A study memo on applied mathematics

Contents

This is a study memo of [1], [3].

1 Integer Programming

1.1 MILP and Branch-and-Bound Method

Definition 1.1 (MILP:Mixed integer linear programming)**.** *Let*

(S1) $A \in M(m, n, \mathbb{Q})$, $G \in M(m, p, \mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

(S2) $S := \{(x, y) \in (\mathbb{Z}_+)^n \times (\mathbb{R}_+)^p | g(x, y) := Ax + Gy \leq b\}$

We call the following problem a MILP.

$$\begin{aligned} &max\ f(x, y) := c^t x + h^t y \\ &subject\ to\ (x, y) \in S \end{aligned}$$

We succeed notations in Definition1.1. And we set

$$S^0 := \{(x, y) \in (\mathbb{R}_+)^n \times (\mathbb{R}_+)^p | Ax + Gy \leq b\}$$

Let us assume the MILP has a opmimal solution (x^*, y^*) and the optimal optimal value z^*. So $S^0 \neq \phi$. Let us fix $(x, y) \in S^0$.

Algorithm Branch-and-Bound Method

Input: $S^0 \neq \phi$

Step 1: Take a $(x^0, y^0) \in S^0$ and $(\underline{x}, \underline{y}, \underline{z}) \leftarrow (x_0, y_0, f(x^0, y^0))$ and $\mathcal{S} \leftarrow S_0$

Step 2: Take $j \in \{1, 2, ..., n\}$. $S_{00} := \{(x, y) \in S | x_j \leq \lfloor x_j^0 \rfloor\}$ and $S_{01} := \{(x, y) \in S | x_j \geq \lceil x_j^0 \rceil\}$ and
MILP$_{00}$: $max f(S_{00})$ and MILP$_{01}$: $max f(S_{01})$.
Delete S_0 from $\mathcal{S}$ and add S_{00} and S_{01} to $\mathcal{S}$.

Step 3: **for** $S_\alpha \in \mathcal{S}$ **do**
    Solve $LP_\alpha : max f(S_\alpha)$.
    if LP_α is not feasible **then**
        Delete S_α from $\mathcal{S}$.
    else
        We set (x^α, y^α) which is a optimal solution and z^α which is its optimal value.
        Delete S_α from $\mathcal{S}$.
        if $x^\alpha \in \mathbb{Z}_+^n$ **then**
            if $z^\alpha > \underline{z}$ **then**
                $(\underline{x}, \underline{y}, \underline{z}) \leftarrow (x^\alpha, y^\alpha, f(x^\alpha, y^\alpha))$.
            end if
        else$z^\alpha > \underline{z}$
            Take $j \in \{1, 2, ..., n\}$. $S_{\alpha 0} := \{(x, y) \in S_\alpha | x_j \leq \lfloor x_j^\alpha \rfloor\}$ and $S_{\alpha 1} := \{(x, y) \in S_\alpha | x_j \geq \lceil x_j^\alpha \rceil\}$.
            Add $S_{\alpha 0}$ and $S_{\alpha 1}$ to $\mathcal{S}$.
        end if
    end if
end for

Output: $(\underline{x}, \underline{y}, \underline{z})$.

1.2 Meyer's Fundamental Theorem

1.2.1 Main result

The propositions shown in this subsection will not be presented with proofs in this subsection, but will be presented with proofs in the subsections that follow.

Definition 1.2 (Polyhedron)**.** *Let* $A \in M(m, n, \mathbb{R}), b \in \mathbb{R}^m$. *We call*

$$P := \{x \in \mathbb{R}^n | Ax \leq b\}$$

a Polyhedron in $\mathbb{R}^n$. *If* $A \in M(m, n, \mathbb{Q}), b \in \mathbb{Q}^m$ *then* P *is a rational polyhedron.*

Definition 1.3 (Recession cone). *Let P be a nonempty polyhedron. We call*

$$rec(P) := \{r \in \mathbb{R}^n | x + \lambda r \in P, \lambda \in \mathbb{R}_+\}$$

the recession cone of P.

Notation 1.1. *Let*

(S1) $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

We set

$$P(A,G,b) := \{(x,y) \in (\mathbb{R}_+)^n \times (\mathbb{R}_+)^p | g(x,y) := Ax + Gy \leq b\}$$

Definition 1.4 (Convex, Convex combination). *Let $A \subset \mathbb{R}^n$. We say A is convex if $\sum_{i=1}^n \lambda_i a_i \in A$ for $a_1, ..., a_n \in A$ and $\lambda_1, ..., \lambda_n \subset [0,1]$ such that $\sum_{i=1}^n \lambda_i = 1$. We call the sum*

$$\sum_{i=1}^n \lambda_i a_i$$

convex combination of $a_1, ..., a_n$.

Proposition 1.1. *Let*

(S1) $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

(S2) $S := \{(x,y) \in (\mathbb{Z}_+)^n \times (\mathbb{R}_+)^p | g(x,y) := Ax + Gy \leq b\}$

Then

(i)

$$sup\{c^t x + h^t y | (x,y) \in S\} = sup\{c^t x + h^t y | (x,y) \in conv(S)\}$$

Furthermore, there is $(x,y) \in S$ such that $c^t x + h^t y = sup\{c^t x + h^t y | (x,y) \in S\}$ $\iff$ there is $(x,y) \in conv(S)$ such that $c^t x + h^t y = sup\{c^t x + h^t y | (x,y) \in S\}$

(ii) $ex(conv(S)) \subset S$

Theorem 1.1 (Meyer(1974)[2] Fundamental Theorem). *Here are the settings and assumptions.*

(S1) $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $b \in \mathbb{Q}^m$.

(S2) $S := \{(x,y) \in P(A,G,b) | x \in (\mathbb{Z}_+)^n\}$.

Then there are $A' \in M(m,n,\mathbb{Q})$, $G' \in M(m,p,\mathbb{Q})$, $b' \in \mathbb{Q}^m$ such that

$$conv(S) = P(A', G', b')$$

By Proposition1.1 and Theorem1.1, MILP

$$\begin{aligned} &\max\ f(x,y) := c^t x + h^t y \\ &\text{subject to } (x,y) \in S \end{aligned}$$

is equal to a pure LP

$$\begin{aligned} &\max\ f(x,y) := c^t x + h^t y \\ &\text{subject to } (x,y) \in P(A', G', b') \end{aligned}$$

We set

$$\tilde{A} := \begin{pmatrix} A \\ A' \end{pmatrix}, \tilde{G} := \begin{pmatrix} G \\ G' \end{pmatrix}, \tilde{b} := \begin{pmatrix} b \\ b' \end{pmatrix},$$

Then clearly

$$S = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^p | (x,y) \in P(\tilde{A}, \tilde{G}, \tilde{b}), x \in \mathbb{Z}^n\}$$

and MILP

$$\begin{aligned} &\max\ f(x,y) := c^t x + h^t y \\ &\text{subject to } (x,y) \in S \end{aligned}$$

has a continuous ralaxation

$$\max\ f(x,y) := c^t x + h^t y$$
$$\text{subject to } (x,y) \in P(\tilde{A},\tilde{G},\tilde{b})$$

whose optimal value is equal to the one of the original MILP. And we can effectively find an optimal solution of this continuas ralaxation which is contained in S.

From the above discussion, the following can be shown.

Proposition 1.2. *Here are the settings and assumptions.*

(S1) $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

(S2) $S := \{(x,y) \in P(A,G,b) | x \in (\mathbb{Z}_+)^n\}$.

Then there is $M \in \mathbb{N}$ *and are* $\tilde{A} \in M(M,n,\mathbb{Q})$, $\tilde{G} \in M(M,p,\mathbb{Q})$, $\tilde{b} \in \mathbb{Q}^M$ *such that*

$$S = P(\tilde{A},\tilde{G},\tilde{b}) \cap \mathbb{Z}_+^n \times \mathbb{R}_+^p$$

and

$$conv(S) = P(\tilde{A},\tilde{G},\tilde{b})$$

1.2.2 Fourier elimination and Farkas Lemma

Definition 1.5 (Conic combination). *Let* $v_1,...,v_m \in \mathbb{R}^n$. *For every* $\lambda_1,...,\lambda_m \geq 0$, *we call* $\sum_{i=1}^m \lambda_i v_i$ *a conic combination of* $v_1,...,v_m$.

