ii) For $x \in \mathbb{Q}_+^n$, we say \bar{x} is a basic feasible solution of

$$Ax = b, x \geq 0$$

if and only if x is a basic solution of $Ax = b$.

Proposition 1.11. *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $b \in \mathbb{Q}^m$.

(S2) x is a solution of $Ax \leq b, x \geq 0$.

(S3) $z = (z_1, \dots, z_m)$ are nonzero slack variables for $Ax + z = b, x, z \geq 0$.

(S4) $I := \{i \in \{1, 2, \dots, m\} | a_i^T x = b_i\}$. Here a_i is the i -th row vector of A .

(S5) $J := \{j \in \{1, 2, \dots, n\} | x_j \neq 0\}$.

Then (x, z) is a basic feasible solution iff $\{a_{i,j}\}_{i \in I}\}_{j \in J}$ are linear independent.

Proof. We set $I' := \{i \in \{1, 2, \dots, m\} | a_i^T x < b_i\}$. (x, z) is a basic feasible solution iff $\{a^j\}_{j \in J} \cup \{e_i\}_{i \in I'}$ are linear independent. Here a^j is the j -th column of A . This is equivalent to $\{a^j - \sum_{i \in I'} a_{i,j} e_i\}_{j \in J} \cup \{e_i\}_{i \in I'}$ are linear independent. So, (x, z) is a basic feasible solution iff $\{a_{i,j}\}_{i \in I}\}_{j \in J}$ are linear independent. \square

Definition 1.15 (Locally ideal). *Here are the settings and assumptions.*

$$(S1) \ A \in M(m, n, \mathbb{Q}), \ G \in M(m, p, \mathbb{Q}), \ B \in M(m, t, \mathbb{Q}), \ b \in \mathbb{Q}^m.$$

$$(S2) \ S \subset \mathbb{Q}^n.$$

$$(S3) \ T(A, G, B, b) := \{(x, y, z) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Z}^t | Ax + Gy + Bz \leq b\}.$$

$$(S4) \ \tilde{S} := \{w \in \mathbb{Q}^M | Cw = c, w \geq 0\} \text{ is a standard equity form of } S \text{ and } \Phi \text{ is the bijection from } S \text{ to } \tilde{S} \text{ in Proposition 1.10.}$$

We say (A, G, B, b) is a locally ideal MILP formulation for S if and only if \tilde{S} has at most one basic feasible solution and for any basic feasible solution of \tilde{S} w , $\Phi^{-1}(w) \in \mathbb{Q}^{n+p} \times \mathbb{Z}^t$.

We will show an example of MILP formulation which is not locally ideal but sharp.

Example 1.1. *Here are the settings and assumptions.*

$$(S1) \ S = \cup_{i=1}^n P_i. \ P_i := \{x \in \mathbb{Q}^n | |x_i| \leq 1, x_j = 0 \ (j \neq i)\} \ (i = 1, 2, \dots, n).$$

Then

(i) *The following is a MILP formulation for S .*

$$y_j - 1 \leq x_i \leq 1 - y_j \ (i = 1, 2, \dots, n, j \neq i), \tag{1.3.1}$$

$$y_i \geq 0, \ (i = 1, 2, \dots, n), \tag{1.3.2}$$

$$\sum_{i=1}^n y_i = 1 \tag{1.3.3}$$

$$y \in \mathbb{Z}^n$$

$$(ii) \ \text{conv}(S) = \{x \in \mathbb{Q}^n | \sum_{i=1}^n |x_i| \leq 1\}$$

(iii) *Equalities and Inequalities in (i) and the following is a sharp MILP formulation for S .*

$$\sum_{i=1}^n r_i x_i \leq 1 \ (r \in \{-1, 1\}^n) \tag{1.3.4}$$

(iv) *If $n = 3$, the formulation in (iii) is not locally ideal.*

(v) *The following is a sharp and locally ideal MILP formulation for S .*

$$-y_i \leq x_i \leq y_i \ (i = 1, 2, \dots, n), \tag{1.3.5}$$

$$y_i \geq 0, \ (i = 1, 2, \dots, n), \tag{1.3.6}$$

$$\sum_{i=1}^n y_i = 1 \tag{1.3.7}$$

$$y \in \mathbb{Z}^n$$

Proof of (i). It is clear. \square

Proof of (ii). The part of \subset is clear. Let us fix any x in the right side. We take $s \geq 1$ such that $\sum_{i=1}^n s|x_i| = 1$. Then

$$x = \sum_{i=1}^n r|x_i| \frac{\text{sign}(x_i)}{r} e_i$$

So, $x \in \text{conv}(S)$. \square

Proof of (iii). We set $T := \{(x, y) \in \mathbb{Q}^n \times \mathbb{Q}^n | (x, y) \text{ satisfies equalities and inequalities of (i)}\}$. Clearly $p_1(T) \subset \text{conv}(S)$. Clearly T is convex. Because $P_i \times \{e_i\} \subset T \ (\forall i)$, $S \subset p_1(T)$. So, $\text{conv}(S) \subset T$. \square

