A Study Note on Applied Mathematics

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Contents

This is a study memo of [8], [12].

1 Preliminaries

1.1 Basic Notations

Notation 1.1 (The set of all probability measures on (R))**.** *Denote the set of all borel sets on $\mathbb{R}$ by $\mathscr{B}(\mathbb{R})$. Denote the set of all probability measures on $\mathscr{B}(\mathbb{R})$ by $\mathscr{P}(R)$.*

Notation 1.2 (order relation in $\mathbb{R}^n$)**.** *Let $x, y \in \mathbb{R}^n$. Denote $x \le y$ $(x < y)$ if $x_i \le y_i$ $(x_i < y_i)$ $(\forall i)$.*

Definition 1.3 (A distribution of random variables)**.** *Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $X = (X_1, X_2, ..., X_n)$ be random variables on Ω. We define $P_X : \mathscr{B}(\mathbb{R}^n) \ni A \mapsto P(X^{-1}(A)) \in [0,1]$. We denote the distribution of X by P_X.*

Definition 1.4 (A distribution function of a probability measure)**.** *Let $\mu \in \mathscr{R}(\mathbb{R}^n)$. We define $F_\mu : \mathbb{R}^n \ni x \mapsto \mu((-\infty, x_1] \times (-\infty, x_2] ... \times (-\infty, x_n]) \in \mathbb{R}$ and we call F_μ the destribution function of μ.*

Notation 1.5 (Fourier transform)**.** *Let $f \in L^1(\mathbb{R}^n)$. Denote fourier transformation of f by $\mathscr{F}(f)$ and denote fourier inverse transformation of f by $\mathscr{F}^{-1}(f)$.*

Definition 1.6 (Weakly convergence of probability measures)**.** *Let*

(S1) $(\Omega, \mathscr{F}, P)$ is a probability space.

(S2) Let $\{\mu_n\}_{n=1}^\infty \in \mathcal{P}(\mathbb{R}^N)$.

(S3) Let $\mu \in \mathcal{P}(\mathbb{R}^N)$.

$\{\mu_n\}_{n=1}^\infty$ is weakly converges to μ if $\lim_{n\to\infty} F_{\mu_n}(x) = F_\mu(x)$ for any point x at which F_μ is continuous. Denote this by $\mu_n \Longrightarrow \mu$ $(n \to \infty)$

Definition 1.7 (Characteristic function of probability measure)**.** *Let*

(S1) $(\Omega, \mathscr{F}, P)$ is a probability space.

(S2) Let $\mu \in \mathcal{P}(\mathbb{R}^n)$.

then call $\varphi_\mu : \mathbb{R}^n \ni t \mapsto \int_{\mathbb{R}^n} exp(itx) d\mu(x) \in \mathbb{C}$ is the characteristic function of μ. Bellow, assume the characteristic function of μ denotes φ_μ unless otherwise noted.

Definition 1.8 (Characteristic function of random variables)**.** *Let*

(S1) $(\Omega, \mathscr{F}, P)$ is a probability space.

(S2) Let $X = (X_1, X_2, ..., X_n)$ be a vector of random variables on $(\Omega, \mathscr{F}, P)$.

then call $\varphi_X : \mathbb{R} \ni t \mapsto \int_\Omega exp(itX) dP \in \mathbb{C}$ is the characteristic function of X. Bellow, assume the characteristic function of X denotes φ_X unless otherwise noted.

Definition 1.9 (Tightness of probability measures)**.** *Let*

(S1) $(\Omega, \mathscr{F}, P)$ is a probability space.

(S2) Let $\{\mu_n\}_{n=1}^\infty \in \mathcal{P}(\mathbb{R}^N)$.

$\{\mu_n\}_{n=1}^\infty$ is tight if for any $\epsilon > 0$ there is a $M > 0$ such that

$$\mu_n(\{x \in \mathbb{R}^N | |x| \le M\}) \ge 1 - \epsilon \tag{1.1.1}$$

Definition 1.10 (Weakly compactness of probability measures)**.** *Let*

(S1) $(\Omega, \mathscr{F}, P)$ is a probability space.

(S2) Let $\{\mu_n\}_{n=1}^\infty \subset \mathcal{P}(\mathbb{R}^N)$.

$\{\mu_n\}_{n=1}^\infty$ is weakly compact if for any subsequence $\{\mu_{\alpha(n)}\}_{n=1}^\infty$ of $\{\mu_n\}_{n=1}^\infty$ there is a subsequence of $\{\mu_{\alpha(n)}\}_{n=1}^\infty$ which weakly comverges to a probability measure.

Definition 1.11 (Outer measure)**.** *Let*

(S1) X is a set.

$\Gamma : 2^X \to [0, \infty]$ is an outer measure on X if the followings hold.

(i) $\Gamma(\phi) = 0$

(ii) If $A \subset B$ then $\Gamma(A) \le \Gamma(B)$

(iii) If $\{A_i\}_{i=1}^\infty \subset 2^X$ then $\Gamma(\cup_{i=1}^\infty A_i) \le \Sigma_{i=1}^\infty \Gamma(A_i)$

1.2 Finite measures on metric space

We introduce several definitions and propositions for only Section2.2.2.

1.3 several facts on metric space

The following three definitions are from [2].

Definition 1.12 (Elementary function family). *Let*

(S1) (X, d) *is a metric space.*

$\mathscr{E} \subset Map(X, [0, \infty))$ *is called a family of elementary functions if the followings holds.*

(i) *if* $f, g \in \mathscr{E}$ *then* $f + g \in \mathscr{E}$.

(ii) *if* $f, g \in \mathscr{E}$ *and* $f \geq g$ *then* $f - g \in \mathscr{E}$.

(iii) *if* $f, g \in \mathscr{E}$ *then* $min\{f, g\} \in \mathscr{E}$.

Definition 1.13 (Elementary integral). *Let*

(S1) (X, d) *is a metric space.*

(S2) $\mathscr{E} \subset Map(X, [0, \infty))$ *is a elementary function family.*

$l : \mathscr{E} \mapsto [0, \infty]$ *is an elementary integral on* $\mathscr{E}$ *if the followings hold.*

(i) *if* $f, g \in \mathscr{E}$ *then* $l(f + g) = l(f) + l(g)$

(ii) *if* $f, g \in \mathscr{E}$ *and* $f \leq g$ *then* $l(f) \leq (g)$

Definition 1.14 (Complete elementary integral). *Let*

(S1) (X, d) *is a metric space.*

(S2) $\mathscr{E} \subset Map(X, [0, \infty))$ *is a elementary function family.*

(S3) $l : \mathscr{E} \mapsto [0, \infty]$ *is an elementary integral.*

l *is a complete elementary integral if for any* $\{f_n\}_{\{n=1\}}^{\infty}$ *such that* $\lim_{n\to\infty} f_n = f$ *(pointwise convergence) and* $f_n \leq f_{n+1}$ $(\forall n \in \mathbb{R})$ *satisfies* $\lim_{n\to\infty} l(f_n) = l(f)$

Definition 1.15 (Functional from elementary integral). *Let*

(S1) (X, d) *is a metric space.*

(S2) $\mathscr{E} \subset Map(X, [0, \infty))$ *is a elementary function family.*

(S3) $l : \mathscr{E} \mapsto [0, \infty]$ *is an elementary integral.*

We define

$$L : \{\varphi : X \to [0, \infty)\} \ni \varphi \mapsto inf\{\Sigma_{i=1}^{\infty} l(\varphi_i) | \varphi_i \in \mathscr{E}\ (\forall i), \varphi \leq \Sigma_{i=1}^{\infty} \varphi_i\} \in [0, \infty] \tag{1.3.1}$$

Proposition 1.16. *Let*

(S1) (X, d) *is a metric space.*

(S2) $\mathscr{E} \subset Map(X, [0, \infty))$ *is a elementary function family.*

(S3) $l : \mathscr{E} \mapsto [0, \infty]$ *is an elementary integral.*

(A1) $[0, \infty)\mathscr{E} \subset \mathscr{E}$.

For any $\alpha > 0$ *and* $f \in \mathscr{E}$

$$l(\alpha f) = \alpha l(f) \tag{1.3.2}$$

Proof. Let us fix $q_1 \in (\alpha, \infty) \cap \mathbb{Q}$ and $q_2 \in (0, \alpha) \cap \mathbb{Q}$. $q_2 l(f) = l(q_2 f) \leq l(\alpha f) \leq l(q_1 f) = q_1 l(f)$. So $l(\alpha f) = \alpha l(f)$ □

Proposition 1.17 (Outer measure from elementary integral). *Let*

(S1) (X, d) *is a metric space.*

(S2) $\mathscr{E} \subset Map(X, [0, \infty))$ *is a elementary function family.*

(S3) $l : \mathscr{E} \mapsto [0, \infty]$ *is an elementary integral.*

(S4) L is the functional in Definition1.15.

(S5) We set $\Gamma : 2^X \ni A \mapsto L(\chi_A)$.

then Γ is outer measure on X.

Proof. It is easy to show terms except (iii) in Definition1.11. So we will show only (iii) in Definition1.11. Let us fix $A_{i\,i=1}^{\infty} \subset 2^X$.

Let us fix $\epsilon > 0$.