Theorem 1.2 (Fourier Elimination). *Let*

(S1) $A \in M(m,n,\mathbb{R})$, $b \in \mathbb{R}^m$.

(S2) $I^+ := \{i | a_{i,n} > 0\}$, $I^- := \{i | a_{i,n} < 0\}$, $I^0 := \{i | a_{i,n} = 0\}$.

(S3) $a'_{i,k} := \dfrac{a_{i,k}}{|a_{i,n}|}$ $(i \in I^+ \cup I^-, k \in \{1,2,...,n-1\})$, $b'_i := \dfrac{b_i}{|a_{i,n}|}$ $(i \in I^+ \cup I^-)$.

(S4) $\tilde{A} := (A,b) \in M(m,n+1,\mathbb{R})$.

(S5) We set $\tilde{A}_{n-1} \in M(\#I^+ * \#I^- + \#I^0, n, \mathbb{R})$ *and* $b' \in \mathbb{R}^{(\#I^+ * \#I^- + \#I^0)}$ *by*

$$(kq\text{-th row of } \tilde{A}_{n-1}) = \frac{1}{|a_{k,n}|}(k\text{-th row of } \tilde{A}) + \frac{1}{|a_{q,n}|}(q\text{-th row of } \tilde{A})\ (\forall k \in I^+, \forall q \in I^-)$$

and

$$((\#I^+ * \#I^- + j)\text{-th row of } \tilde{A}') = (j\text{-th row of } \tilde{A})\ (j = 1,2,...,\#I^0)$$

(S6) $x^i := (x_1,...,x_i)$ $(x \in \mathbb{R}^n)$

Then

(i) $Ax \leq b, x \in \mathbb{R}^n$ *is feasible if and only if*

$$\sum_{i=1}^{n-1}(a'_{k,i} + a'_{q,i})x_i \leq b'_k + b'_q\ (\forall k \in I^+, \forall q \in I^-),$$
$$\sum_{i=1}^{n-1} a_{p,i}x_i \leq b_p\ (\forall p \in I^0)$$

(ii) If $A \in M(m,n,\mathbb{Q})$ *and* $b \in \mathbb{Q}^m$, *then* $a'_{k,i}, a'_{q,i}, b'_k, b'_q \in \mathbb{Q}$ $(\forall k \in I^+, \forall i \in \{1,2,...,n-1\}, \forall q \in I^-)$.

(iii) $\{x \in \mathbb{R}^n | Ax \leq b\} \neq \phi \iff \{x \in \mathbb{R}^{n+1} | \tilde{A}(x^t,-1)^t \leq 0\} \neq \phi \iff \{x \in \mathbb{R}^n | \tilde{A}_{n-1}((x^{n-1})^t,-1)^t \leq 0\} \neq \phi$.

(iv) For each $i \in \{0,1,...,n-1\}$, *there is* $m_i \in \mathbb{N}$ *and* $\tilde{A}_i \in M(m_i, i+1, \mathbb{R}$ *such that every row of* $\tilde{A}_i$ *is a conic combination of rows of* $\tilde{A}$ *and*

$$\{x \in \mathbb{R}^n | Ax \leq b\} \neq \phi \iff \{x \in \mathbb{R}^i | \tilde{A}_i((x^i)^t,-1)^t \leq 0\}$$

(v) If $\tilde{A} \in M(m,n+1,\mathbb{Q})$ *then* $\tilde{A}_i \in M(m_i, i+1, \mathbb{Q})$ $i \in \{0,1,...,n-1\}$.

(vi) $\{x \in \mathbb{R}^n | Ax \leq b\} \neq \phi \iff \tilde{A}_0 \leq 0$.

Proof of the 'only if' part in (i). Let us assume $x \in \mathbb{R}^n$ such that $Ax \leq b$. Then

$$\sum_{i=1}^{n-1} a'_{k,i} x_i + x_n \leq b'_k \ (\forall k \in I^+)$$

and

$$\sum_{i=1}^{n-1} a'_{q,i} x_i - x_n \leq b'_q \ (\forall q \in I^-)$$

So, by adding the left and right sides of these two inequalities, respectively, the following holds.

$$\sum_{i=1}^{n-1} (a'_{k,i} + a'_{q,i}) x_i \leq b'_k + b'_q \ (\forall k \in I^+, \forall q \in I^-),$$
$$\sum_{i=1}^{n-1} a_{p,i} x_i \leq b_p \ (\forall p \in I^0)$$

□

Proof of the 'if' part in (i). Let us assume

$$\sum_{i=1}^{n-1} (a'_{k,i} + a'_{q,i}) x_i \leq b'_k + b'_q \ (\forall k \in I^+, \forall q \in I^-),$$
$$\sum_{i=1}^{n-1} a_{p,i} x_i \leq b_p \ (\forall p \in I^0)$$

Then

$$\sum_{i=1}^{n-1} a'_{k,i} x_i - b'_k \leq -(\sum_{i=1}^{n-1} a'_{q,i} - b'_q) \ (\forall k \in I^+, \forall q \in I^-)$$

We set

$$x_n := \min\{-(\sum_{i=1}^{n-1} a'_{k,i} - b'_k) | k \in I^+\}$$

Then

$$x_n \geq \max\{(\sum_{i=1}^{n-1} a'_{q,i} - b'_q) | q \in I^-\}$$

So, $Ax \leq b$. □

Proof of (ii)-(iv). These are followed by (i). □

Theorem 1.3 (Farkas Lemma I)**.** *Let*

(S1) $A \in M(m, n, \mathbb{R})$, $b \in \mathbb{R}^m$.

Then

$$\{x \in \mathbb{R}^n | Ax \leq b\} = \phi \iff \{v \in \mathbb{R}^m | A^t v = 0, b^t v < 0, v \geq 0\} \neq \phi$$

Proof of 'only if' part. By Fourier elimination method (iv), there are $m_0 \in \mathbb{N}$ and $U \in M(m_0, n, \mathbb{R})$ such that $U \geq 0$ and $U\tilde{A} = (O_{m_i,n-1}, b^0)$ and $b^0 \not\geq 0$. Then there is $u \in \mathbb{R}^{m_0}$ such that $u^t b^0 < 0$. We set

$$v := (u^t U)^t$$

Then $v \geq 0$ and $Av = 0$ and $v^t b < 0$. □

Proof of 'if' part. Let us assume $\exists v \in \mathbb{R}^m$ such that $v^t A = 0$ and $v^t b < 0$ and $v \geq 0$. For any $x \in \mathbb{R}^n$, $v^t A x = 0$. So, $Ax \not\leq b$. □

Theorem 1.4 (Farkas Lemma II)**.** *Let*

(S1) $A \in M(m, n, \mathbb{R})$, $b \in \mathbb{R}^m$.