Proof of (iv). Clearly $x_1 = x_2 = y_1 = y_2 = \frac{1}{2}, x_3 = y_3 = 0$ is a feasible solution. We will show this is a basic feasible solution. By Proposition 1.11, it is enough to show the column vectors of

$$\begin{array}{rcccc} & x_1 & x_2 & y_1 & y_2 \\ x_1 \leq 1 - y_1 & 1 & 0 & 0 & 1 \\ x_2 \leq 1 - y_2 & 0 & 1 & 1 & 0 \\ y_1 + y_2 = 1 & 0 & 0 & 1 & 1 \\ x_1 + x_2 = 1 & 1 & 1 & 0 & 0 \end{array}$$

are linear independent. Because this matrix is nonsingular, the column vectors of this matrix are linear independent. \square

Proof of (v). By the same argument as the proof of (iii), we can show this formulation is sharp. For locally ideal property, it is enough to show for any basic feasible solution (x^+, x^-, y, z) there is $\#\{i|y_i \neq 0\} = 1$. Because $\sum_{i=1}^n y_i = 1$, $\#\{i|y_i \neq 0\} \geq 1$. For aiming contradiction, let us assume $\#\{i|y_i \neq 0\} > 1$. So, there are $i_1 \neq i_2$ such that $y_{i_1}, y_{i_2} > 0$. We can assume $i_1 = 1, i_2 = 2$. We will show in each case of the followings.

case1 $|x_1| < y_1$ or $|x_2| < y_2$.

case2 $|x_1| = y_1$ and $|x_2| = y_2$.

In case1, we can assume $|x_1| < y_1$. If $|x_2| < y_2$, then By Proposition 1.11, the columns vectors of the following matrix are linear independent.

$$\begin{array}{rcc} & y_1 & y_2 \\ * & 0 & 0 \\ \dots & \dots & \dots \\ * & 0 & 0 \\ \sum_i y_i = 1 & 1 & 1 \end{array}$$

This is contradiction. So, $|x_{i_2}| = y_{i_2}$. By Proposition 1.11, the columns vectors of the following matrix are linear independent.

$$\begin{array}{rcccc} & y_1 & y_2 & x_2^* \\ * & 0 & 0 & 0 \\ \dots & \dots & \dots & 0 \\ * & 0 & 0 & 0 \\ q_2 y_2 + r_2 x_2 \leq 0 & 0 & q_2 & r_2 \\ \sum_i y_i = 1 & 1 & 1 & 0 \end{array}$$

Here, $q_2 r_2 \neq 0$. So, the columns vectors of the following matrix are linear independent.

$$\begin{array}{rcccc} & y_1 & y_2 & x_2^* \\ * & 0 & 0 & 0 \\ \dots & \dots & \dots & 0 \\ * & 0 & 0 & 0 \\ q_2 y_2 + r_2 x_2 \leq 0 & 0 & 0 & r_2 \\ \sum_i y_i = 1 & 1 & 0 & 0 \end{array}$$

This is contradiction.

In case2, By Proposition 1.11, the columns vectors of the following matrix are linear independent.

$$\begin{array}{rccccc} & y_1 & y_2 & x_1^* & x_2^* \\ * & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ * & 0 & 0 & 0 & 0 \\ q_1 y_1 + r_1 x_1 \leq 0 & q_1 & 0 & r_1 & 0 \\ q_2 y_2 + r_2 x_2 \leq 0 & 0 & q_2 & 0 & r_2 \\ \sum_i y_i = 1 & 1 & 1 & 0 & 0 \end{array}$$

Here, $q_1 r_1 q_2 r_2 \neq 0$. So, the columns vectors of the following matrix are linear independent.

$$\begin{array}{rccccc} & y_1 & y_2 & x_1^* & x_2^* \\ * & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ * & 0 & 0 & 0 & 0 \\ q_1 y_1 + r_1 x_1 \leq 0 & 0 & 0 & r_1 & 0 \\ q_2 y_2 + r_2 x_2 \leq 0 & 0 & 0 & 0 & r_2 \\ \sum_i y_i = 1 & 1 & 1 & 0 & 0 \end{array}$$

This is contradiction.

Consequently, $\#\{i|y_i \neq 0\} \leq 1$. □

Memo 1.1. We measured execution times in three formulations in Example1.1. Here are the settings.

Version of SCIP: SCIP9.0.0.0

Target Machine: Ubuntu Desktop 22.04

Host Machine: Windows10

CPU:Inter Core i7-6700T@2.8GHz

DRAM: 32GB

Here are the problem.

$$\text{maximize } \sum_{i=1} c_i x_i, x \in S$$

For $n = 1000$, the execution times are below.

Simple formulation(i) : 119sec

Locally ideal formulation(v) : 0.2sec

For sharp formulation(iii) and $n = 17$, the execution time is 229sec. See [5] for sample code.

Proposition 1.12. Here are the settings and assumptions.

(S1) $A \in M(m, n, \mathbb{Q}), b \in \mathbb{Q}^m$.

(S2) $S := \{x \in \mathbb{Q}^{n_1} \times \mathbb{Q}^{n_2} | Ax \leq b\}$.

(S3) $T(A, G, B, b) := \{(x, y, z) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Z}^t | Ax + Gy + Bz \leq b\}$.