For each $i \in \mathbb{N}$, there are $\{\varphi_{i,j}\}_{j=1}^{\infty} \subset \mathscr{E}$ such that $\chi_{A_i} \leq \Sigma_{j=1}^{\infty}\varphi_{i,j}$ and $\Sigma_{j=1}^{\infty} l(\varphi_{i,j}) \leq \Gamma(A_i) + \frac{\epsilon}{2^i}$

So $\chi_{\cup_{i=1}^{\infty}A_i} \leq \Sigma_{i=1,j=1}^{\infty}\varphi_{i,j}$.

$\Gamma(\cup_{i=1}^{\infty}A_i) \leq \Sigma_{i=1,j=1}^{\infty} l(\varphi_{i,j}) \leq \Sigma_{i=1}^{\infty}\Gamma(A_i) + \epsilon$

Consequently, (iii) holds. □

Proposition 1.18. *Let*

(S1) (X, d) is a metric space.

(S2) $\mathscr{E} \subset Map(X, [0, \infty))$ is a elementary function family.

(S3) $l : \mathscr{E} \mapsto [0, \infty]$ is an elementary integral.

(S4) L is the functional in Definition1.15.

(S5) Γ is the outer measure in Proposition1.17.

(S6) $\mathfrak{M}_\Gamma$ is the σ-algebra in Proposition2.9.

(A1) $C_+(X) \subset \mathscr{E}$.

(A2) If A, B are borel sets and $d(A, B) > 0$ then $\mu(A) + \mu(B) = \mu(A \cup B)$.

then $\mathscr{B}(X) \subset \mathfrak{M}_\Gamma$.

Proof. Because $\mathfrak{M}_\Gamma$ is σ-algebra, it is enough to show that all closed sets are contained in $\mathfrak{M}_\Gamma$.

Let us fix closed set A. Let us subset B and C such that $A \subset B$ and $C \subset A^c$.

Because A is closed set, $C \subset \{x | d(x, A) > 0\}$.

For each $n \in \mathbb{N}$ we set $C_n := \{x \in C | d(x, A) > \frac{1}{n}\}$ and $D_n := \{x \in C | \frac{1}{n-1} \geq d(x, A) > \frac{1}{n}\}$.

The followings holds.

$$C = \cup_{n=1}^{\infty} D_n \tag{1.3.3}$$

$$C_N = \cup_{n=1}^{N} D_n \ (\forall N) \tag{1.3.4}$$

We assume $\Sigma_{n=1}^{\infty}\Gamma(D_n) < \infty$. Let us fix $\epsilon > 0$.

There is n_0 such that $\Sigma_{n=n_0}^{\infty}\Gamma(D_n) < \epsilon$.

Because $d(A, C_{n_0}) > 0$,

$$\begin{aligned} \Gamma(A) + \Gamma(C) &= \Gamma(A) + \Gamma(C_{n_0} \cup \cup_{n=n_0}^{\infty} D_n) \\ &\leq \Gamma(A) + \Gamma(C_{n_0}) + \epsilon \\ &\leq \Gamma(A) + \Gamma(C_{n_0}) + \epsilon \\ &= \Gamma(A \cup C_{n_0}) + \epsilon \\ &\leq \Gamma(A \cup C) + \epsilon \end{aligned} \tag{1.3.5}$$

So if $\Sigma_{n=1}^{\infty}\Gamma(D_n) < \infty$ then $\Gamma(A) + \Gamma(C) = \Gamma(A \cup C)$.

We assume $\Sigma_{n=1}^{\infty}\Gamma(D_n) = \infty$. Then $\Sigma_{n=1}^{\infty}\Gamma(D_{2n}) = \infty$ or $\Sigma_{n=1}^{\infty}\Gamma(D_{2n-1}) = \infty$. We assume $\Sigma_{n=1}^{\infty}\Gamma(D_{2n}) = \infty$.

If $n_1 \neq n_2$ then $d(D_{n_1}, D_{n_2}) > 0$. So $\Gamma(C) \geq \Gamma(\cup_{n=1}^{\infty} D_{2n}) \geq \Sigma_{n=1}^{\infty}\Gamma(D_{2n}) = \infty$. So if $\Sigma_{n=1}^{\infty}\Gamma(D_{2n}) = \infty$ then $\Gamma(B) + \Gamma(C) = \Gamma(A \cup C) = \infty$.

Similary, if $\Sigma_{n=1}^{\infty}\Gamma(D_{2n-1}) = \infty$ then $\Gamma(B) + \Gamma(C) = \Gamma(A \cup C) = \infty$. □

Proposition 1.19. *Let*

(S1) (X, d) is a metric space.

(S2) $\mathscr{E} \subset Map(X, [0, \infty))$ is a elementary function family.

(S3) $l : \mathscr{E} \mapsto [0, \infty]$ is an elementary integral.

(S4) $\{f_n\}_{n=1}^{\infty} \subset \mathscr{E}$ and $f_n \geq f_{n+1}$ on X $(\forall n)$.

(A1) There is $f \in \mathscr{E}$ such that $\lim_{n\to\infty} ||f_n - f||_\infty = 0$

(A2) $\mathbb{R}\mathscr{E} \subset \mathscr{E}$

then

$$\lim_{n\to\infty} l(f_n) = l(f) \tag{1.3.6}$$

Proof. $|l(f) - l(f_n)| = l(f - f_n) \leq ||f - f_n||_\infty l(1) \to 0 \ (n \to \infty)$ □

Proposition 1.20. *Let*

(S1) (X, d) *is a metric space.*

(S2) $l : \mathscr{E} \mapsto [0, \infty]$ *is an elementary integral on* $\mathscr{E} := \{f | f$ *is nonnegative borel measurable on* X $\}$.

(S3) L *is the functional in Definition1.15.*

(S4) $h_1, h_2 \in \mathscr{E}$.

(A1) $d(supp(h_1), supp(h_2)) > 0$.

then $L(h_1 + h_2) = L(h_1) + L(h_2)$

Proof. Let us fix arbitary $\epsilon > 0$. Let us fix f and g in Proposition**??**.

Let us fix $\{\varphi_i\} \subset \mathscr{E}$ such that $h_1 + h_2 \leq \Sigma_{i=1}^\infty \varphi_i$ and $\Sigma_{i=1}^\infty l(\varphi_i) \leq L(h_1 + h_2) + \epsilon$.

By definition of f and g,

$$h_1 + h_2 \leq (f + g)\Sigma_{i=1}^\infty \varphi_i \tag{1.3.7}$$

and

$$h_1 \leq f\Sigma_{i=1}^\infty \varphi_i \tag{1.3.8}$$

and

$$h_2 \leq g\Sigma_{i=1}^\infty \varphi_i \tag{1.3.9}$$

So

$$\begin{aligned} L(h_1 + h_2) + \epsilon &\geq \Sigma_{i=1}^\infty l(\varphi_i) \\ &\geq \Sigma_{i=1}^\infty (l(f\varphi_i) + \Sigma_{i=1}^\infty l(g\varphi_i)) \\ &\geq L(h_1) + L(h_2) \end{aligned} \tag{1.3.10}$$

Consequently

$$L(h_1) + L(h_2) \leq L(h_1 + h_2) \tag{1.3.11}$$

□

Proposition 1.21. *Let*

(S1) (X, d) *is a metric space.*

(S2) $l : \mathscr{E} \mapsto [0, \infty]$ *is an elementary integral on* $C_+(X)$.

(S3) L *is the functional in Definition1.15.*

(S4) Γ *is the outer measure in Proposition1.17.*

(S5) $\mathfrak{M}_\Gamma$ *is the* σ*-algebra in Proposition2.9.*

then $\mathscr{B}(X) \subset \mathfrak{M}_\Gamma$.

Proof. Let us fix arbitary borel sets A, B such that $d(A, B) > 0$.

By Proposition1.20, $\Gamma(A \cup B) = L(\chi_{A\cup B}) = L(\chi_A + \chi_B) = L(\chi_A) + L(\chi_B) = \Gamma(A) + \Gamma(B)$.

By Proposition1.18, $\mathscr{B}(X) \subset \mathfrak{M}_\Gamma$. □

1.4 several facts on compact metric spaces

Proposition 1.22. *Let*

(S1) (X,d) is a compact metric space.

(S2) l is an elementary integral on $C_+(X)$. $C_+(X) := \{f \in C(X) | f \geq 0\}$

then there is an unique measure μ on $(X, \mathscr{B}(X))$ such that for any $f \in C_+(X)$

$$l(f) = \int_X f\mu \tag{1.4.1}$$

Existence. Let us fix $f \in C_+(X)$.

By replacing f by $||f||_\infty - f$, it is enough to show

$$\int_X f d\mu l(f) \leq l(f) \tag{1.4.2}$$

By an argument similar to one in the proof of Proposition2.17, there are $a_{m,i_{1\leq m\leq\infty,1\leq i\leq\varphi(m)}} \subset \mathbb{R}$ such that

$$0 = a_{m,1} \leq a_{m,2} \leq ... \leq a_{m,\varphi(m)} > ||f||_\infty \ (\forall m \in \mathbb{N}) \tag{1.4.3}$$

$$|a_{m,i} - a_{m,i+1}| \leq \frac{1}{2^m} \ (\forall m, \forall i) \tag{1.4.4}$$

$$\mu(\{f = a_{m,i}\}) = 0 \ (\forall m, \forall i) \tag{1.4.5}$$

We set

$$h_m := \Sigma_{i=1}^{\varphi(m)} a_{m,i} \chi_{[a_{m,i}, a_{m,i+1})} \ (m \in \mathbb{N}) \tag{1.4.6}$$

and

$$h_{m,n} := \Sigma_{i=1}^{\varphi(m)} a_{m,i} \chi_{(a_{m,i}+\frac{1}{n}, a_{m,i+1}-\frac{1}{n})} \ (m \in \mathbb{N}, \ 1 \leq i \leq \varphi(m)) \tag{1.4.7}$$

Let us fix $\epsilon > 0$.