Then

$$\{x \in \mathbb{R}^n | Ax = b, x \geq 0\} \neq \phi \iff \{u \in \mathbb{R}^m | A^t u \leq 0\} \subset \{u \in \mathbb{R}^m | u^t b \leq 0\}$$

Proof of '$\implies$'. Let us fix $x \in \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$. Let us fix any $u \in \{u \in \mathbb{R}^m | A^t u \leq 0\}$. So, $b^t u \leq 0$. □

Proof of '$\impliedby$'. Let us assume

$$\{x \in \mathbb{R}^n | Ax = b, x \geq 0\} = \phi$$

Then

$$\{x \in \mathbb{R}^n | Ax \leq b, -Ax \leq -b, x \geq 0\} = \{x \in \mathbb{R}^n | Bx \leq c\} = \phi$$

Here,

$$B := \begin{pmatrix} A \\ -A \\ -I_n \end{pmatrix}, c := \begin{pmatrix} b \\ -b \\ O_{n,1} \end{pmatrix}$$

and I_n is the n-th unit matrix. By Farkas Lemma I, there are $v \in \mathbb{R}^m_+$ and $v' \in \mathbb{R}^m_+$ and $w \in \mathbb{R}^n_+$ such that

$$B^t \begin{pmatrix} v \\ v' \\ w \end{pmatrix} = 0, \begin{pmatrix} v \\ v' \\ w \end{pmatrix}^t c < 0$$

This implies

$$A(-(v - v')) = -w, -(v - v')^t b > 0$$

We set $u := -(v - v')$. Then

$$u \in \{u \in \mathbb{R}^m | A^t u \leq 0\} \setminus \{u \in \mathbb{R}^m | u^t b \leq 0\}$$

□

1.2.3 Polyhedron and Minkowski Weyl Theorem

Definition 1.6 (Polytope)**.** *We say $A \subset \mathbb{R}^n$ is a polytope if there are finite vectors $v_1, ..., v_m \in \mathbb{R}^n$ such that $A = conv(v_1, ..., v_m)$. We call $v_1, ..., v_m$ vertices of A. If $v_1, ..., v_m \in \mathbb{Q}^n$, we call A is a rational polytope.*

Definition 1.7 (Cone)**.** *We say $C \subset \mathbb{R}^n$ is a cone if $0 \in C$ and for every $x \in C$ and $\lambda \in \mathbb{R}_+$ $\lambda x \in C$.*

By the definition of cone, the following holds.

Proposition 1.3. *Any cone containing nonzero vector is not bounded.*

Definition 1.8 (Convex Cone)**.** *We say $C \subset \mathbb{R}^n$ is a convex cone if C is cone and every conic combination of finite vectors of C is contained in C.*

Because every intersection of convex cones is also convex cone, the following holds.

Proposition 1.4 (Convex Cone generated by a set)**.** *Let us assume A is any subset of $\mathbb{R}^n$. Then there is the minimum convex cone containing A. We denote this convex cone by $cone(A)$.*

Definition 1.9 (Polyhedral cone)**.** *Let*

(S1) $A \in M(m, n, \mathbb{Q})$.

We call

$$P := \{x \in \mathbb{R}^n | Ax \leq 0\}$$

a Polyhedral cone.

Theorem 1.5 (Minkowski Weyl Theorem for cones)**.** *Let*

(S1) $C \subset \mathbb{R}^n$.

Then C is a Polyhedral cone if and only if C is finite generated cone.

STEP1. Proof of 'if' part. Let us assume C is finite generated cone. Then there is $r_1, ..., r_k \in \mathbb{R}$ such that $C = cone(r_1, ..., r_k)$. We set $R = (r_1, ..., r_k)$.

By applying Fourier elimination method k times to the the following inequality

$$-\mu \leq 0, R\mu \leq x, -R\mu \leq -x$$

and Fourier elimination method (vi), there is $A \in M(m, n, \mathbb{R})$ such that the above inequality is equivalent to

$$Ax \leq 0$$

So, $C = \{x \in \mathbb{R}^n | Ax \leq 0\}$.

□

STEP2. Proof of 'only if' part. Let us assume C is a Polyhedral cone. So, there is $A \in M(m, n, \mathbb{R})$ such that $C = \{x \in \mathbb{R}^n | Ax \leq 0\}$. We set $C^* := \{y \in \mathbb{R}^n | \exists \nu \in \mathbb{R}^m_+ \text{ such that } A^t v = y\}$. Then

$$C^* = cone(a^1, ..., a^m)$$

Here, $a^i \in \mathbb{R}^n$ is the i-th row vector of A $(i = 1, 2, ..., m)$. By STEP1, there is $R \in M(n, k, \mathbb{R})$ such that

$$C^* = \{y \in \mathbb{R}^n | R^t y \leq 0\}$$

We denote the i-th column vector of R by r^i $(i = 1, 2, ..., k)$. We will show

$$C = cone(r_1, ..., r_k)$$

Let us fix any $x \in cone(r_1, ..., r_k)$. Then there are $\nu_1, ..., \nu_k \in \mathbb{R}_+$ such that $x = R\nu$. Because $a_i = A^t e_i$ $(i = 1, 2, ..., m)$, $a_i \in C^*$ $(i = 1, 2, ..., m)$. So, $AR \leq 0$. This implies $Ax = AR\nu \leq 0$. This means $x \in C$. We have shown $cone(r_1, ..., r_k) \subset C$.

Let us fix any $\bar{x} \in cone(r_1, ..., r_k)^c$. So, $\{\nu \in \mathbb{R}^k | R\nu = \bar{x}, \nu \geq 0\} = \phi$. By Farkas Lemma II, there is $y \in \mathbb{R}^n$ such that $R^t y \leq 0$ and $y^t \bar{x} > 0$. So, $y \in C^*$. Then there are $\nu \in \mathbb{R}^m_+$ such that $y = A^t \nu$. So, $\nu^t A\bar{x} > 0$. Because $\nu \in \mathbb{R}^m_+$, this implies $A\bar{x} \not\leq 0$. This means $\bar{x} \in C^c$. Consequently $C \subset cone(r_1, ..., r_k)$. □

Definition 1.10 (Minkowski sum)**.** *Let $A, B \subset \mathbb{R}^n$. We call*

$$A + B$$

the Minkowski sum of A and B.

Proposition 1.5. *Let*

(i) Minkowski sum of any two convex set is convex.

(ii) For any two subset $A, B \subset \mathbb{R}^n$,

$$conv(A + B) = conv(A) + conv(B)$$

Proof of (i). Let $A, B \subset \mathbb{R}^n$ be convex. For any $a_1, ..., a_m \in A$ and $b_1, ..., b_m \in B$ and $\lambda_1, ..., \lambda_m \subset [0, 1]$ such that $\sum_{i=1}^m \lambda_i = 1$,

$$\sum_{i=1}^m \lambda_i(a_i + b_i) = \sum_{i=1}^m \lambda_i a_i + \sum_{i=1}^m \lambda_i b_i \in A + B$$

So, $A + B$ is convex. □

Proof of (ii). By (i), $conv(A)+conv(B)$ is convex. And $A+B \subset conv(A)+conv(B)$. So, $conv(A+B) \subset conv(A)+conv(B)$. Let us fix any $a_1, .., a_k \in A$ and $b_a, ..., b_l \in B$ and $\lambda_1, ..., \lambda_k, \mu_1, ..., \mu_l \in [0, 1]$ such that $\sum_{i=1}^k \lambda_i = 1$ and $\sum_{i=1}^l \mu_i = 1$. Then

$$\sum_{i=1}^k \lambda_i a_i + \sum_{j=1}^l \mu_j b_j = \sum_{j=1}^l \mu_j (\sum_{i=1}^k \lambda_i a_i + b_j) = \sum_{j=1}^l \mu_j (\sum_{i=1}^k \lambda_i (a_i + b_j)) = \sum_{i,j} \lambda_i \mu_j (a_i + b_j) \in conv(A + B)$$

□

Theorem 1.6 (Minkowski-Weyl Theorem)**.** *A subset $P \subset \mathbb{R}^n$ is a Polyhedron if and only if there is a polytope Q a finite generated cone C such that*

$$P = Q + C$$

Proof of 'only if' part. Let us fix $A \in M(m, n, \mathbb{R})$ and $b \in \mathbb{R}^m$ such that $P = \{x \in \mathbb{R}^n | Ax \leq b\}$. We set

$$C_P := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} | Ax - yb \leq 0, y \leq 0\}$$