(S4) $\tilde{S} := \{w \in \mathbb{Q}^M | Cw = c, w \geq 0\}$ is a standard equity form of S and Φ is the bijection from S to \tilde{S} in Proposition1.10.

Then

(i) For any $x \in \text{ex}(S)$, $\Phi(x)$ is a basic feasible solution.

(ii) For any $x \in S \setminus \{0\}$ such that $\Phi(x)$ is a basic feasible solution, $\Phi(x) \in \text{ex}(\tilde{S})$.

(iii) Let us assume $S \subset [0, \infty)^{n_1+n_2}$. Then for any $x \in S$ such that $\Phi(x)$ is a basic feasible solution, $x \in \text{ex}(S)$.

Proof of (i). Let z is a slack variable for $Ax \leq b$. $I_n := \{i|x_i \neq 0\}$. $J_0 := \{j|z_j = 0\}$. If $J_0 = \emptyset$, then $Ax < b$. So, there is $x^1, x^2 \in \mathbb{Q}^n$ $t \in (0, 1)$ such that $Ax^1 Ax^2 < b$ and $x = tx^1 + (1-t)x^2$. This is contradiction. So, $J_0 \neq \emptyset$. If $x = 0$, $\Phi(x)$ is clearly basic feasible solution. So, we can assume $x \neq 0$. It is enough to show $\{a_{i,j}\}_{i \in I_n, j \in J_0}$ are linear independent. For aiming contradiction, let us assume $\{a_{i,j}\}_{i \in I_n, j \in J_0}$ are linear dependent. We set $k := \#I_n$ and

$$A' := \{a_{i,j}\}_{i \in I_n, j \in \{1,2,\dots,n\}}, A'' := \{a_{i,j}\}_{i \notin I_n, j \in \{1,2,\dots,n\}}, b' := \{b_i\}_{i \in I_n}, b'' := \{b_i\}_{i \notin I_n}$$

Then there is a $c \in \mathbb{Q}^n \setminus \{0\}$ such that

$$c_i = 0 \ (\forall i \notin I_n), x + sc \text{ is a solution of } A'x = b' \ (\forall s \in \mathbb{R})$$

Because $A''x < b''$, there is $s > 0$ such that $A''(x + sc) < b''$ and $A''(x - sc) < b''$. This means that $x \notin \text{ex}(S)$. This is contradiction. □

Proof of (ii). Let us fix any $x \in S \setminus \{0\}$ such that $\Phi(x)$ is a basic feasible solution. We can assume $|x_1|, \dots, |x_k| > 0, x_{k+1} = \dots = x_n = 0$ and

$$\begin{aligned} a_{1,1}x_1 + \dots + a_{1,k}x_k &= b_1 \\ &\dots \\ a_{l,1}x_1 + \dots + a_{l,k}x_k &= b_l \end{aligned}$$

and

$$\text{rank} \begin{pmatrix} a_{1,1} & \dots & a_{1,k} \\ \dots & \dots & \dots \\ a_{l,1} & \dots & a_{l,k} \end{pmatrix} = k$$

So, the equation

$$\begin{aligned} a_{1,1}\bar{x}_1 + \dots + a_{1,k}\bar{x}_k &= b_1 \\ &\dots \\ a_{l,1}\bar{x}_1 + \dots + a_{l,k}\bar{x}_k &= b_l \end{aligned}$$

has the unique solution.

For aiming contradiction, let us assume that $\Phi(x) \notin \text{ex}(S)$. Then there are $x^1 := (x^{1,+}, x^{1,-}, z^1)$ and $x^2 := (x^{2,+}, x^{2,-}, z^2)$ and $t \in (0, 1)$ such that $x = tx^1 + (1-t)x^2$. So, $x_i^{j,+} = x_i^{j,-} = 0$ ($\forall i > k, j = 1, 2$) and $x_i^{j,+} = \delta_{+, \text{sign}(x_i)} x_i^{j,+}$ ($\forall i \leq k, j = 1, 2$) and $x_i^{j,-} = \delta_{+, \text{sign}(x_i)} x_i^{j,-}$ ($\forall i \leq k, j = 1, 2$). This implies $(x_1^{1, \text{sign}(x_1)}, \dots, x_k^{1, \text{sign}(x_1)})$ and $(x_1^{2, \text{sign}(x_2)}, \dots, x_k^{2, \text{sign}(x_2)})$ satisfies the equation

$$\begin{aligned} a_{1,1}\bar{x}_1 + \dots + a_{1,k}\bar{x}_k &= b_1 \\ &\dots \\ a_{l,1}\bar{x}_1 + \dots + a_{l,k}\bar{x}_k &= b_l \end{aligned}$$

This is contradiction. □

Proof of (iii). (iii) is followed by the same argument of the proof of (ii). □

Definition 1.16 (Affine combination, Affine independent).

(i) For $x_1, \dots, x_m \in \mathbb{Q}^n$,

$$\sum_{i=1}^m \lambda_i x_i, \lambda_1, \dots, \lambda_m \in \mathbb{Q}, \sum_{i=1}^m \lambda_i = 1$$

is called an affine combination of x_1, \dots, x_m .