By Proposition2.10, $f \in C_u(X)$.

By (1.4.5), there is m, n such that

$$|\int_X f d\mu - \int_X h_{m,n} d\mu| < \epsilon \tag{1.4.8}$$

Because $f \in C_u(X)$, if $i \neq j$ then $d(f^{-1}((a_{m,i} + \frac{1}{n}, a_{m,i+1} - \frac{1}{n})), f^{-1}((a_{m,j} + \frac{1}{n}, a_{m,j+1} - \frac{1}{n}))) > 0$.

So

$$l(f) \geq L(h_{m,n} \geq \int_X h_{m,n} d\mu \tag{1.4.9}$$

Therefore,

$$\int_X f d\mu - \epsilon \leq l(f) \tag{1.4.10}$$

Consequently,

$$\int_X f d\mu \leq l(f) \tag{1.4.11}$$

□

Uniqueness. Let us fix arbitary $\mu_1 \in \mathscr{P}(X)$ and arbitary $\mu_2 \in \mathscr{P}(X)$ such that

$$\int_X f d\mu_1 = \int_X f d\mu_2 \ (\forall f \in C_+(X)) \tag{1.4.12}$$

We set $\mathscr{B} := A \in \mathscr{B}(X) | \mu_1(A) = \mu_2(A)$. Clearly $\mathscr{B}$ is σ-algebra.

Let us fix closed set A.

By Proposition**??**, there are $\{f_m\}_{m=1}^{\infty} \subset C_+(X)$ such that

$$||f_m||_\infty \leq 1 \ (\forall m) \tag{1.4.13}$$

and

$$lim_{m\to\infty} f_m = \chi_A \ (pointwize\ convergence) \tag{1.4.14}$$

By Lebesugue's convergence theorem, $\mu_1(A) = \mu_2(A)$.
So $A \in \mathscr{B}$.
Consequently $\mathscr{B} \subset \mathscr{B}(X)$.

□

2 Probability

2.1 Some Facts Used Without Proofs

In this note, we use the following propositions without proofs.

Proposition 2.1. *Let*

> *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
>
> *(S2)* X *is a* N*-dimensional vector of random variables on* $(\Omega, \mathscr{F})$.
>
> *(S3) Let* μ_X *be a probability distribution of* X.
>
> *(S4)* $f \in L^1(\Omega) \cup L^\infty(\Omega)$

then

$$\int_{\mathbb{R}^N} f d\mu_X = \int_\Omega f \circ X dP \tag{2.1.1}$$

Proposition 2.2. *For any* $\eta > 0$,

$$\mathscr{F}(exp(-\eta(\cdot)^2) = \frac{1}{\sqrt{2\eta}} exp(-\frac{(\cdot)^2}{4\eta}). \tag{2.1.2}$$

Proposition 2.3. *Let* Σ *be a positive definite symmetric matrix.*

$$\varphi_{N(0,\Sigma)}(\boldsymbol{t}) = exp(-\frac{\boldsymbol{t}^T \Sigma^2 \boldsymbol{t}}{2}) \tag{2.1.3}$$

Proposition 2.4. *Let*

> *(S1) Arbitrarily take* $M > 0$ *and fix it.*
>
> *(S2) Let* $f_n : \overline{D(0, M)} \ni z \mapsto (1 + \frac{z}{n})^n \in \mathbb{C}$, *where* $\overline{D(0, M)} := \{z \in \mathbb{C} | \ |z| \leq M\}$, $(n = 1, 2, ...)$.

then $\{f_n\}_{n=1}^\infty$ *uniformly converges to exp on* $\overline{D(0, M)}$.

Proposition 2.5. *Let*

> *(A1) Let* $F : \mathbb{R} \mapsto \mathbb{R}$ *is monotone increasing.*

then $\{x \mid F \text{ is not continuous at } x\}$ *is at most countable.*

Proposition 2.6. *Let*

> *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
>
> *(S2) Let* $\{\mu_n\}_{n=1}^\infty \subset \mathcal{P}(\mathbb{R})$.
>
> *(A1) Let* $\mu \in \mathscr{P}(\mathbb{R})$ *such that* $\mu_n \Longrightarrow \mu$ $(n \to \infty)$.

then for any bounded continuous function $f : \mathbb{R} \mapsto \mathbb{R}$

$$lim_{n\to\infty} \int_\mathbb{R} f(x) d\mu_n(x) = \int_\mathbb{R} f(x) d\mu(x) \tag{2.1.4}$$

Proposition 2.7. *Let*

> *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
>
> *(S2)* μ *is a probability measure on* $\mathbb{R}$.
>
> *(A1)* $E[\mu] = 0$ *and* $V[\mu] = 1$.

then $\varphi_\mu(s) = 1 - \frac{s^2}{2} + o(s^2)$ $(s \to 0)$

The following propositions are used for only Section1.2 and Subsection2.2.2.

Proposition 2.8. *Let*

> *(S1)* (X, d) *is a metric space.*

then there is a complete metric space $(\tilde{X}, \tilde{d})$ *and an isometry mapping* $i : (X, d) \to (\tilde{X}, \tilde{d})$ *such that* $i(X)$ *is dense in* $\tilde{X}$. *We call* $(\tilde{X}, \tilde{d})$ *is a completion of* (X, d).

Proposition 2.9. *Let*

(S1) X *is a set.*

(S2) Γ *is an outer measure on* X.

(S3) $\mathfrak{M}_\Gamma := \{A \subset X| \text{ if } B \subset A \text{ and } C \subset A^c \text{ then } \mu(B) + \mu(C) = \mu(B \cup C)\}$.

then the followings holds.

(i) $\mathfrak{M}_\Gamma$ *is a* σ*-algebra.*

(ii) Γ *is a measure on* $\mathfrak{M}_\Gamma$.

Proposition 2.10. *Let*

(S1) (X, d) *is a compact metric space.*

then $C(X) \subset C_u(X)$.

Proposition 2.11. *Let*

(S1) $(X, d1)$ *is a compact metric space.*

(S2) $(Y, d2)$ *is a compact metric space.*

(A1) $f \in C(X, Y)$.

then $f(X)$ *is compact in* Y.

Proposition 2.12. $C_c(\mathbb{R}^n)$ *is dense in* $L^1(\mathbb{R}^n)$.

2.2 Weak convergence of probability distributions

2.2.1 The Case of Single Variable

Proposition 2.13 (Helly's selection theorem)**.** *Let*

(S1) $(\Omega, \mathscr{F}, P)$ *is a probability space.*

(S2) *Let* $\{\mu_n\}_{n=1}^\infty \subset \mathcal{P}(\mathbb{R})$ *and denote* F_{μ_n} *by* F_n $(n = 1, 2, 3, ...)$.

Then there is a subsequence $\{F_{\alpha(n)}\}_{n=1}^\infty$ *and* $F : \mathbb{R} \to [0, \infty)$ *such that* F *is monotone increasing and right continuous, and* $F_{\alpha(n)}(x) \to F(x)$ *for any point* x *at which* F *is continuous.*

Proof. There is $\{x_n\}_{n=1}^\infty \subset \mathbb{R}$ such that $\overline{\{x_n\}_{n=1}^\infty} = \mathbb{R}$. Let fix such $\{x_n\}_{n=1}^\infty$. Because $0 \leq F_n(x_m) \leq 1$ (for any m, n in $\mathbb{N}$), there is a subsequence $\{\alpha(n)\}_{n=1}^\infty \subset \mathbb{N}$ and $\{F(x_n)\}_{n=1}^\infty \subset [0, 1]$ such that $F_{\alpha(m)}(x_n) \to F(x_n)$ $(m \to \infty)$. We fix such $\{\alpha(n)\}_{n=1}^\infty$ and $F(x_n)_{n=}^\infty$ We define $F(x) := inf_{m\in\{k|x\leq x_k\}}F(x_m)$. By the definition of F, F is right continuous and monotone increasing. Arbitrarily take $x \in \mathbb{R}$ at which F is continuous and fix it. Arbitrarily take $\epsilon > 0$ and fix it. Let pick $x_{\alpha(m1)}$ and $x_{\alpha(m2)}$ such that $x_{\alpha(m1)} \leq x \leq x_{\alpha(m2)}$ and $(F(x_{\alpha(m2)}) - F(x_{\alpha(m1)})) \leq \frac{\epsilon}{8}$. There is a $n_0 \in \mathbb{N}$ such that $|F_n(x_{\alpha(m1)}) - F(x_{\alpha(m1)})| \leq \frac{\epsilon}{8}$ and $|F_n(x_{\alpha(m2)}) - F(x_{\alpha(m2)})| \leq \frac{\epsilon}{8}$ for any $n \geq n_0$. Let fix such n_0 and $m1$ and $m2$. For any $n \geq n_0$

$$\begin{aligned} |F_n(x_{\alpha(m1)}) - F(x)| &\leq |F_n(x_{\alpha(m1)}) - F(x_{\alpha(m1)})| + |F(x_{\alpha(m1)}) - F(x)| \\ &\leq \frac{\epsilon}{4} \end{aligned} \tag{2.2.1}$$

and

$$\begin{aligned} |F_n(x_{\alpha(m2)}) - F(x)| &\leq |F_n(x_{\alpha(m1)}) - F(x_{\alpha(m1)})| + |F(x_{\alpha(m1)}) - F(x)| \\ &\leq \frac{\epsilon}{4} \end{aligned} \tag{2.2.2}$$