Then clearly

$$P = \{x \in \mathbb{R}^n | (x, 1) \in C_P\}$$

By Minkowski Weyl Theorem for cones, there are $r^1, r^2, ..., r^K \mathbb{R}^{n+1}$ such that

$$C_P := cone(r^1, r^2, ..., r^K)$$

Because C_P is a cone, we can assume $r^i_{n+1} = 0$ or 1 ($\forall i$). So, there are $u_1, ..., u_k \in \mathbb{R}^n$ and $v_1, ..., v_l \in \mathbb{R}^n$ such that

$$C_P = cone(\begin{pmatrix} u_1 \\ 1 \end{pmatrix}, ... \begin{pmatrix} u_k \\ 1 \end{pmatrix}, \begin{pmatrix} v_1 \\ 0 \end{pmatrix}, ..., \begin{pmatrix} v_l \\ 0 \end{pmatrix})$$

So,

$$P = conv(u^1, ..., u^k) + cone(v^1, ..., v^l)$$

□

Proof of 'if' part. We assume we can get

$$P = conv(u^1, ..., u^k) + cone(v^1, ..., v^l)$$

Then

$$P = cone(\begin{pmatrix} u^1 \\ 1 \end{pmatrix}, ..., \begin{pmatrix} u^k \\ 1 \end{pmatrix}, \begin{pmatrix} v^1 \\ 0 \end{pmatrix}, ..., \begin{pmatrix} v^l \\ 0 \end{pmatrix}) \cap \mathbb{R}^n \times \{1\}$$

Because $cone(\begin{pmatrix} u^1 \\ 1 \end{pmatrix}, ..., \begin{pmatrix} u^k \\ 1 \end{pmatrix}, \begin{pmatrix} v^1 \\ 0 \end{pmatrix}, ..., \begin{pmatrix} v^l \\ 0 \end{pmatrix})$ is a Polyhedral cone, P is a Polyhedron. □

Proposition 1.6. *Let*

(i) Bounden Polyhedron is polytone.

(ii) If $A \in M(m, n, \mathbb{Q})$ and $b \in \mathbb{Q}^m$, then there are $v_1, ..., v_k \in \mathbb{Q}^n$ and $r_1, ..., r_l \in \mathbb{Z}^n$ such that

$$P := \{x \in \mathbb{R}^n | Ax \leq b\} = conv(v_1, ..., v_k) + cone(r_1, ..., r_l)$$

If P is bounded, P is a rational polytope.

(iii) $P \subset \mathbb{R}^n$ is a rational polyhedron if and only if P is a minkowski sum of a rational polytope and a convex cone generated by finite rational vectors.

Proof of (i). By Proposition1.3, (i) holds. □

Proof of (ii). By the proof of Theorem1.5, (ii) holds. □

Proof of (iii). By the proof of Theorem1.5, (iii) holds. □

1.2.4 Perfect formulation and Meyer's Foundamental theorem

Proposition 1.7. *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $G \in M(m, p, \mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

(S2) $S := \{(x, y) \in (\mathbb{Z}_+)^n \times (\mathbb{R}_+)^p | g(x, y) := Ax + Gy \leq b\}$

Then

(i)

$$sup\{c^t x + h^t y | (x, y) \in S\} = sup\{c^t x + h^t y | (x, y) \in conv(S)\}$$

Furthermore, there is $(x, y) \in S$ such that $c^t x + h^t y = sup\{c^t x + h^t y | (x, y) \in S\} \iff$ there is $(x, y) \in conv(S)$ such that $c^t x + h^t y = sup\{c^t x + h^t y | (x, y) \in S\}$

(ii) $ex(conv(S)) \subset S$

Proof of the first part of (i). Because $S \subset conv(S)$,

$$sup\{c^tx + h^ty|(x,y) \in S\} \leq sup\{c^tx + h^ty|(x,y) \in conv(S)\}$$

We can assume $z^* = sup\{c^tx + h^ty|(x,y) \in S\} < \infty$. Let us set $H := \{(x,y) \in \mathbb{R}^{n+p}|c^tx + h^ty \leq z^*\}$. Because H is convex and $S \subset H$, $conv(S) \subset H$. So,

$$sup\{c^tx + h^ty|(x,y) \in S\} \geq sup\{c^tx + h^ty|(x,y) \in conv(S)\}$$

□

Proof of the last part of (i). The part of $\Longrightarrow$ is clear. We set $d := (c,h)$. Let us assume there is $\bar{z} = (\bar{x}, \bar{y})$ such that $d^t\bar{z} = sup\{c^tx + h^ty|(x,y) \in conv(S)\}$. Then there are $\lambda_1, ..., \lambda_k > 0$ and $z_1, ..., z_k \in S$ such that $\bar{z} = \sum_{i=1}^k \lambda_i z_i$. Clearly $d^tz_i \leq d^t\bar{z}$ $(\forall i)$. Because $d^t\bar{z} = \sum_{i=1}^k d^t\lambda_i z_i$, there is i such that $d^tz_i \geq d^t\bar{z}$. So, $d^tz_i = sup\{c^tx + h^ty|(x,y) \in conv(S)\}$. □

Proof of (ii). Let us fix any $v \in ex(conv(S))$. Because $ex(conv(S)) \subset conv(S)$, there are $\lambda_1, ..., \lambda_m \in (0,1]$ and $v_1, ..., v_m \in S$ such that $v = \sum_{i=1} \lambda_i v^i$. We can assume $m > 1$. We set $v' := \sum_{i=2}^m \frac{\lambda_i}{1-\lambda_1} v^i$. Then $v' \in conv(S)$. Because $v = \lambda_1 v_1 + (1-\lambda_1)v'$ and $v \in ex(conv(S))$, $v = v_1 \in S$. □

Proposition 1.8. *Let $r^1, ..., r^K \in \mathbb{R}^n$. Then*

$$conv(\sum_{i=1}^K \mathbb{Z}_+ r^i) = cone(r_1, ..., r^K)$$

Proof. We will show this by Mathematical induction. If $K = 1$, then this proposition holds. Let us fix any $k \in \mathbb{N}$ and assume this proposition holds for every $K \leq k$.

We set $C := conv(\sum_{i=1}^{k+1} \mathbb{Z}_+ r^i)$. Clearly $C \subset cone(r^1, ..., r^{k+1})$. Let us fix $x \in cone(r^1, ..., r^{k+1})$. Then there are $\mu_1, ..., \mu_{k+1} \in \mathbb{R}_+$ such that $x = \sum_{i=1}^{k+1} \mu_i r^i$. We can assume $\mu_{k+1} > 0$. We set $\lambda := \frac{2\mu_{k+1}}{\lceil 2\mu_{k+1} \rceil}$. Because $0 \in C$, $2\mu_{k+1}r^{k+1} = (1-\lambda)0 + \lambda\lceil 2\mu_{k+1}\rceil r^{k+1} \in C$. By Mathematical induction assumption, $\sum_{i=1}^k 2\mu_i r^i \in C$. So,

$$\sum_{i=1}^{k+1} \mu_i r^i = \frac{1}{2}(2\mu_{k+1}r^{k+1} + \sum_{i=1}^k 2\mu_i r^i) \in C$$

So, $cone(r^1, ..., r^{k+1}) \subset C$. □

Theorem 1.7 (Meyer(1974)[2] Fundamental Theorem)**.** *Here are the settings and assumptions.*

(S1) $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

(S2) $S := \{(x,y) \in P(A,G,b)|x \in \mathbb{Z}^n\}$.