(ii) We say $x_1, \dots, x_m \in \mathbb{Q}^n$ are affinely independent if for any $\lambda_1, \dots, \lambda_m \in \mathbb{Q}$ such that $\sum_{i=1}^m \lambda_i = 0$ and $\sum_{i=1}^m \lambda_i x_i = 0$, $\lambda = 0$.

Definition 1.17 (Dimension). For $S \subset \mathbb{R}^n$,

$$\dim(S) := \max\{\#A \mid A \text{ is a finite subset of } S \text{ and } A \text{ is affinely independent}\} - 1$$

Definition 1.18 (Pointed set). We say convex subset $S \subset \mathbb{R}^n$ is pointed if and only if $\text{ex}(S) \neq \emptyset$.

Proposition 1.13. Here are the settings and assumptions.

(S1) $A \in M(m, n, \mathbb{Q})$, $b \in \mathbb{Q}^m$.

(S2) $P := \{x \in \mathbb{Q}_+^n \mid Ax \leq b\}$.

(A1) $P \neq \emptyset$.

Then P is pointed.

Proof. For $y \in P$, We set

$$N(y) := \#\{i \mid y_i \neq 0\}, M(y) := \#\{j \mid a_j^T y = b_j\}$$

Here, a_j is the j -th row vector of A . We set

$$K := \max\{N(y) \mid y \in P\}$$

If $K = n$, clearly $0 \in \text{ex}(P)$. So, we can assume $K < n$. We set

$$L := \max\{M(y) \mid y \in P, N(y) = K\}$$

Because $K < n$, $L > 0$. There is $x \in P$ such that $N(x) = K, M(x) = L$. We set $k := n - K$. We can assume $x_1, \dots, x_k > 0, x_{k+1} = \dots = x_n = 0$ and

$$A'x' = b'$$

Here, $A' := \{a_{i,j}\}_{i=1,\dots,L,j=1,2,\dots,n}$, $x' := (x_1, \dots, x_k)$, $b' := (b_1, \dots, b_L)^T$. For aiming contradiction, let us assume $\text{rank}(A') < n$. there is $r' \in \mathbb{Q}^k$ such that

$$A'(x' + tr) = b' \quad (\forall t \in \mathbb{R})$$

So, there is $y \in P$ such that $N(y) > N(x)$ or $M(x) < M(y)$. This is contradiction. So, $\text{rank}(A') = n$. From this, $x \in \text{ex}(P)$. \square

Definition 1.19 (Edge). *Let P be a nonempty polyhedron in \mathbb{R}^n . We call a face of P whose dimension is 1 an edge of P .*

Proposition 1.14. *Here are the settings and assumptions.*

(S1) $a_1, \dots, a_k \in \mathbb{Q}^n$.

(A1) For any i , $a_i \notin \text{conv}(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_k)$.

Then

$$\text{ex}(\text{conv}(a_1, \dots, a_k)) = \{a_1, \dots, a_k\}$$

Proof. By Proposition 1.7, it is enough to show *subset* part. Let us assume there is i such that $a_i \notin \text{ex}(\text{conv}(a_1, \dots, a_k))$. We can assume $i = k$. So there are $\lambda_1, \dots, \lambda_k, \eta_1, \dots, \eta_k, t \in (0, 1)$ such that $\sum_{i=1}^k \lambda_i = 1$ and $\sum_{i=1}^k \eta_i = 1$ and $a_k = t \sum_{i=1}^k \lambda_i a_i + (1-t) \sum_{i=1}^k \eta_i a_i$ and $\sum_{i=1}^k \lambda_i a_i \neq a_k$ and $\sum_{i=1}^k \eta_i a_i \neq a_k$. So, $t\lambda_k + (1-t)\eta_k < 1$. This implies

$$a_k = \sum_{i=1}^{k-1} \frac{t\lambda_i + (1-t)\eta_i}{1 - t\lambda_k - (1-t)\eta_k} a_i$$

So, $a_k \in \text{conv}(a_1, \dots, a_{k-1})$. This is contradiction. \square

Proposition 1.15. *Here are the settings and assumptions.*

(S1) $P \subset \mathbb{R}^n$ is a Polyhedron.

(S2) $a_1, \dots, a_k, r_1, \dots, r_l \in \mathbb{Q}^n$ such that $P = \text{conv}(a_1, \dots, a_k) + \text{cone}(r_1, \dots, r_l)$.

(A1) For any i , $a_i \notin \text{conv}(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_k)$.

Then

$$\text{ex}(P) \subset \{a_1, \dots, a_k\}$$

Proof. By Proposition 1.7, it is enough to show $\text{ex}(P) \subset \text{conv}(a_1, \dots, a_k)$. Let us fix any $x \in P \setminus \text{conv}(a_1, \dots, a_k)$. There are $y \in \text{conv}(a_1, \dots, a_k)$ and $z \in \text{cone}(r_1, \dots, r_l) \setminus \{0\}$ such that $x = y + z$. Because $2z, 0 \in \text{cone}(r_1, \dots, r_l)$, $y, y + 2z \in P$. So, $z = \frac{1}{2}(y + y + 2z)$. This means $x \notin \text{ex}(P)$. Consequently, $\text{ex}(P) \subset \text{conv}(a_1, \dots, a_k)$. \square

Proposition 1.16. *Here are the settings and assumptions.*

(S1) $P \subset \mathbb{R}^n$ is a Polyhedron.