So for any $n \geq n_0$

$$|F_n(x_{\alpha(m1)}) - F_n(x_{\alpha(m2)})| \leq \frac{\epsilon}{2} \tag{2.2.3}$$

Arbitrarily take $n \geq n_0$ and fix it. Because $F_n(x_{m1}) \leq F_n(x) \leq F_n(x_{m2})$,

$$max\{|F_n(x_{\alpha(m1)}) - F_n(x)|, |F_n(x_{\alpha(m2)}) - F_n(x)|\} \leq \frac{\epsilon}{2} \tag{2.2.4}$$

By (2.2.1) and (2.2.2) and (2.2.4),

$$|F_n(x) - F(x)| \leq \epsilon \tag{2.2.5}$$

□

Proposition 2.14. *Let*

(S1) $(\Omega, \mathscr{F}, P)$ *is a probability space.*

(S2) Let $\{\mu_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathbb{R})$.

If $\{\mu_n\}_{n=1}^{\infty}$ *is tight then* $\{\mu_n\}_{n=1}^{\infty}$ *is weakly compact.*

Proof. By Proposition2.13, there is $F : \mathbb{R} \to [0, \infty)$ such that F is monotone increasing and right continuous, and for any point x at which F is continuous

$$F_{\alpha(n)}(x) \to F(x)\ (n \to \infty) \tag{2.2.6}$$

Here we denote F_{μ_n} by F_n. Because of tightness of $\{\mu_n\}_{n=1}^{\infty}$, $limit_{x\to\infty}(F(x) - F(-x)) = 1$. So there is a probability measure μ such that F is a distribution function of μ. By (2.2.6), $\mu_n \Longrightarrow \mu\ (n \to \infty)$. □

Proposition 2.15. *Let*

(S1) $(\Omega, \mathscr{F}, P)$ *is a probability space.*

(S2) Let $\{\mu_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathbb{R})$. *and* $\mu \in \mathcal{P}(\mathbb{R})$

(A1) $\mu_n \Longrightarrow \mu\ (n \to \infty)$.

(A2) Let f *be an arbitary bouded continuous function on* $\mathbb{R}$.

then

$$\lim_{n\to\infty} \int f d\mu_n(x) = \int f d\mu(x) \tag{2.2.7}$$

Proof. Let us fix arbitary $f \in C_b(\mathbb{R})$ and $\epsilon > 0$.

Because $\mu(\mathbb{R}) = 1$ and $\mathbb{R} = \cup_{a\in\mathbb{R}} a$, for each $n \in \mathbb{N}$ $\{a \in \mathbb{R} | \mu(a) > \frac{1}{n}\}$ is finite. So $\{a \in \mathbb{R} | \mu(a) > 0\}$ is at most coutable.

So there is $r_1 > 0$ and $r_2 > 0$ such that

$$1 - \mu((-r_1, r_2)) < \frac{\epsilon}{3(||f||_\infty + 1)} \tag{2.2.8}$$

and $\mu(-r_1) = 0$ and $\mu(-r_2) = 0$.

Because f is uniformly continuous on X,

So there are $a_{m,i_{1\le m\le\infty, 1\le i\le\varphi(m)}} \subset \mathbb{R}$ such that

$$-r_1 = a_{m,1} \le a_{m,2} \le ... \le a_{m,\varphi(m)} = r_2\ (\forall m \in \mathbb{N}) \tag{2.2.9}$$

and

$$|a_{m,i} - a_{m,i+1}| \le \frac{1}{2^m}\ (\forall m, \forall i) \tag{2.2.10}$$

and

$$\mu(\{a_{m,i}\}) = 0\ (\forall m, \forall i) \tag{2.2.11}$$

For each $m \in \mathbb{N}$, set $f_m := \Sigma_{i=1}^{\varphi(m)} f(a_i)\chi_{[a_i, a_{i+1})}$.

Because $\lim_{m\to\infty} f_m = f$ (pointwize convergence), by Lebesugue's convergence theorem there is $m \in \mathbb{N}$ such that

$$|\int_{-r_1}^{r_2} f_m\mu - \int_{-r_1}^{r_2} f\mu| < \frac{\epsilon}{3} \tag{2.2.12}$$

Because

$$\int_{-r_1}^{r_2} f_m\mu = \Sigma_{i=1}^{\varphi(m)} f(a_i)\mu([a_i, a_{i+1})) \tag{2.2.13}$$

and

$$\int_{-r_1}^{r_2} f_m\mu_n = \Sigma_{i=1}^{\varphi(m)} f(a_i)\mu_n([a_i, a_{i+1}))\ (\forall n) \tag{2.2.14}$$

So there is n_0 such that

$$|\int_{-r_1}^{r_2} f_m\mu_n - \int_{-r_1}^{r_2} f_m\mu| < \frac{\epsilon}{3}\ (\forall n \ge n_0) \tag{2.2.15}$$

By (2.2.8) and (2.2.12) and (2.2.15),

$$|\int_{\mathbb{R}} f\mu_n - \int_{\mathbb{R}} f\mu| < \epsilon\ (\forall n \ge n_0) \tag{2.2.16}$$

□

2.2.2 The Case of Multi Variables

Definition 2.16 (Weak convergence(in general metric space)). *Let*

- *(S1)* (X, d) *is a metric space.*
- *(S2)* $\{\mu_n\}_{n=1}^{\infty} \subset \mathscr{P}(X)$.
- *(S3)* $\mu \in \mathscr{P}(X)$.

We say $\{\mu_n\}_{n=1}^{\infty}$ *weakly converges to* μ *if for any borel set* A *such that* $\mu(\partial(A)) = 0$ $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ *Denote* $\mu_n \Longrightarrow \mu$ *by weak convergence.*

The following proposition gives the equivalent definition of weak convergence.

Proposition 2.17. *Let*

- *(S1)* (X, d) *is a metric space.*
- *(S2)* $\{\mu_n\}_{n=1}^{\infty} \subset \mathscr{P}(X)$.
- *(S3)* $\mu \in \mathscr{P}(X)$.

then the followings are equivalent.

- *(i)* $\mu_n \Longrightarrow \mu$.
- *(ii) Set* $C_b(X) := \{f \in C(X) | ||f||_\infty < \infty\}$. *For any* $f \in C_b(X)$

$$\lim_{n\to\infty} \int f d\mu_n = \int f d\mu \tag{2.2.17}$$

- *(iii) Set* $C_u(X) := \{f \in C(X) |\ f$ *is uniformly continuous on* $X\}$. *For any* $f \in C_b(X) \cap C_u(X)$

$$\lim_{n\to\infty} \int f d\mu_n = \int f d\mu \tag{2.2.18}$$

- *(iv) For any closed set* A

$$\overline{\lim_{n\to\infty}} \mu_n(A) \le \mu(A) \tag{2.2.19}$$

- *(v) For any closed set* U

$$\underline{\lim_{n\to\infty}} \mu_n(U) \ge \mu(U) \tag{2.2.20}$$

(i) $\Longrightarrow$ *(ii):* Let fix arbitary $f \in C_b(X)$. Because $\cup_{a\in\mathbb{R}}\{f = a\} = X$ and $\mu(X) = 1$, for any $n \in \mathbb{N}$ $\{a \in \mathbb{R} | \mu(\{f = a\}) > \frac{1}{n}\}$ is a finite set. So $\{a \in \mathbb{R} | \mu(\{f = a\}) > 0\} = \cup_{n=1}^{\infty}\{a \in \mathbb{R} | \mu(\{f = a\}) > \frac{1}{n}\}$ is at most countable.

So there are $a_{m,i_{1\le m\le\infty, 1\le i\le\varphi(m)}} \subset \mathbb{R}$ such that

$$-||f||_\infty > a_{m,1} \le a_{m,2} \le ... \le a_{m,\varphi(m)} > ||f||_\infty\ (\forall m \in \mathbb{N}) \tag{2.2.21}$$

$$|a_{m,i} - a_{m,i+1}| \le \frac{1}{2^m}\ (\forall m, \forall i) \tag{2.2.22}$$

$$\mu(\{f = a_{m,i}\}) = 0\ (\forall m, \forall i) \tag{2.2.23}$$

For $m \in \mathbb{N}$ set

$$g_m := \Sigma_{i=1}^{\varphi(m)} a_{m,i+1} \chi_{\{a_{m,i} \le f \le a_{m,i+1}\}} \tag{2.2.24}$$

and

$$h_m := \Sigma_{i=1}^{\varphi(m)} a_{m,i} \chi_{\{a_{m,i} \le f \le a_{m,i+1}\}} \tag{2.2.25}$$

Because for any m and i $\partial\{a_{m,i} \le f \le a_{m,i+1}\} \subset \{f = a_{m,i}\} \cup \{f = a_{m,i+1}\}$, for any m and i

$$\mu(\partial\{a_{m,i} \le f \le a_{m,i+1}\}) = 0 \tag{2.2.26}$$

Let fix arbitary $\epsilon > 0$.