Then there are $A' \in M(m,n,\mathbb{Q})$, $G' \in M(m,p,\mathbb{Q})$, $b' \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$ such that

$$conv(S) = P(A', G', b')$$

STEP1. Decomposition of S. We can assume $S \neq \phi$. Then by Proposition1.6, there are $v^1, ..., v^t \subset \mathbb{Q}^{n+p}$ and $r^1, ..., r^q \subset \mathbb{Z}^{n+p}$ such that

$$P := P(A,G,b) = conv(v^1, ..., v^t) + cone(r^1, ..., r^q)$$

We set

$$T := \{\sum_{i=1}^s \lambda_i v^i + \sum_{j=1}^q \mu_j r^j | 0 \leq \lambda_i, \mu_j \leq 1 \ (\forall i,j), \sum_{i=1}^s \lambda_i = 1\} = conv(v^1, ..., v^t) + \sum_{j=1}^q [0,1]r_j$$

Then T is bounded. There is $M \in \mathbb{N}$ and $D \in M(M, n+p, \mathbb{Q})$ such that

$$T = \{z \in \mathbb{R}^{n+p} | \exists\lambda \in \mathbb{R}^n_+, \exists\mu \in \mathbb{R}^p_+ \text{ s.t. } D\begin{pmatrix}\lambda \\ \mu\end{pmatrix} \leq z, \sum_{i=1}^s \lambda_i \leq 1, -\sum_{i=1}^s \lambda_i \leq -1, \mu \leq 1\}$$

By Fourier elimination method, there are $C \in M(M,n,\mathbb{R})$ and $d \in \mathbb{Q}^n$ such that $T = \{x \in \mathbb{R}^n | Cx \leq d\}$. So, by Proposition1.6, T is a rational polytope.

Let

$$T_I := \{(x,y) \in \mathbb{Z}^n \times \mathbb{R}^p | (x,y) \in T\}, R_I := \{\sum_{j=1}^{q} \mu_j r^j | \mu_j \in \mathbb{Z}_+ \ (\forall j)\}$$

We will show

$$S = T_I + R_I$$

Because $T_I + R_I \subset T$ and i-th component of $T_I + R_I$ is integer for every $i \in \{1, 2, ..., s\}$, $T_I + R_I \subset S$.

Let us fix any $(x,y) \in \mathbb{Z}^n \times \mathbb{R}^p$ such that $(x,y) \in S$. Then there are $\lambda_1, ..., \lambda_s, \mu_1, ..., \mu_q \in [0,1]$ such that $\sum_{i=1}^{s} \lambda_i = 1$ and

$$(x,y) = \sum_{i=1}^{s} \lambda_i v^i + \sum_{j=1}^{q} \mu_j r^j$$

We set

$$(x', y') := \sum_{i=1}^{s} \lambda_i v^i + \sum_{j=1}^{q} (\mu_j - \lfloor \mu_j \rfloor) r^j, r := \sum_{j=1}^{q} \lfloor \mu_j \rfloor r^j$$

Then $(x', y') \in T_I$ and $r \in R_I$. So, $(x,y) \in T_I + R_I$. Consequently, $S = T_I + R_I$. □

STEP2. Proof that $conv(S)$ is a rational polyhedron. By Proposition1.5 and STEP1,

$$conv(S) = conv(T_I) + conv(R_I)$$

Because $conv(R_I) = conv(r^1, ..., r^q)$, by Proposition1.8, $conv(R_I)$ is a rational polyhedral cone. So, it is enough to show

$$conv(T_I) \text{ is a rational polytope}$$

Since T is bounded, $X := \{x \in \mathbb{Z}^n | \exists y \in \mathbb{R}^p \text{ such that } (x,y) \in T_I\}$ is bounded and so is a finite set.

For each $x \in X$, we set $T_x := \{(x,y) | \exists y \in \mathbb{R}^p \text{ such that } (x,y) \in T_I\}$. For any $\bar{x} \in X$,

$$T_{\bar{x}} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^p | x = \bar{x} \text{ and } (x,y) \in T\}$$

Because T is a rational polytope, $T_{\bar{x}}$ is a rational polytope. We denote th set of all vertices of $T_{\bar{x}}$ by $V_{\bar{x}}$ for any $\bar{x} \in X$. We set $V := \cup_{x \in X} V_x$. V is a finite set. We will show

$$conv(T_I) = conv(V)$$

Because $T_I = \cup_{x \in X} T_x = \cup_{x \in X} conv(V_x) \subset conv(V)$, $conv(T_I) \subset conv(V)$. Because $V = \cup_{x \in X} V_x \subset \cup_{x \in X} conv(V_x) = \cup_{x \in X} T_x = conv(T_I)$, $conv(V) \subset conv(T_I)$. So, $conv(T_I) = conv(V)$. Consequently, $conv(T_I)$ is a rational polytope. □

1.2.5 Sharp MILP Formulation

Definition 1.11 (MILP Formulation)**.** *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$*,* $G \in M(m, p, \mathbb{Q})$*,* $B \in M(m, t, \mathbb{Q})$*,* $b \in \mathbb{Q}^m$*.*

(S2) $S \subset \mathbb{Q}^n$*.*

(S3) $T(A, G, B, b) := \{(x, y, z) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Z}^t | Ax + Gy + Bz \leq b\}$*.*

We say (A, G, B, b) is a MILP formulation for S if and only if S is equal to the image of

$$p_n : T(A, G, B, b) \ni (x, y, z) \mapsto x \in \mathbb{Q}^n$$

Clearly the following holds.

Proposition 1.9. *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$*,* $G \in M(m, p, \mathbb{Q})$*,* $b \in \mathbb{Q}^m$*,* $c \in \mathbb{R}^n$*,* $h \in \mathbb{R}^p$*.*

(S2) $S := \{(x, y) \in P(A, G, b) | x \in (\mathbb{Z}_+)^n\}$*.*

(S3) We set

$$\tilde{A} := \begin{pmatrix} A \\ E_n \\ O_{p,n} \\ O_{n,n} \end{pmatrix}, \tilde{G} := \begin{pmatrix} G \\ O_{n,p} \\ -E_p \\ O_{n,p} \end{pmatrix}, \tilde{B} := \begin{pmatrix} B \\ -E_n \\ O_{p,n} \\ -E_n \end{pmatrix}, \tilde{b} := \begin{pmatrix} b \\ 0_n \\ 0_p \\ 0_n \end{pmatrix}$$

Then $(\tilde{A}, \tilde{G}, \tilde{B}, \tilde{b})$ is a MILP formultation for S.

Definition 1.12 (Sharp MILP Formulation)**.** *Here are the settings and assumptions.*

- *(S1)* $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $B \in M(m,t,\mathbb{Q})$, $b \in \mathbb{Q}^m$.
- *(S2)* $S \subset \mathbb{Q}^n$.
- *(Aq)* (A,G,B,b) *is a MILP formulation for* S.

We say (A,G,B,b) is sharp MILP formulation for S if and only if $conv(S)$ is equal to the image of

$$p_n : \tilde{T}(A,G,B,b) \ni (x,y,z) \mapsto x \in \mathbb{Q}^n$$

Here, $\tilde{T}(A,G,B,b)$ is a LP relaxation of $T(A,G,B,b)$.

Theorem 1.8. *Here are the settings and assumptions.*

- *(S1)* $S \subset \mathbb{Q}^n$.
- *(A1) There are* $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $B \in M(m,t,\mathbb{Q})$, $b \in \mathbb{Q}^m$ *such that* (A,G,B,b) *is a MILP formulation for* S.

Then there there are $M \in \mathbb{N}$ and $\tilde{A} \in M(M,n,\mathbb{Q})$, $\tilde{G} \in M(M,p,\mathbb{Q})$, $\tilde{B} \in M(M,t,\mathbb{Q})$, $\tilde{b} \in \mathbb{Q}^M$ such that $(\tilde{A}, \tilde{G}, \tilde{B}, \tilde{b})$ is a sharp MILP formulation for S.