(A1) P is pointed.

Then there are $a_1, \dots, a_k, r_1, \dots, r_l \in \mathbb{Q}^n$

$$P = \text{conv}(a_1, \dots, a_k) + \text{cone}(r_1, \dots, r_l), \text{ex}(P) = \{a_1, \dots, a_k\}, 0 \notin \text{cone}(r_1, \dots, r_l)$$

Proof. By Minkowski-Weyl Theorem, Then there are $a_1, \dots, a_k, r_1, \dots, r_l \in \mathbb{Q}^n$

$$P = \text{conv}(a_1, \dots, a_k) + \text{cone}(r_1, \dots, r_l)$$

For aiming contradiction, let us assume $0 \notin \text{ex}(\text{cone}(r_1, \dots, r_l))$. Then there are $z_1 \neq z_2 \in \text{cone}(r_1, \dots, r_l)$ such that $0 = \frac{1}{2}(z_1 + z_2)$. For any i ,

$$a_i = \frac{1}{2}((a_i + z_1) + (a_i + z_2))$$

This implies $a_i \notin \text{ex}(P)$. By Proposition 1.15, $\text{ex}(P) = \emptyset$. This is contradiction. So, $0 \in \text{ex}(\text{cone}(r_1, \dots, r_l))$.

By dropping some elements if necessary, we can assume that for each i

$$a_i \notin \text{conv}(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_k) + \text{cone}(r_1, \dots, r_l)$$

For aiming contradiction, let us assume $a_i \notin \text{ex}(P)$. We can assume $i = k$. Then there are $y_1, y_2 \in \text{conv}(a_1, \dots, a_k)$ and $z_1, z_2 \in \text{cone}(r_1, \dots, r_l)$, $t \in (0, 1)$ such that

$$a_k = t(y_1 + z_1) + (1-t)(y_2 + z_2), y_1 + z_1 \neq a_k, y_2 + z_2 \neq a_k$$

There are $\lambda_1, \dots, \lambda_k, \eta_1, \dots, \eta_k, t \in (0, 1)$ such that $\sum_{i=1}^k \lambda_i = 1$ and $\sum_{i=1}^k \eta_i = 1$ and $y_1 = \sum_{i=1}^k \lambda_i a_i$ and $y_2 = \sum_{i=1}^k \eta_i a_i$. If $y_1 = y_2 = a_k$, $0 = tz_1 + (1-t)z_2$. This contradicts with $0 \notin \text{ex}(P)$. So, $y_1 \neq a_k$ or $y_2 \neq a_k$. This implies $t\lambda_k + (1-t)\eta_k < 1$. So,

$$a_k = \frac{1}{1 - t\lambda_k - (1-t)\eta_k} \left(\sum_{i=1}^k (t\lambda_i + (1-t)\eta_i) a_i + tz_1 + (1-t)z_2 \right)$$

This means $a_k \in \text{conv}(a_1, \dots, a_{k-1}) + \text{cone}(r_1, \dots, r_l)$. This is contradiction. \square

Proposition 1.17. *Here are the settings and assumptions.*

(S1) $S \subset \mathbb{Q}^n$.

(A1) (A, B, D, b) is a locally ideal MILP formulation for S .

(S2) We set

$$P := \{(x, u, y) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Q}^t \mid Ax + Bu + Dy \leq b\}, P_I := \{(x, u, y) \in P \mid y \in \mathbb{Z}^t\}$$

$$p : P \ni (x, u, y) \mapsto x \in \mathbb{Q}^n$$

(A2) P is pointed.

then (A, B, D, b) is a sharp formulation for S .

Proof. By Proposition 1.16, there are $a_1, \dots, a_k, r_1, \dots, r_l \in \mathbb{Q}^n$

$$P = \text{conv}(a_1, \dots, a_k) + \text{cone}(r_1, \dots, r_l), \text{ex}(P) = \{a_1, \dots, a_k\}, 0 \notin \text{cone}(r_1, \dots, r_l),$$

$$a_i = (\hat{x}^i, \hat{u}^i, \hat{y}^i) \ (i = 1, 2, \dots, k), r_j = (\tilde{x}^j, \tilde{u}^j, \tilde{y}^j) \in \mathbb{Z}^n \times \mathbb{Z}^s \times \mathbb{Z}^t \ (j = 1, 2, \dots, l)$$

By the assumption of locally idealness and Proposition 1.12, $\hat{y}^i \in \mathbb{Z}^t \ (\forall i)$.