By Lebesugue's convergence theorem, there is $m \in \mathbb{N}$ such that $\int g_m d\mu - \int h_m d\mu \le \epsilon$.

By (i),

$$\begin{aligned}\int f d\mu - \epsilon &\leq \int h_m d\mu \\ &= \lim_{n\to\infty} \int h_m d\mu_n \\ &\leq \varliminf_{n\to\infty} \int f d\mu_n\end{aligned} \tag{2.2.27}$$

and

$$\begin{aligned}\int f d\mu + \epsilon &\geq \int g_m d\mu \\ &= \lim_{n\to\infty} \int g_m d\mu_n \\ &\geq \varlimsup_{n\to\infty} \int f d\mu_n\end{aligned} \tag{2.2.28}$$

Consequently, $\int f d\mu = \lim_{n\to\infty} \int f d\mu_n$.

□

(ii) $\Longrightarrow$ (iii): It's trivial. □

(iii) $\Longrightarrow$ (iv): Let fix arbitary closed set A. We set

$$f_n(x) := |1 - min(1, d(x,A))|^n \ (n \in \mathbb{N}, x \in x) \tag{2.2.29}$$

$f_n \in C_b(X) \cap C_u(X)$ $(\forall n)$ and $\lim_{n\to\infty} f_n \to \chi_A$ $(pointwise convergence)$ and

$$\int f_m d\mu_n \geq \mu_n(A) \tag{2.2.30}$$

By Lebesugue's convergence theorem,

$$\mu(A) \geq \varlimsup_{n\to\infty} \mu_n(A) \tag{2.2.31}$$

□

(iv) $\Longleftrightarrow$ (v): It's trivial. □

(iv) and (v) $\Longrightarrow$ (i): Let $A \in \mathscr{B}(X)$ and $\mu(\partial A) = 0$. By (iv),

$$\begin{aligned}\varlimsup_{n\to\infty} \mu_n(A) &\leq \varlimsup_{n\to\infty} \mu_n(\overline{A}) \\ &\leq \mu(\overline{A}) \\ &= \mu(\overline{A} \setminus A) + \mu(A) \\ &\leq \mu(\partial) + \mu(A) \\ &= \mu(A)\end{aligned} \tag{2.2.32}$$

In the same way as above we obtain

$$\varliminf_{n\to\infty} \mu_n(A) \quad \geq \quad \mu(A) \tag{2.2.33}$$

Consequently

$$\lim_{n\to\infty} \mu_n(A) = \mu(A) \tag{2.2.34}$$

□

The following is the definition of a metric of $\mathscr{P}(\mathbb{R})$.

Proposition 2.18. *Let*

(S1) (X,d) *is a compact metric space.*

(S2) $\{f_n\}_{n=1}^{\infty}$ *is a dense subset of* (X,d)*. By Proposition*??*, such subsets always exist.*

(S3) $\tau(\mu_1,\mu_2) := \Sigma_{n=1}^{\infty}|\int f_n d\mu_1 - \int f_n d\mu_2|$ $(\mu_1,\mu_2 \in \mathscr{P}(\mathbb{R}))$.

then the followings hold.

(i) τ *is a metric on* $\mathscr{P}(\mathbb{R})$.

(ii) for any $\{\mu_n\}_{n=1}^{\infty} \subset \mathscr{P}(\mathbb{R})$ *and* $\mu \in \mathscr{P}(\mathbb{R})$, $\mu_n \Longrightarrow \mu$ $(n \to \infty)$ *is equivalent to* $\tau(\mu_n,\mu) \to 0$ $(n \to \infty)$.

(i): Let fix $\mu_1 \in \mathscr{P}(X)$ and $\mu_2 \in \mathscr{P}(X)$ such that $\tau(\mu_1,\mu_2) = 0$. It is enough to show $\mu_1 = \mu_2$ for showing (i). By (S2), for any $f \in C_+(X)$ $\int f d\mu_1 = \int f d\mu_2$. By uniqueness in Proposition1.22, $\mu_1 = \mu_2$. □

(ii): Let us assume $\tau(\mu_n,\mu) \to 0$ $(n \to \infty)$. Let us fix arbitary $\epsilon > 0$. There is $m \in \mathbb{N}$ such that $||f - f_m||_\infty < \frac{\epsilon}{3}$. There is $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$

$$|\int_X f_m d\mu_n - \int_X f_m d\mu| < \frac{\epsilon}{3}. \tag{2.2.35}$$

For any $n \geq n_0$

$$\begin{aligned} |\int_X f d\mu_n - \int_X f d\mu| &< |\int_X f d\mu_n - \int_X f_m d\mu_n| \\ &\quad +|\int_X f_m d\mu_n - \int_X f_m d\mu| + |\int_X f_m d\mu - \int_X f_m d\mu| \\ &< \epsilon \end{aligned} \tag{2.2.36}$$

Consequently, $\mu_n \Longrightarrow \mu$ $(n \to \infty)$.
The inverse is clear. □

Proposition 2.19. $(\mathscr{P}(X),\tau)$ *is a compact metric space.*

Proof. By Proposition??, it is enough to show $(\mathscr{P}(X),\tau)$ is sequencially compact.
Let us fix arbitary $\mu_n{}_{n=1}^{\infty} \subset \mathscr{P}(X)$.
For any $m \in \mathbb{N}$, $\{\int f_m \mu_n\}_{n=1}^{\infty}$ is bounded.
For each $m \in \mathbb{N}$, there is $\{\varphi(m,n)\}_{n=1}^{\infty}$ such that $l(f_m) := \lim_{n\to\infty} \int f_m d\mu_{\varphi(m,n)}$ exists and $|l(f_m) - \int f_m d\mu_{\varphi(m,n)}| < \frac{1}{m}$ $(\forall n \geq m)$.
We set $\psi(m) := \varphi(m,m)$ $(m \in \mathbb{N})$.
By the definition of ψ, for any $m \in \mathbb{N}$ $l(f_m) = \lim_{n\to\infty} \int f_m d\mu_{\psi(n)}$.
Let us fix arbitary $f \in C_b(X)$ and $\epsilon > 0$. There is $k \in \mathbb{N}$ such that $||f - f_k|| < \frac{\epsilon}{3}$.
There is $n_0 \in \mathbb{N}$ such that for any $m \geq n_0$ and any $n \geq n_0$ $|\int f_k d\mu_{\psi(m)} - \int f_k d\mu_{\psi(m)}| < \frac{\epsilon}{3}$
So for any $m \geq n_0$ and any $n \geq n_0$ $|\int f d\mu_{\psi(m)} - \int f d\mu_{\psi(m)}| < \epsilon$.
So $l(f) := \lim_{m\to\infty} \int f d\mu_{\psi(m)}$ exists.
Clearly l is an elementary integral on $C_+(X)$.
So by Proposition1.22, there is $\mu \in \mathscr{P}(X)$ such that

$$l(f) = \int_X f d\mu \ (\forall f \in C_+(X)) \tag{2.2.37}$$

Clearly $\mu_{\psi(n)} \Longrightarrow \mu$ $(n \to \infty)$. □

Proposition 2.20. *Let*

(S1) (X,d) *is a separable metric space.*

(A1) $\{\mu_n\}_{n=1}^{\infty} \subset \mathscr{P}(X)$ *is tight.*

There is a subsequence $\mu_\varphi(n)_{\{n=1\}}^{\infty}$ *and* $\mu \in \mathscr{P}(X)$ *such that* $\mu_\varphi(n) \Longrightarrow \mu$ $(n \to \infty)$.

Proof. Let $(\tilde{X},\tilde{d})$ be a compact metric space in Proposition?? and $i : X \to \tilde{X}$ in Proposition??. By Proposition1.22, for each $n \in \mathbb{N}$ there is a measure $\tilde{\mu}_n$ such that for any $g \in C_+(\tilde{X})$ and $n \in \mathbb{N}$

$$\int_X g \circ i d\mu_n = \int_{\tilde{X}} g d\tilde{\mu}_n \tag{2.2.38}$$

There is an increasing sequence of compact sets $\{K_n\}_{n=1}^{\infty}$ such that

$$\mu_m(K_n) > 1 - \frac{1}{n} \tag{2.2.39}$$

$(\forall m \in \mathbb{N}, \forall n \in \mathbb{N})$

Let $K := \cup_{n=1}^{\infty} K_n$. By (2.2.39), for any $m \in \mathbb{N}$

$$\mu_m(K) = \tilde{\mu}_m(i(K)) = 1 \tag{2.2.40}$$

For $n \in \mathbb{N}$ and $x \in \tilde{X}$, $g_{m,n}(x) := (1 - min1, d(x, K_m))^n$. $\int_{\tilde{X}} g_{m,n} d\tilde{\mu}_l \geq \tilde{\mu}_m(K_m) \geq 1 - \frac{1}{m}$. By reaching $n \to \infty$, $\mu_m(K_m) = \tilde{\mu}(i(K_m)) \geq 1 - \frac{1}{m}$. By reaching $m \to \infty$,

$$\tilde{\mu}(i(K)) = 1 \tag{2.2.41}$$

By Proposition, there is a subsequence $\{\tilde{\mu}_{\varphi(n)}\}_{n=1}^{\infty}$ and $\tilde{\mu} \in \mathscr{P}(\tilde{X})$ such that $\tilde{\mu}_n \Longrightarrow \tilde{\mu}$ $(n \to \infty)$.