Proof. We set

$$T_I := \{(x,y,z) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Z}^t | Ax + Gy + Bz \leq b\}$$

and $p_1 : T_I \ni (x,y,z) \mapsto x \in \mathbb{Q}^n$. Because (A,G,B,b) is a MILP formulation for S,

$$p_1(T_I) = S$$

By Theorem1.2.4, there are $M \in \mathbb{N}$ and $\tilde{A} \in M(M,n,\mathbb{Q})$, $\tilde{G} \in M(M,p,\mathbb{Q})$, $\tilde{B} \in M(M,t,\mathbb{Q})$, $\tilde{b} \in \mathbb{Q}^M$ such that

$$T_I = \{(x,y,z) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Z}^t | \tilde{A}x + \tilde{G}y + \tilde{B}z \leq \tilde{b}\}$$

$$conv(T_I) = \{(x,y,z) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Q}^t | \tilde{A}x + \tilde{G}y + \tilde{B}z \leq \tilde{b}\}$$

Because $conv(S) = conv(p_1(T_I)) = p_1(conv(T_I))$,

$$conv(S) = p_1(conv(T_I))$$

So, $(\tilde{A}, \tilde{G}, \tilde{B}, \tilde{b})$ is a sharp MILP formulation for S. □

1.2.6 Review

Meyer theorem states that the convex hull of the feasible region of MILP is a rational polyhedron. So, the feasibility and the optimal value of MILP are equivalent to the feasibility and the optimal value of some LP, respectively. By methods such as simplex method, we can find this LP solution in extreme points of feasible reasion. By Proposition1.7, this extreme point is a solution of original MILP problem.

I think there are the following three ideas that are important in the proof of Meyer theorem.

1. Fourier elimination method
2. Expressing the feasible region of MILP or LP in terms of the Minkowski sum of bounded and unbounded parts
3. Going back and forth between integer and continuous parts of a polyhedron

Fourier elimination method plays an important role throughout this section. Fourier elimination method is a method of solving linear inequalities

$$Ax \leq b \tag{1.2.1}$$

focusing on the sign of the coefficients of a certain variable and using only non-negative multipliers to eliminate the variable. (1.2.1) corresponds to another two linear inequalities. If there is a solution of (1.2.1), then there is $U \in M(m_0, n, \mathbb{R})$ such that $U \geq 0$ and $UA = 0$ and

$$0 \leq Ub \tag{1.2.2}$$

By focusing on row vectors of U, if there is no solutions of (1.2.1), then there is $u \in \mathbb{R}^n_+$ such that

$$A^t u = 0, u^t b < 0, u \geq 0 \tag{1.2.3}$$

Correspondance between (1.2.1) and (1.2.3) is stated by Farkas Lemma.

For idea2 on LP feasible reasion P, we state this idea as Minkowski Weyl Theorem.

$$P = conv(v^1, ..., v^s) + cone(r^1, ..., r^q) \tag{1.2.4}$$

By increasing the dimension of the solution space of the simultaneous inequalities by one as follows, Minkowski Weyl Theorem is boil down to the case in P is a polyhedral cone.

$$P = \tilde{P} \cap \mathbb{R}^n \times \{1\}, \tilde{P} := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} | (A, -b) \begin{pmatrix} x \\ y \end{pmatrix} \leq 0\} \tag{1.2.5}$$

By Fourier elimination method and Farkas Lemma, any polyhedral cone is equivalent to finite generated convex cone.

Meyer theorem is the following.

Theorem 1.9. *Here are the settings and assumptions.*

- *(S1)* $A \in M(m, n, \mathbb{Q})$, $G \in M(m, p, \mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.
- *(S2)* $S := \{(x, y) \in P(A, G, b) | x \in \mathbb{Z}^n\}$.

Then $conv(S)$ *is a rational polyhedron.*

In the proof of Meyer theorem, we focus on Polyhedron $P := P(A, G, b)$ which is containing S. By Minkowski Weyl Theorem, we get

$$P = conv(v^1, ..., v^s) + cone(r^1, ..., r^q)$$

We focus a bounded part of P

$$T = conv(v^1, ..., v^s) + \sum_{j=1}^{q} [0, 1] r_j$$

We denote a integer part of T by T_I and denote a integer part of $cone(r^1, ..., r^q)$ by R_I. Then we get

$$S = T_I + R_I$$

So,

$$conv(S) = conv(T_I) + conv(R_I)$$

Because $conv(T_I)$ is a rational polytope and $conv(R_I)$ is a rational polyhedral cone, $conv(S)$ is a rational polyhedron.

1.3 MILP formulation

1.3.1 Locally ideal formulation

Proposition 1.10 (Standard equity form for LP)**.** *Here are the settings and assumptions.*

- *(S1)* $A \in M(m, n, \mathbb{Q})$, $b \in \mathbb{Q}^m$.
- *(S2)* $S := \{x \in \mathbb{Q}^n | Ax \leq b\}$.
- *(S3) We set for* $x \in S$,

$$\Phi(x) := (y^+, y^-, z)$$

Here,

$$y_i^+ := \max\{x_i, 0\} \ (i = 1, 2, ..., n)$$
$$y_i^- := \max\{-x_i, 0\} \ (i = 1, 2, ..., n)$$
$$z_j := (a_j, x) - b_j \ (j = 1, 2, ..., m)$$

- *(S4)* $\tilde{S} := \{(y^+, y^-, z) \in \mathbb{Q}^n_+ | A(y^+ - y^-) + z \leq b\}$.

Then Φ *is a bijective from* S *to* $\tilde{S}$*. We call* $\tilde{S}$ *the standard equity form of* S*. We call each* z_j *a slack variable.*

Definition 1.13 (Basic feasible solution for *LP*.)**.** *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $b \in \mathbb{Q}^m$.

Then

(i) *For* $x \in \mathbb{Q}^n$, *we say* $\bar{x}$ *is a basic solution of* $Ax = b$ *if and only if* $\{a_i | a_i$ *is the i-th column of* A *and* $\bar{x}_i > 0\}$ *are linear independent.*

(ii) *For* $x \in \mathbb{Q}^n_+$, *we say* $\bar{x}$ *is a basic feasible solution of*

$$Ax = b, x \geq 0$$

if and only if x *is a basic solution of* $Ax = b$.

Proposition 1.11. *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $b \in \mathbb{Q}^m$.

(S2) x *is a solution of* $Ax \leq b, x \geq 0$.

(S3) $z = (z_1, .., z_m)$ *are nonzero slack variables for* $Ax + z = b, x, z \geq 0$.

(S4) $I := \{i \in \{1, 2, ..., m\} | a_i^T x = b_i\}$. *Here* a_i *is the i-th row vector of* A.

(S5) $J := \{j \in \{1, 2, ..., n\} | x_j \neq 0\}$.

Then (x, z) *is a basic feasible solution iff* $\{\{a_{i,j}\}_{i \in I}\}_{j \in J}$ *are linear independent.*

Proof. We set $I' := \{i \in \{1, 2, ..., m\} | a_i^T x < b_i\}$. (x, z) is a basic feasible solution iff $\{a^j\}_{j \in J} \cup \{e_i\}_{i \in I'}$ are linear independent. Here a^j is the j-th column of A. This is equivalent to $\{a^j - \sum_{i \in I'} a_{i,j} e_i\}_{j \in J} \cup \{e_i\}_{i \in I'}$ are linear independent. So, (x, z) is a basic feasible solution iff $\{\{a_{i,j}\}_{i \in I}\}_{j \in J}$ are linear independent. □

Definition 1.14 (Locally ideal)**.** *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $G \in M(m, p, \mathbb{Q})$, $B \in M(m, t, \mathbb{Q})$, $b \in \mathbb{Q}^m$.

(S2) $S \subset \mathbb{Q}^n$.

(S3) $T(A, G, B, b) := \{(x, y, z) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Z}^t | Ax + Gy + Bz \leq b\}$.

(S4) $\tilde{S} := \{w \in \mathbb{Q}^M | Cw = c, w \geq 0\}$ *is a standard equity form of* S *and* Φ *is the bijection from* S *to* $\tilde{S}$ *in Proposition1.10.*

We say (A, G, B, b) *is a locally ideal MILP formulation for* S *if and only if* $\tilde{S}$ *has at most one basic feasible solution and for any basic feasible solution of* $\tilde{S}$ w, $\Phi^{-1}(w) \in \mathbb{Q}^{n+p} \times \mathbb{Z}^t$.