Let us fix any $(x, u, y) \in P$. There are $\lambda_1, \dots, \lambda_k, \eta_1, \dots, \eta_l \in [0, 1] \in \mathbb{Q}$ such that $\sum_{i=1}^k \lambda_i = 1$ and $\sum_{i=1}^l \eta_i = 1$ and

$$(x, u, y) = \sum_{i=1}^k \lambda_i (\hat{x}^i, \hat{u}^i, \hat{y}^i) + \sum_{j=1}^l \eta_j (\tilde{x}^j, \tilde{u}^j, \tilde{y}^j)$$

We set

$$(x^1, u^1, y^1) := \sum_{i=1}^k \lambda_i (\hat{x}^i, \hat{u}^i, \hat{y}^i)$$

Because $y^1 \in \mathbb{Z}^t$ and (A, B, D, b) is a MILP formulation for S , $x^1 \in S$. Without loss of generality, we can assume $\lambda_1 > 0$.

There is $\alpha \in \mathbb{Z} \cap (1, \infty)$ such that $\frac{\alpha}{\lambda_1} \sum_{j=1}^l \eta_j \in \mathbb{Z}^t$. We set

$$(x^2, u^2, y^2) := \sum_{i=1}^k \lambda_i (\hat{x}^i, \hat{u}^i, \hat{y}^i) + \alpha \sum_{j=1}^l \eta_j (\tilde{x}^j, \tilde{u}^j, \tilde{y}^j)$$

Then

$$(x^2, u^2, y^2) = \lambda_1 ((\hat{x}^1, \hat{u}^1, \hat{y}^1) + \frac{\alpha}{\lambda_1} \sum_{j=1}^l \eta_j (\tilde{x}^j, \tilde{u}^j, \tilde{y}^j)) + \sum_{i=2}^k \lambda_i (\hat{x}^i, \hat{u}^i, \hat{y}^i) \in \text{conv}(a_1, \dots, a_k)$$

So, $x^2 \in \text{conv}(p(P))$. So, $x = (1 - \frac{1}{\alpha})x^1 + \frac{1}{\alpha}x^2 \in \text{conv}(p(P))$. Consequently, (A, B, D, b) is a sharp formulation. \square

Proposition 1.18. *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $b \in \mathbb{Q}^m$.

(S2) $S := \{x \in \mathbb{Q}_+^{n_1} \times \mathbb{Z}_+^{n_2} \mid Ax \leq b\}$.

(S3) $\tilde{A} := \begin{pmatrix} A_1 & A_2 \\ O_{n_2, n_1} & E_{n_2} \end{pmatrix}$. Here $A_1 := (a^1, \dots, a^{n_1})$ and $A_2 := (a^{n_1+1}, \dots, a^n)$ and each a^i is the i -th column vector of A . $B := \begin{pmatrix} O_{m, n_2} \\ -E_{n_2} \end{pmatrix}$, $\tilde{b} := \begin{pmatrix} b \\ 0_{n_2} \end{pmatrix}$
(A1) $Q := \{x \in \mathbb{Q}_+^{n_1} \times \mathbb{Q}_+^{n_2} \mid Ax \leq b\}$ has at least one basic feasible solution.

Then $(\tilde{A}, B, \tilde{b})$ is a locally ideal formulation for S iff $(\tilde{A}, B, \tilde{b})$ is a sharp formulation for S .

Proof. Proposition 1.13 and Proposition 1.17, it is enough to show ‘if’ part. Let us assume $(\tilde{A}, B, \tilde{b})$ is a sharp formulation for S . Then

$$\text{conv}(S) = p(\{(x', x'', y) \in \mathbb{Q}^{n_1} \times \mathbb{Q}^{n_2} \times \mathbb{Q}^n \mid (x', x'') \in Q, x'' = y\}) = Q$$

By Theorem 1.8, there are $\mathbf{a}_1, \dots, \mathbf{a}_k \in S$ and $\mathbf{r}_1, \dots, \mathbf{r}_l \in \mathbb{Z}^n$ such that

$$Q = \text{conv}(\mathbf{a}_1, \dots, \mathbf{a}_k) + \text{cone}(\mathbf{r}_1, \dots, \mathbf{r}_l)$$

We can assume $\mathbf{a}_1, \dots, \mathbf{a}_k$ are distinct. Let us fix any x which is a basic feasible solution of Q . By Proposition 1.12, $x \in \text{ex}(Q)$. We will show $x \in S$. There are $\lambda_1, \dots, \lambda_k, \eta_1, \dots, \eta_l \in [0, 1]$ such that $\sum_{i=1}^k \lambda_i = \sum_{j=1}^l \eta_j = 1$ and

$$x = \sum_{i=1}^k \lambda_i \mathbf{a}_i + \sum_{j=1}^l \eta_j \mathbf{r}_j$$

For aiming contradiction, let us assume there is j such that $\eta_j > 0$. We can assume $j = 1$. Then We set

$$x^1 = \sum_{i=1}^k \lambda_i \mathbf{a}_i + \frac{1}{2} \eta_1 \mathbf{r}_1 + \sum_{j=2}^l \eta_j \mathbf{r}_j, x^2 = \sum_{i=1}^k \lambda_i \mathbf{a}_i + \frac{3}{2} \eta_1 \mathbf{r}_1 + \sum_{j=2}^l \eta_j \mathbf{r}_j$$