Because for any $n \in \mathbb{N}$ $i(K_n)$ is compact, $i(K_n) \in \mathscr{B}(\tilde{X})$. So $i(K) \in \mathscr{B}(\tilde{X})$.

We will show

$$\mathscr{B}(X) \subset \mathscr{B} := \{A \subset X | i(A \cap K)\mathscr{B}(\tilde{X})\} \tag{2.2.42}$$

Because i is injective, if $\{A_n\}_{n=1}^{\infty} \subset \mathscr{B}$ then $\cup_{n=1}^{\infty} A_n \in \mathscr{B}$. And if $A\mathscr{B}$ then $i(A^c \cap K) = i(K) \cap i(A \cap K)^c \in \mathscr{B}$ So $\mathscr{B}$ is a σ-algebra. For any closed set A, $A \in \mathscr{B}$. So (2.2.42) holds.

For $A \in \mathscr{B}(X)$, we define

$$\mu(A) := \tilde{\mu}(i(A \cup K)) \tag{2.2.43}$$

By (2.2.41),

$$\mu(K) = 1 \tag{2.2.44}$$

Let me fix arbitary $f \in C_b(X) \cap C_u(X)$. Because $f \in C_u(X)$ and $i(X) is dense in \tilde{X}$, there is $\tilde{f} \in C_b(\tilde{X}) \cap C_u(\tilde{X})$ such that $\tilde{f}|i(X) = f \circ i^{-1}$.

By the definition of $\{\mu_n\}_{n=1}^{\infty}$ and μ,

$$\begin{aligned}
\lim_{n\to\infty} \int_X f d\mu_n &= \lim_{n\to\infty} \int_X \tilde{f} \circ i d\mu_n \\
&= \lim_{n\to\infty} \int_{\tilde{X}} \tilde{f} d\tilde{\mu}_n \\
&= \int_{\tilde{X}} \tilde{f} d\tilde{\mu} \\
&= \int_{i(K)} \tilde{f} d\tilde{\mu} \\
&= \int_{i(K)} f \circ i^{-1} d\tilde{\mu} \\
&= \int_K f d\mu \\
&= \int_X f d\mu
\end{aligned} \tag{2.2.45}$$

□

2.3 Characteristic functions of probability distribution

2.3.1 The Case of Single Variable

By Fubini's theorem, the following holds.

Proposition 2.21. *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S2) Let* $\mu \in \mathcal{P}(\mathbb{R})$.
- *(S3) Let* $f \in L^1(\mathbb{R})$.

then

$$\int_{\mathbb{R}} f(t)\varphi_\mu(t)dt = \int_{\mathbb{R}} \mathscr{F}^{-1}(f)(x)d\mu(x) \tag{2.3.1}$$

Proposition 2.22 (Uniqueness of Characteristic Function)**.** *Let*

(S1) $(\Omega, \mathscr{F}, P)$ *is a probability space.*

(S2) Let $\mu \in \mathcal{P}(\mathbb{R})$ *and* $\mu' \in \mathcal{P}(\mathbb{R})$*.*

If $\varphi_\mu = \varphi_{\mu'}$ *then* $\mu = \mu'$*.*

Proof. Let us arbitary $f \in C_c^\infty(\mathbb{R}^n)$. By Proposition**??**, $\mathscr{F}(f) \in L^1(\mathbb{R}^n)$. By Proposition**??**, $\int_{\mathbb{R}} f(x)d\mu(x) = \int_{\mathbb{R}} f(x)d\mu'(x)$. By Proposition**??**, $\mu = \mu'$. □

This proposition states that convergence of distributions in law is derived from each point convergence of the characteristic function.

Proposition 2.23 (Levy's Continuity Theorem(Single Variable Case))**.** *Let*

(S1) $\{\mu_n\}_{n=1}^\infty \subset \mathcal{P}(\mathbb{R})$

(S2) φ_n *is the characteristic function of* μ_n $(n = 1, 2, ...)$

(A1) $\{\mu_n\}_{n=1}^\infty \subset \mathcal{P}(\mathbb{R})$ *then the followings are equivalent.*

(i) There is a φ *s.t* φ *is a measurable function on* $\mathbb{R}$ *and* φ *is continuous at* 0 *and* $\varphi(0) = 1$ *and* $\varphi_n \xrightarrow[n\to\infty]{} \varphi$ *(converge pointwise). Below, we fix such* φ*.*

(ii) Then there is a $\mu \in \mathcal{P}(\mathbb{R})$ *such that* φ *is the characteristic function of* μ *and* $\mu_n \Longrightarrow \mu$ $(n \to \infty)$*.*

(i) $\Longrightarrow$ *(ii).* The followings are strategy of the proof.
–Memo

(STEP1) Showing $\{\mu_n\}_{n=1}^\infty$ is tight.

(STEP2) Getting μ of the subject.

–
(STEP1)
For each $m \in \mathbb{N}$, there is a measurable function f_m such that f_m continuous at 0 and $f_m(0) = 1$ and $supp(f) \subset [\frac{-1}{m}, \frac{-1}{m}]$ is compact and $f_m \leq 1$ in $\mathbb{R}$ and $\mathcal{F}^{-1}f_m \leq 1$ in $\mathbb{R}$. $\{\chi_{[-\frac{1}{m},\frac{1}{m}]}\}_{m=1}^\infty$ sutisfies the above conditions. Fix such $\{f_m\}_{m=1}^\infty$.
We get

$$\int_{\mathbb{R}} f_m(x)\varphi_n(x)dx = \int_{\mathbb{R}} \mathcal{F}^{-1}f_m(x)d\mu_n(x) \tag{2.3.2}$$

So

$$1 - \frac{m}{2}\int_{\mathbb{R}} f_m(x)\varphi_n(x)dx = 1 - \frac{m}{2}\int_{\mathbb{R}} \mathcal{F}^{-1}f_m(x)d\mu_n(x) \tag{2.3.3}$$

Call the left side of the above (2.3.3) $I_{m,n}$ and call the right side of the above (2.3.3) $J_{m,n}$. Fix any $\varepsilon > 0$.

(STEP1-1)
–Memo
We will show that $I_{m,n} < \varepsilon$ for sufficient large m, n. We will show this statement using the dominated convergence theorem and continuity of φ at 0
–
(STEP1-2)
–Memo
We will show that $J_{m,n} > \mu_n(\{x \in \mathbb{R}|\ |x| \geq m\})$ for sufficient large m, n. We will show this statement using the dominated convergence theorem and continuity of φ at 0
–

The following holds.

$$\mathcal{F}^{-1}f_m(x) = \frac{1}{m}\mathcal{F}^{-1}f_m(\frac{x}{m}) \tag{2.3.4}$$

So

$$\begin{aligned} J_{m,n} &= 1 - \frac{1}{2}\int_{\mathbb{R}} \mathcal{F}^{-1} f_m(\frac{x}{m}) d\mu_n(x) \\ &= \int_{\mathbb{R}} 1 - \frac{1}{2}\mathcal{F}^{-1} f_m(\frac{x}{m}) d\mu_n(x) \\ &= \int_{\{x\in\mathbb{R}|\ |x|\geq m\}} 1 - \frac{1}{2}\mathcal{F}^{-1} f_m(\frac{x}{m}) d\mu_n(x) \end{aligned} \tag{2.3.5}$$

In (2.3.5), we use statement $\mathcal{F}^{-1}f_m \leq 1$ in $\mathbb{R}$ ($\forall m \in \mathbb{N}$).

$$\begin{aligned} 1 - \frac{1}{2}\mathcal{F}^{-1} f_m(\frac{x}{m}) &\geq 1 - \frac{1}{2} max_{y\in supp(|f_m|)} |f_m(y)| \frac{m}{|x|} \\ &\geq \frac{1}{2} \end{aligned} \tag{2.3.6}$$

So

$$J_{m,n} \geq \frac{1}{2}\mu_n(\{x \in \mathbb{R}|\ |x| \geq m\}) \tag{2.3.7}$$

By (STEP1-1) and (2.3.7) for sufficient large m and n we get

$$2\epsilon \geq \mu_n(\{x \in \mathbb{R}|\ |x| \geq m\}) \tag{2.3.8}$$

So We have shown $\{\mu_n\}_{n=1}^{\infty}$ is tight.
(STEP2)
By (STEP1), there is a subsequence $\{\mu_{\psi(n)}\}_{n=1}^{\infty}$ which converges to a μ in law. It is enough to show for any subsequence of $\{\mu_n\}_{n=1}^{\infty}$ the subsequence has some subsequnece of the subsequence which converges to μ in law. Let fix any subsequence $\{\mu_{\omega(n)}\}_{n=1}^{\infty}$. There is a subsequence $\{\mu_{\omega(\alpha(n))}\}_{n=1}^{\infty}$ which converges to μ'. By increasing n to ∞ in (2.3.3) and Proposition2.15, $\phi_\mu = \phi$ and $\phi_{\mu'} = \phi$. By uniqueness of characteristic function, $\mu = \mu'$.