We will show an example of MILP formulation which is not locally ideal but sharp.

Example 1.1. *Here are the settings and assumptions.*

(S1) $S = \cup_{i=1}^n P_i$. $P_i := \{x \in \mathbb{Q}^n || x_i| \leq 1, x_j = 0 \ (j \neq i)\}$ $(i = 1, 2, ..., n)$.

Then

(i) *The following is a MILP formulation for* S.

$$y_j - 1 \leq x_i \leq 1 - y_j \ (i = 1, 2, .., n, j \neq i), \tag{1.3.1}$$
$$y_i \geq 0, \ (i = 1, 2, .., n), \tag{1.3.2}$$
$$\sum_{i=1}^n y_i = 1 \tag{1.3.3}$$
$$y \in \mathbb{Z}^n$$

(ii) $conv(S) = \{x \in \mathbb{Q}^n | \sum_{i=1}^n |x_i| \leq 1\}$

(iii) *Equalities and Inequalities in (i) and the following is a sharp MILP formulation for* S.

$$\sum_{i=1}^n r_i x_i \leq 1 \ (r \in \{-1, 1\}^n) \tag{1.3.4}$$

(iv) *If* $n = 3$, *the formulation in (iii) is not locally ideal.*

(v) The following is a sharp and locally ideal MILP formulation for S.

$$-y_i \le x_i \le y_i \ (i = 1, 2, .., n), \tag{1.3.5}$$
$$y_i \ge 0, \ (i = 1, 2, .., n), \tag{1.3.6}$$
$$\sum_{i=1}^{n} y_i = 1 \tag{1.3.7}$$
$$y \in \mathbb{Z}^n$$

Proof of (i). It is clear. □

Proof of (ii). The part of $\subset$ is clear. Let us fix any x in the right side. We take $s \ge 1$ such that $\sum_{i=1}^{n} s|x_i| = 1$. Then

$$x = \sum_{i=1}^{n} r|x_i| \frac{sign(x_i)}{r} e_i$$

So, $x \in conv(S)$. □

Proof of (iii). We set $T := \{(x, y) \in \mathbb{Q}^n \times \mathbb{Q}^n | (x, y)$ satisfies equalities and inequalities of (i)$\}$. Clearly $p_1(T) \subset conv(S)$. Clearly T is convex. Because $P_i \times \{e_i\} \subset T \ (\forall i)$, $S \subset p_1(T)$. So, $conv(S) \subset T$. □

Proof of (iv). Clearly $x_1 = x_2 = y_1 = y_2 = \frac{1}{2}, x_3 = y_3 = 0$ is a feasible solution. We will show this is a basic feasible solution. By Proposition1.11, it is enough to show the column vectors of

	x_1	x_2	y_1	y_2
$x_1 \le 1 - y_1$	1	0	0	1
$x_2 \le 1 - y_2$	0	1	1	0
$y_1 + y_2 = 1$	0	0	1	1
$x_1 + x_2 = 1$	1	1	0	0

are linear independent. Because this matrix is nonsingular, the column vectors of this matrix are linear independent. □

Proof of (v). By the same argument as the proof of (iii), we can show this formulation is sharp. For locally ideal property, it is enough to show for any basic feasible solution (x^+, x^-, y, z) there is $\#\{i|y_i \ne 0\} = 1$. Because $\sum_{i=1}^{n} y_i = 1$, $\#\{i|y_i \ne 0\} \ge 1$. For aiming contradiction, let us assume $\#\{i|y_i \ne 0\} > 1$. So, there are $i_1 \ne i_2$ such that $y_{i_1}, y_{i_2} > 0$. We can assume $i_1 =, i_2 = 2$. We will show in each case of the followings.

case1 $|x_1| < y_1$ or $|x_2| < y_2$.

case2 $|x_1| = y_1$ and $|x_2| = y_2$.

In case1, we can assume $|x_1| < y_1$. If $|x_2| < y_2$, then By Proposition1.11, the clumns vectors of the following matrix are linear independent.

	y_1	y_2
$*$	0	0
...	...	....
$*$	0	0
$\sum_i y_i = 1$	1	1

This is contradiction. So, $|x_{i_2}| = y_{i_2}$. By Proposition1.11, the clumns vectors of the following matrix are linear independent.

	y_1	y_2	x_2^*
$*$	0	0	0
...	...	...	0
$*$	0	0	0
$q_2 y_2 + r_2 x_2 \le 0$	0	q_2	r_2
$\sum_i y_i = 1$	1	1	0

Here, $q_2 r_2 \ne 0$. So, the clumns vectors of the following matrix are linear independent.

	y_1	y_2	x_2^*
$*$	0	0	0
...	...	...	0
$*$	0	0	0
$q_2 y_2 + r_2 x_2 \le 0$	0	0	r_2
$\sum_i y_i = 1$	1	0	0

This is contradiction.

In case2, By Proposition1.11, the clumns vectors of the following matrix are linear independent.

$$\begin{array}{ccccc} & y_1 & y_2 & x_1^* & x_2^* \\ * & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... \\ * & 0 & 0 & 0 & 0 \\ q_1y_1 + r_1x_1 \leq 0 & q_1 & 0 & r_1 & 0 \\ q_2y_2 + r_2x_2 \leq 0 & 0 & q_2 & 0 & r_2 \\ \sum_i y_i = 1 & 1 & 1 & 0 & 0 \end{array}$$

Here, $q_1r_1q_2r_2 \neq 0$. So, the clumns vectors of the following matrix are linear independent.

$$\begin{array}{ccccc} & y_1 & y_2 & x_1^* & x_2^* \\ * & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... \\ * & 0 & 0 & 0 & 0 \\ q_1y_1 + r_1x_1 \leq 0 & 0 & 0 & r_1 & 0 \\ q_2y_2 + r_2x_2 \leq 0 & 0 & 0 & 0 & r_2 \\ \sum_i y_i = 1 & 1 & 1 & 0 & 0 \end{array}$$

This is contradiction.

Consequently, $\#\{i | y_i \neq 0\} \leq 1$. □

2 Event graph analysis

2.1 Max-plus algebra

Definition 2.1 (Semi-ring)**.** *Here are the settings.*

(S1) R is a set.

(S2) $\oplus, \otimes$ are binomial operators on R.

We say $(R, \oplus, \otimes)$ is a semi ring if

(i) For any $x, y, z \in R$,

$$(x \oplus y) \oplus z = x \oplus (y \oplus z)$$

$$(x \otimes y) \otimes z = x \otimes (y \otimes z)$$

(ii) For any $x, y, z \in R$,

$$x \oplus y = y \oplus x$$

(iii) For any $x, y, z \in R$,

$$x \otimes (y \oplus z) = x \otimes y \oplus x \otimes z$$

(iv) R has the unit element ϵ with respect to $\oplus$.

(v) R has the unit element e with respect to $\oplus$.

(vi) $\epsilon \otimes x = x \otimes \epsilon = \epsilon$.

We say R is commutative if $\otimes$ is commutative. We say R is idempotent if $\otimes$ is idempotent.

Definition 2.2 ($\mathbb{R}_{max}$)**.** *Here are the settings.*

(S1) $\mathbb{R}_{max} := \mathbb{R} \cup \{-\infty\}$. We set $\epsilon := -\infty$ and $e := 0$.

(S2) For $x, y \in \mathbb{R}_{max}$

$$x \oplus y := \max\{x, y\}$$

$$x \otimes y := x + y$$

We call $\mathcal{R}_{max} := (\mathbb{R}_{max}, \oplus, \otimes)$ the max-plus algebra.

Clearly the following holds.