Then $x^1 \neq x^2$ and $x = \frac{1}{2}(x^1 + x^2)$. This contradicts with $x \in \text{ex}(Q)$. So,

$$x = \sum_{i=1}^k \lambda_i \mathbf{a}_i$$

For aiming contradiction, let us assume there is $i_1 \neq i_2$ such that $\lambda_{i_1}, \lambda_{i_2} > 0$. We can assume $i_1 = 1, i_2 = 2$. We set

$$y^1 = (\lambda_1 + \lambda_2) \mathbf{a}_1 + \sum_{i=3}^k \lambda_i \mathbf{a}_i, y^2 = (\lambda_1 + \lambda_2) \mathbf{a}_2 + \sum_{i=3}^k \lambda_i \mathbf{a}_i$$

Because $\mathbf{a}_1 \neq \mathbf{a}_2$, $y^1 \neq y^2$. And $x = \frac{\lambda_1}{\lambda_1 + \lambda_2} y^1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} y^2$. This contradicts with $x \in \text{ex}(Q)$. So, $x \in S$. \square

2 Event graph analysis

2.1 Max-plus algebra

Definition 2.1 (Semi-ring). *Here are the settings.*

(S1) R is a set.

(S2) \oplus, \otimes are binomial operators on R .

We say (R, \oplus, \otimes) is a semi ring if

(i) For any $x, y, z \in R$,

$$(x \oplus y) \oplus z = x \oplus (y \oplus z)$$

$$(x \otimes y) \otimes z = x \otimes (y \otimes z)$$

(ii) For any $x, y, z \in R$,

$$x \oplus y = y \oplus x$$

(iii) For any $x, y, z \in R$,

$$x \otimes (y \oplus z) = x \otimes y \oplus x \otimes z$$

- (iv) R has the unit element ϵ with respect to \oplus .
- (v) R has the unit element e with respect to \otimes .
- (vi) $\epsilon \otimes x = x \otimes \epsilon = \epsilon$.

We say R is commutative if \otimes is commutative. We say R is idempotent if \otimes is idempotent.

Definition 2.2 (\mathbb{R}_{max}). Here are the settings.

(S1) $\mathbb{R}_{max} := \mathbb{R} \cup \{-\infty\}$. We set $\epsilon := -\infty$ and $e := 0$.

(S2) For $x, y \in \mathbb{R}_{max}$

$$x \oplus y := \max\{x, y\}$$

$$x \otimes y := x + y$$

We call $\mathcal{R}_{max} := (\mathbb{R}_{max}, \oplus, \otimes)$ the max-plus algebra.

Clearly the following holds.

Proposition 2.1. \mathcal{R}_{max} is a commutative and idempotent semi ring.

2.2 Petri net and Event graph

Definition 2.3 (Petri net, place, transition). Here are the settings.

(S1) $(\mathcal{N}, \mathcal{A})$ is a directed graph.

We say $(\mathcal{N}, \mathcal{A})$ is a petri net if there is $(\mathcal{P}, \mathcal{Q})$ which is a pair of disjoint subsets of \mathcal{N} satisfying the following two conditions.

- (i) $\mathcal{N} = \mathcal{P} \cup \mathcal{Q}, \mathcal{P} \cap \mathcal{Q} = \emptyset$.
- (ii) $\mathcal{A} \subset \mathcal{P} \times \mathcal{Q} \cup \mathcal{Q} \times \mathcal{P}$.

We denote this petri net by $(\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A})$.

We call each element of \mathcal{P} a place and call each element of \mathcal{Q} a transition. Let us fix $p \in \mathcal{P}$ and $q \in \mathcal{Q}$. We say p is the input place of the transition q and q is the output place of the transition p if $(p, q) \in \mathcal{A}$. We say p is the output place of the transition q and q is the input place of the transition p if $(q, p) \in \mathcal{A}$.

We denote the set of all input place of q by $\pi(q)$ and denote the set of all input transition of p by $\pi(p)$.

We denote the set of all output place of q by $\sigma(q)$ and denote the set of all output transition of p by $\sigma(p)$.

Definition 2.4 (Event graph). Here are the settings.

(S1) $(\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A})$ is a petri net.

We say this petri net is an event graph if for each $p \in \mathcal{P}$ there is the unique $q_1 \in \mathcal{Q}$ such that $(p, q_1) \in \mathcal{A}$ and there is the unique $q_2 \in \mathcal{Q}$ such that $(q_2, p) \in \mathcal{A}$.

Definition 2.5 (Enability and Firing in petri net). Here are the settings.

(S1) $(\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A})$ is a petri net.

(S2) $w : \mathcal{A} \rightarrow \mathbb{N}_{\geq 1}$. We call $w(a)$ is the weight of $a \in \mathcal{A}$.

(S3) $M_1 : \mathcal{P} \rightarrow \mathbb{Z}_{\geq 0}$. For each $p \in \mathcal{P}$, we say p is marked with $M_1(p)$ tokens.

(S4) $q \in \mathcal{Q}$.

Then

- (i) We say q is enable if each input place p of q is marked with at least $w(p, q)$ tokens.
- (ii) Let us assume q is enable. We set for each $p \in \mathcal{P}$

$$M_1(p) := M_0(p) + \chi_{\sigma(q)}(p)w(q, p) - \chi_{\pi(q)}(p)w(p, q)$$

We call M_1 the firing of M_0 with respect to q .