□

(ii) $\implies$ (i). $\varphi_\mu : \mathbb{R} \ni t \mapsto \int_\Omega exp(itx) d\mu$. It is easy to show φ_μ is continuous at 0.
By Proposition2.15,

$$\int_{\mathbb{R}} exp(itx) d\mu(x) = \lim_{n\to\infty} \int_{\mathbb{R}} exp(itx) d\mu_n \ (\forall t) \tag{2.3.9}$$

□

2.3.2 The Case of Multi variables

Proposition 2.24 (Levy's continuity theorem(multi variate case)). *Let*

- *(S1)* $\{\mu_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathbb{R}^N)$
- *(S2)* φ_n *is the characteristic function of* μ_n $(n = 1, 2, ...)$
- *(A1)* $\{\mu_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathbb{R}^N)$
- *(A1) There is a* φ *s.t* φ *is a measurable function on* $\mathbb{R}^N$ *and* φ *is continuous at* 0 *and* $\varphi(0) = 1$ *and* $\varphi_n \xrightarrow[n\to\infty]{} \varphi$ *(converge pointwise). Below, we fix such* φ.

Then there is a $\mu \in \mathcal{P}(\mathbb{R}^N)$ *such that* φ *is the characteristic function of* μ *and* $\mu_n \Longrightarrow \mu$ $(n \to \infty)$.

Proof. By an argument which is similar to the proof of Proposition2.23, we can show that $\{\mu_n\}_{n=1}^{\infty}$ is tight.
By Proposition2.20 and uniqueness of fourier transformation in $\mathbb{R}^N$ and Proposition2.17, there is $\mu \in \mathscr{P}(R)^N$ such that $\mu_n \Longrightarrow \mu$ $(n \to \infty)$ and $\varphi_\mu = \varphi$.

□

2.4 Central limit theorem

2.4.1 The Case of Single Variable

Theorem 2.25 (Central limit theorem)**.** *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S2)* $\{X_i\}_{i=1}^{\infty}$ *is a sequence of random variables on* $(\Omega, \mathscr{F}, P)$*.*
- *(A1)* $\exists \mu$ *such that* $X_i \sim \mu$ $(\forall i)$*. Bellow, we fix such* μ*.*
- *(A2)* $\{X_i\}_{i=1}^{N}$ *are independent for any* $N \in \mathbb{N}$*.*
- *(A3)* $E[\mu] = \nu$ *and* $V[\mu] = \sigma^2$ *and* $\sigma > 0$*.*

then $P_{\sqrt{n}(\bar{X}-\nu)}$ *weakly converges to* $N(0, \sigma)$*.*

Proof. We can assume $\nu = 0$ and $\sigma = 1$. Bellow, we assume that.
Let $Y_{i,n} := \frac{X_i}{\sqrt{n}}$ $(i = 1, 2, ..., n)$ and $Y_n := \sum_{i=1}^{n} Y_{i,n}$ $(n = 1, 2, ...)$. By $(A1)$, $\varphi_{Y_{i,n}} = \varphi_{Y_{1,n}}$ $(\forall i, \forall n)$. Let $\varphi_n := \varphi_{Y_n}$ and $\psi_n := \varphi_{Y_{1,n}}$ $(n = 1, 2, ...)$. And let $\psi_\mu : \mathbb{R} \ni s \mapsto \int_{\mathbb{R}} exp(isx) d\mu(x)$. Then $\varphi_n = (\psi_n)^n$ and $\psi_n(t) = \psi_\mu(\frac{t}{\sqrt{n}})$ and $(\forall t \in \mathbb{R})$. We will show the following equation. By Proposition2.7,

$$\varphi_{Y_{1,n}}(t) = 1 - \frac{t^2}{2n} + o(\frac{1}{n})(n \to \infty) \tag{2.4.1}$$

By the above equation and Proposition2.4,

$$\varphi_n(t) = (1 - \frac{t^2}{2n} + o(\frac{1}{n}))^n \to exp(-\frac{t^2}{2})\ (n \to \infty) \tag{2.4.2}$$

By Proposition2.23, there is a $\mu_0 \in \mathscr{P}(\mathbb{R})$ such that $P_{\sqrt{n}\bar{X}}$ converges to μ_0 in law and $\varphi_{\mu_0} = exp(-\frac{(\cdot)^2}{2})$. Because $\varphi_{N(0,1)} = exp(-\frac{(\cdot)^2}{2})$ and uniqueness of characteristic function, $P_{\sqrt{n}\bar{X}}$ converges to $N(0, 1)$ □

2.4.2 The Case of Multi Variables

Theorem 2.26 (Central Limit Theorem(Multi Variables Case))**.** *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S2)* $\{X_i\}_{i=1}^{\infty}$ *is a sequence of* N*-dimensional vectors of random variables on* $(\Omega, \mathscr{F}, P)$*.*
- *(A1)* $\exists \mu$ *such that* $X_i \sim \mu$ $(\forall i)$*. Bellow, we fix such* μ*.*
- *(A2)* $\{X_i\}_{i=1}^{n}$ *are independent for any* $n \in \mathbb{N}$*.*
- *(A3)* $E[\mu] = \nu$ *and* $cov[\mu] = \sigma^2$ *and* σ *is* N*-by-*N *positive definite symmetric matrix.*

then $P_{\sqrt{n}(\bar{X}-\nu)}$ *weakly converges to* $N(0, \Sigma)$*.*

Proof. Let us fix arbitary $\boldsymbol{t} \in \mathbb{R}^N$ and $s \in \mathbb{R}$. Let us set $Y_n := s\boldsymbol{t}^T(X_n - \nu)$.
The following holds.

$$\varphi_{\sqrt{n}(\bar{X}-\nu)}(s\boldsymbol{t}) = E(exp(\sqrt{n} is\boldsymbol{t}^T(\bar{X} - \nu))) = \varphi_{\sqrt{n}(\bar{Y}-\nu)}(s) \tag{2.4.3}$$

By Theorem2.25 and Proposition2.23 and Proposition2.3,

$$\lim_{n\to\infty} \varphi_{\sqrt{n}(\bar{Y}-\nu)}(s) = exp(-\frac{s^2\boldsymbol{t}^T\Sigma^2\boldsymbol{t}}{2}) \tag{2.4.4}$$

By setting $s = 1$,

$$\lim_{n\to\infty} \varphi_{\sqrt{n}(\bar{X}-\nu)}(s\boldsymbol{t}) = exp(-\frac{\boldsymbol{t}^T\Sigma^2\boldsymbol{t}}{2}) \tag{2.4.5}$$

By Proposition2.24 and Proposition2.3, $P_{\sqrt{n}(\bar{X}-\nu)}$ weakly converges to $N(0, \Sigma)$.

□

2.5 Law of large numbers

Proposition 2.27 (Weak law of large numbers)**.** *Let*

(S1) $(\Omega, \mathscr{F}, P)$ *is a probability space.*

(A1) $\{X_i\}_{i=1}^{\infty}$ *is a sequence of independent random variables on* $(\Omega, \mathscr{F}, P)$*.*

(A2) There is a $\mu \in \mathscr{P}(\mathbb{R})$ *such that* $X_i \sim \mu(\forall i)$*.*

(A3) $E[\mu] = \nu$ *and* $V[\mu] = \sigma^2$ *exist.*

then the followings hold.

(i) $\{X_i\}_{i=1}^{\infty}$ *stochastic converges to* μ*, i.e., for any* $\epsilon > 0$

$$\lim_{n\to\infty} \mu(|\bar{X} - \mu| \geq \epsilon) = 0 \tag{2.5.1}$$

Hereafter we denote stochastic convergence by $\xrightarrow[N\to\infty]{\mathrm{p}}$ *or* plim.

(ii) For any $\epsilon > 0$*,*

$$\mu(|\bar{X} - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2} \tag{2.5.2}$$

A proof using Chebyshev's inequality. For any $n \in \mathbb{N}$,

$$\begin{aligned} \mu(|\bar{X} - \mu| \geq \epsilon) &= \frac{\epsilon^2 \mu(|\bar{X} - \mu|^2 \geq \epsilon^2)}{\epsilon^2} \\ &\leq \frac{1}{\epsilon^2} \int_{\{|\bar{X}-\mu|^2 \geq \epsilon^2\}} \epsilon^2 dP \\ &\leq \frac{1}{\epsilon^2} V[\bar{X}] = \frac{\sigma^2}{n\epsilon^2} \end{aligned} \tag{2.5.3}$$

This implies the above equation. □

A proof using Central limit theorem. By resetting $X_i \to \dfrac{X_i - \mu}{\sigma}$, we can assume $\mu = 0$ and $\sigma = 1$. Let us fix arbitary $\epsilon > 0$ and $\delta > 0$. There is $a > 0$ such that

$$N(0,1)((-\infty, -a) \cup (a, \infty)) < \delta \tag{2.5.4}$$

By Central limit theorem, there is $n_0 \in \mathbb{N}$ such that

$$\frac{a}{\sqrt{n_0}} < \delta \tag{2.5.5}$$

and for any $n \geq n_0$

$$|\mu(|\sqrt{n}\bar{X}| \geq a) - N(0,1)((-\infty, -a) \cup (a, \infty))| < \delta \tag{2.5.6}$$

So for any $n \geq n_0$

$$\begin{aligned} \mu(|\bar{X}| \geq \epsilon) &\leq \mu(|\bar{X}| \geq \frac{a}{\sqrt{n}}) = \mu(\sqrt{n}|\bar{X}| \geq a) \\ &\leq 2\delta \end{aligned} \tag{2.5.7}$$

So $\overline{\lim_{n\to\infty}} \mu(|\bar{X}| \geq \epsilon) \leq 2\delta$. Consequently, $\lim_{n\to\infty} \mu(|\bar{X}| \geq \epsilon) = 0$. □

2.6 Multivariate normal distribution

Remark 2.28. *Let*

(S1) $(\Omega, \mathscr{F}, P)$ *is a probability space.*

(S2) $X := (X_1, ..., X_n)$ *is a vector of random variables.*

(S3) A *is a* (m, n) *matrix.*

(A1) $(X_1, ..., X_n) \sim N(0, E_n)$.

then $cov(AX) = AA^T$.