Proposition 2.1. *$\mathcal{R}_{max}$ is a commutative and idempotent semi ring.*

2.2 Petri net and Event graph

Definition 2.3 (Petri net, place, transition)**.** *Here are the settings.*

(S1) $(\mathcal{N}, \mathcal{A})$ *is a directed graph.*

We say $(\mathcal{N}, \mathcal{A})$ *is a petri net if there is* $(\mathcal{P}, \mathcal{Q})$ *which is a pair of disjoint subsets of* $\mathcal{N}$ *satisfying the following two conditions.*

(i) $\mathcal{N} = \mathcal{P} \cup \mathcal{Q}, \mathcal{P} \cap \mathcal{Q} = \phi$.

(ii) $\mathcal{A} \subset \mathcal{P} \times \mathcal{Q} \cup \mathcal{Q} \times \mathcal{P}$.

We denote this petri net by $(\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A})$.

We call each element of $\mathcal{P}$ *a place and call each element of* $\mathcal{Q}$ *a transistion. Let us fix* $p \in \mathcal{P}$ *and* $q \in \mathcal{Q}$. *We say* p *is the input place of the transition* q *and* q *is the output place of the transition* p *if* $(p, q) \in \mathcal{A}$. *We say* p *is the output place of the transition* q *and* q *is the input place of the transition* p *if* $(p, q) \in \mathcal{A}$.

We denote the set of all input place of q *by* $\pi(q)$ *and denote the set of all input transition of* p *by* $\pi(p)$.

We denote the set of all output place of q *by* $\sigma(q)$ *and denote the set of all output transition of* p *by* $\sigma(p)$.

Definition 2.4 (Event graph)**.** *Here are the settings.*

(S1) $(\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A})$ *is a petri net.*

We say this petri net is an event graph if for each $p \in \mathcal{P}$ *there is the unique* $q_1 \in \mathcal{Q}$ *such that* $(p, q_1) \in \mathcal{A}$ *and there is the unique* $q_2 \in \mathcal{Q}$ *such that* $(q_2, p) \in \mathcal{A}$.

Definition 2.5 (Enability and Firing in petri net)**.** *Here are the settings.*

(S1) $(\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A})$ *is a petri net.*

(S2) $w : \mathcal{A} \to \mathbb{N}_{\geq 1}$. *We call* $w(a)$ *is the weight of* $a \in \mathcal{A}$.

(S3) $M_1 : \mathcal{P} \to \mathbb{Z}_{\geq 0}$. *For each* $p \in \mathcal{P}$, *we say* p *is marked with* $M_1(p)$ *tokens.*

(S4) $q \in \mathcal{Q}$.

Then

(i) We say q *is enable if each input place* p *of* q *is marked with at least* $w(p, q)$ *tokens.*

(ii) Let us assume q *is enable. We set for each* $p \in \mathcal{P}$

$$M_1(p) := M_0(p) + \chi_{\sigma(q)}(p)w(q, p) - \chi_{\pi(q)}(p)w(p, q)$$

We call M_1 *the firing of* M_0 *with respect to* q.

Definition 2.6 (Liveness, Autonomous, Time event graph)**.** *Here are the settings.*

(S1) $G := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_0)$ *is an event graph with weight and token.*

Then

(i) We say G *is liveness if for any cycle* c *of* G *there is* $p \in \mathcal{P}$ *whose output transition is enable.*

(ii) For each $q \in \mathcal{Q}$, q *is a supplier transition if* $\pi(q) = \phi$.

(iii) We say G *is autonomous if* G *is no supplier transitions.*

(iv) Let $\tau : \mathcal{P} \to \mathbb{Z}_{\geq 0}$ *and* $\gamma : \mathcal{A} \cap \mathcal{P} \times \mathcal{Q} \to \mathbb{Z}_{\geq 0}$ *such that*

$$\gamma(p, q) \leq \tau(p)$$

Then (G, τ, γ) *with time event graph.*

Definition 2.7 (Enability and Firing in Time event graph)**.** *Here are the settings.*

(S1) $G := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_0, \tau, \gamma_0)$ *is a time event graph.*

(A1) For any $q_1, q_2 \in \mathcal{Q}$, *there is at most one* $p \in \mathcal{P}$ *such that* $(q, p), (p, q) \in \mathcal{A}$.

(A2) $w = 1$ *on* $\mathcal{A}$.

(S2) $q \in \mathcal{Q}$.

Then

(i) *We say q is enable if each input place p of q is marked with at least $w(p,q)$ tokens and $\tau(p) \leq \gamma(p,q)$. We denote the all enable transitions by $E(G)$.*

(ii) *Let us assume q is enable. We set for each $p \in \mathcal{P}$*

$$M_1(p) := M_0(p) + \chi_{\sigma(q)}(p)w(p,q) - \chi_{\pi(q)}(p)w(p,q), \gamma_1(p) := 0$$

We call (M_1, γ_1) the firing of (M_0, γ_0) with respect to q.

Clearly the following holds.

Proposition 2.2. *Here are the settings.*

(S1) $G_0 := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_0, \tau, \gamma_0)$ *is a time event graph.*

(A1) For any $q_1, q_2 \in \mathcal{Q}$, there is at most one $p \in \mathcal{P}$ such that $(q,p),(p,q) \in \mathcal{A}$.

(A2) $w = 1$ on $\mathcal{A}$.

(S3) We set

$$M_1(p) := M_0(p) + \chi_{E(G_0)}(q_1) - \chi_{E(G_0)}(q_2)$$

Here $q_1 \in \pi(p)$ and $q_2 \in \sigma(p)$. And

$$\gamma_1(p,q) := \begin{cases} \gamma_0(p,q) + 1 & M_0(p) > 0 \text{ and } q \text{ is not enable} \\ 0 & \text{otherwise} \end{cases}$$

(S4) We set $G_1 := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_1, \tau, \gamma_1)$.

Then G_1 is a time event graph.

Definition 2.8 (Firing time)**.** *Here are the settings.*

(S1) $G_0 := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_0, \tau, \gamma_0)$ *is a time event graph.*

(A1) For any $q_1, q_2 \in \mathcal{Q}$, there is at most one $p \in \mathcal{P}$ such that $(q,p),(p,q) \in \mathcal{A}$.

(A2) $w = 1$ on $\mathcal{A}$.

(S3) We define $\{G_t\}_{t=0}^{\infty}$ inductively by the procedure defined in Proposition2.2.

Then

$$x_q(k) := \{t_0 \in \mathbb{Z}_{\geq 0} | k = \#\{t \leq t_0 | q \in E(G_t)\}\} \ (q \in \mathcal{Q}, k \in \mathbb{N}_{\geq 1})$$

We call $x_q(k)$ the k-th firing time of q. We set

$$x(k) := (x_{q_1}(k), ..., x_{q_{\#\mathcal{Q}}})^T \ (k \in \mathbb{N}_{\geq 1})$$

Definition 2.9 (System Matrix)**.** *Here are the settings.*

(S1) $\{G_t := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_t, \tau, \gamma_t)\}_{t \in \mathbb{Z}_{\geq 0}}$ is a sequence of time event graphs by the procedure defined in Proposition2.2.

(S2) $\{x(k)\}_{k=1}^{\infty}$ is the sequence by Definition2.8.

(S3) We denote the maximum number of tokens at any one place in $\{G_t\}_{t \in \mathbb{Z}_{\geq 0}}$ by M.

Then for each $m \in \{0, 1, ..., M\}$

$$[A_m]_{j,l} := \begin{cases} a_{j,l} & p_{j,l} \text{ exists and } p_{j,l} \text{ has } m \text{ tokens in } G_0 \\ \epsilon & \text{otherwise} \end{cases} \quad (j, l = 1, 2, ..., \#\mathcal{Q})$$

Here $p_{j,l}$ is the place such that $(q_j, p_{j,l}), (p_{j,l}, q_l) \in \mathcal{A}$.

Proposition 2.3. *We succeed notations in Definition2.9. And let us assume any G_t is autonomous. Then*

$$x(k) = A_0 \otimes x(k) \oplus A_1 \otimes x(k-1) \oplus ... \oplus A_M \otimes x(k-M) \ (k = M+1, M+2, ...)$$

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