Definition 2.6 (Liveness, Autonomous, Time event graph). Here are the settings.

(S1) $G := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_0)$ is an event graph with weight and token.

Then

- (i) We say G is liveness if for any cycle c of G there is $p \in \mathcal{P}$ whose output transition is enable.
- (ii) For each $q \in \mathcal{Q}$, q is a supplier transition if $\pi(q) = \phi$.
- (iii) We say G is autonomous if G is no supplier transitions.
- (iv) Let $\tau : \mathcal{P} \rightarrow \mathbb{Z}_{\geq 0}$ and $\gamma : \mathcal{A} \cap \mathcal{P} \times \mathcal{Q} \rightarrow \mathbb{Z}_{\geq 0}$ such that

$$\gamma(p, q) \leq \tau(p)$$

Then (G, τ, γ) with time event graph.

Definition 2.7 (Enability and Firing in Time event graph). Here are the settings.

- (S1) $G := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_0, \tau, \gamma_0)$ is a time event graph.
- (A1) For any $q_1, q_2 \in \mathcal{Q}$, there is at most one $p \in \mathcal{P}$ such that $(q, p), (p, q) \in \mathcal{A}$.
- (A2) $w = 1$ on \mathcal{A} .
- (S2) $q \in \mathcal{Q}$.

Then

- (i) We say q is enable if each input place p of q is marked with at least $w(p, q)$ tokens and $\tau(p) \leq \gamma(p, q)$. We denote the all enable transitions by $E(G)$.
- (ii) Let us assume q is enable. We set for each $p \in \mathcal{P}$

$$M_1(p) := M_0(p) + \chi_{\sigma(q)}(p)w(p, q) - \chi_{\pi(q)}(p)w(p, q), \gamma_1(p) := 0$$

We call (M_1, γ_1) the firing of (M_0, γ_0) with respect to q .

Clearly the following holds.

Proposition 2.2. Here are the settings.

- (S1) $G_0 := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_0, \tau, \gamma_0)$ is a time event graph.
- (A1) For any $q_1, q_2 \in \mathcal{Q}$, there is at most one $p \in \mathcal{P}$ such that $(q, p), (p, q) \in \mathcal{A}$.
- (A2) $w = 1$ on \mathcal{A} .
- (S3) We set

$$M_1(p) := M_0(p) + \chi_{E(G_0)}(q_1) - \chi_{E(G_0)}(q_2)$$

Here $q_1 \in \pi(p)$ and $q_2 \in \sigma(p)$. And

$$\gamma_1(p, q) := \begin{cases} \gamma_0(p, q) + 1 & M_0(p) > 0 \text{ and } q \text{ is not enable} \\ 0 & \text{otherwise} \end{cases}$$

- (S4) We set $G_1 := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_1, \tau, \gamma_1)$.

Then G_1 is a time event graph.

Definition 2.8 (Firing time). Here are the settings.

- (S1) $G_0 := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_0, \tau, \gamma_0)$ is a time event graph.
- (A1) For any $q_1, q_2 \in \mathcal{Q}$, there is at most one $p \in \mathcal{P}$ such that $(q, p), (p, q) \in \mathcal{A}$.
- (A2) $w = 1$ on \mathcal{A} .
- (S3) We define $\{G_t\}_{t=0}^{\infty}$ inductively by the procedure defined in Proposition 2.2.

Then

$$x_q(k) := \{t_0 \in \mathbb{Z}_{\geq 0} | k = \#\{t \leq t_0 | q \in E(G_t)\}\} \quad (q \in \mathcal{Q}, k \in \mathbb{N}_{\geq 1})$$

We call $x_q(k)$ the k -th firing time of q . We set

$$x(k) := (x_{q_1}(k), \dots, x_{q_{\#\mathcal{Q}}}(k))^T \quad (k \in \mathbb{N}_{\geq 1})$$

Definition 2.9 (System Matrix). *Here are the settings.*

(S1) $\{G_t := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_t, \tau, \gamma_t)\}_{t \in \mathbb{Z}_{\geq 0}}$ is a sequence of time event graphs by the procedure defined in Proposition 2.2.

(S2) $\{x(k)\}_{k=1}^{\infty}$ is the sequence by Definition 2.8.

(S3) We denote the maximum number of tokens at any one place in $\{G_t\}_{t \in \mathbb{Z}_{\geq 0}}$ by M .

Then for each $m \in \{0, 1, \dots, M\}$

$$[A_m]_{j,l} := \begin{cases} a_{j,l} & p_{j,l} \text{ exists and } p_{j,l} \text{ has } m \text{ tokens in } G_0 \\ \epsilon & \text{otherwise} \end{cases} \quad (j, l = 1, 2, \dots, \#\mathcal{Q})$$

Here $p_{j,l}$ is the place such that $(q_j, p_{j,l}), (p_{j,l}, q_l) \in \mathcal{A}$.

Proposition 2.3. *We succeed notations in Definition 2.9. And let us assume any G_t is autonomous. Then*

$$x(k) = A_0 \otimes x(k) \oplus A_1 \otimes x(k-1) \oplus \dots \oplus A_M \otimes x(k-M) \quad (k = M+1, M+2, \dots)$$

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