The following Proposition3.6.4 is used to prove the Proposition3.42 discussed later.

Proposition 2.29. *Let*

(A1) $X := (X_1, X_2, ..., X_p)^T \sim N(\gamma, BB^T)$*, where* B *is a* (p, q) *matrix.*

(S1) Let $s \in [1, p-1] \cap \mathbb{N}$ *and* $X^{(1)} := (X_1, ..., X_s)$ *and* $X^{(2)} := (X_{s+1}, ..., X_p)$.

(A2) $cov(X^{(1)}, X^{(2)}) = 0$.

then $X^{(1)}$ *and* $X^{(2)}$ *are independent.*

Proof. The following proof consists of two steps.
STEP1. General case

In this step, we will show that it is enough to show the Proposition when $r := rank(B) = p \leq q$. For each $i \in \mathbb{N} \cap [1, p]$, let b_i be the i-th row vector of B. Let V_1 be the vector space generated from $b_1, b_2, ..., b_s$ and let V_2 be the vector space generated from $b_{s+1}, b_{s+2}, ..., b_p$. We can take $\{b_{\sigma(i)}\}_{i=1}^{r_1}$ is a basis of V_1 and $\{b_{\tau(i)}\}_{i=1}^{r_2}$ is a basis of V_2. Since $V_1 \perp V_2$, $\{b_{\sigma(i)}\}_{i=1}^{r_1} \cap \{b_{\tau(i)}\}_{i=1}^{r_2} = \phi$ and $\{b_{\sigma(i)}\}_{i=1}^{r_1} \cup \{b_{\tau(i)}\}_{i=1}^{r_2}$ are linear independent. So it is enough to show $\{b_{\sigma(i)}\}_{i=1}^{r_1}$ and $\{b_{\tau(i)}\}_{i=1}^{r_2}$ are independent when $rank(B)$ is the number of rows of B.

STEP2. Case when $rank(B) = p \leq q$

Let W be the orthogonal complement of the vector space generated from $b_1, b_2, ..., b_p$. We can take $c_1, ..., c_{(q-p)}$ which is an orthonormal basis of W and let

$C := \begin{bmatrix} c_1 \\ c_2 \\ ... \\ c_{(q-p)} \end{bmatrix}$, and let $D := \begin{bmatrix} B \\ C \end{bmatrix}$. By (A1), there are random variables $\{\epsilon\}_{i=1}^{p}$ on $(\Omega, \mathscr{F})$ and random variables $\{\ Y\}_{i=1}^{q-p}$ on $(\Omega, \mathscr{F})$ such that $\boldsymbol{\epsilon} := \{\epsilon\}_{i=1}^{q}$ are $i.i.d$ and $\epsilon_i \sim N(0, 1)$ $(\forall i)$

and $Z := \begin{bmatrix} X \\ Y \end{bmatrix} = D\boldsymbol{\epsilon} + \gamma$ and $cov(Z) = DD^T$.

The distribution of Z has the density function $f_q : \mathbb{R}^q \ni x \mapsto c \cdot exp(x^T DD^T x) \in \mathbb{R}$, where c is a constant. By (A2) and the definition of C,

$DD^T = \begin{bmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Sigma_2 & 0 \\ 0 & 0 & E_{(q-p)} \end{bmatrix}$, where Σ_1 and Σ_2 are symmetric positive definite matrixies. So the distribution of X has the density function $f_p : \mathbb{R}^p \ni x \mapsto d \cdot exp({x^{(1)}}^T \Sigma_1 x^{(1)}) \cdot exp({x^{(2)}}^T \Sigma_1 x^{(2)}) \in \mathbb{R}$, where d is a constant and $x^{(1)} = (x_1, ..., x_s)$ and $x^{(2)} = (x_{s+1}, ..., x_p)$. By the format of f_p, $X^{(1)}$ and $X^{(2)}$ are independent. □

3 Statistics

3.1 Popular Probability Distributions

3.1.1 General Topics on Random Variables

By the definition of independence, the following clearly holds.

Proposition 3.1. *Let*

- *(S1)* $(\S_i, \mathscr{S}, P_i)$ $(i = 1, 2, ..., N)$ *is a sequence of probability spaces.*
- *(S2)* $(\Omega, \mathscr{F}, P)$ *is the probability spaces which is direct product of* $(\S_i, \mathscr{S}, P_i)$ $(i = 1, 2, ..., N)$
- *(S3)* X_i *is a random variable on* S_i $(i = 1, 2, ..., N)$.
- *(S3) We set* $Y_i := X_i \circ \pi_i$ $(i = 1, 2, ..., N)$.

then $Y_1, ..., Y_N$ *is a sequence of independent random variables.*

The following clearly holds.

Proposition 3.2. *Let* P *is probability measure on* $(\Omega := \mathbb{N} \cup \{0\}, 2^{\Omega})$. *Then* id_Ω *is random variable on* Ω *and* $id_\Omega \sim P$.

By Fubini's theorem(see [5]), the following two propositions clearly holds.

Proposition 3.3 (Marginal distribution)**.** *Let*

- *(S1)* $(\Omega_i, \mathscr{F}_i, P_i)$ *is a probability spaces* $(i = 1, 2)$.
- *(A1)* $P_1 \times P_2$ *has a density function* f_{P_1,P_2}.

Then for almost everywhere $x \in \mathbb{R}$, $f_{P_1,P_2}(x, \cdot)$ *is measurable and*

$$f_{P_1}(x) := \int_{\mathbb{R}} f_{P_1,P_2}(x, y) dP_2(y)$$

exists and f_{P_1} *is measurable and*

$$\int_{\mathbb{R}} f_{P_1}(x) dP_1(x) = 1$$

Proposition 3.4 (Conditional probability density function)**.** *Let*

- *(S1)* $(\Omega_i, \mathscr{F}_i, P_i)$ *is a probability spaces* $(i = 1, 2)$.
- *(A1)* $P_1 \times P_2$ *has a density function* $f_{X,Y}$.
- *(S2)* $x \in \mathbb{R}$ *such that* $f_{X,Y}(x, \cdot)$ *is measurable and* $f_X(x) > 0$.
- *(S3) Set*

$$f_{P_2|P_1(x)}(y) := \frac{f_{P_1,P_2}(x, y)}{f_{P_1}(x)} \ (y \in \mathbb{R})$$

We call $f_{P_2|P_1(x)}$ *the conditional probability density function of* P_2 *given the occurrence of the value* x *of* P_1.

Then

$$\int_{\mathbb{R}} f_{P_2|P_1(x)}(y) dP_2(y) = 1$$

The following definitions are based on [6].

Definition 3.5 (Probability model, True distribution, Prior probability)**.** *The followings are settings and assumptions.*

- *(A1)* Q *is a probability borel measure on* $\mathbb{R}^N$ *and* Q *has the density function* q. *We call* q *a true distribution.*
- *(S1)* W *is a Borel set of* $\mathbb{R}^d$.
- *(A2)* Φ *is a probability borel measure on* W *that has the density function* ϕ. *We call* ϕ *a prior probability.*
- *(A3)* $Q \times \Phi$ *has the densition function* p.
- *(S2) We set* $p(\cdot_1|\cdot_2)$ *by for* $w \in W$ *such that* $\phi(w) > 0$

$$p(x|w) := p_{Q|\Phi(w)}(x) \ (x \in \mathbb{R}^N)$$

We call $p(\cdot_1|\cdot_2)$ *the a probability model. Or, we denote* $p(\cdot_1|\cdot_2)$ *by* $p(x|w)$.

Definition 3.6 (Exponential family)**.** *The followings are settings and assumptions.*

(S1) (Q, q, W, Φ, ϕ, p) *is a pair of true distribution, prior probability, probability model.*

We say p *is in exponential family if there are* v, g, f *such that* f *is a borel measurable map from* W *to* $\mathbb{R}^J$ *and* g *are borel measurable maps from* $\mathbb{R}^N$ *to* $\mathbb{R}^J$ *and* v *is a borel measurable function on* W *and for any* $x \in \mathbb{R}^N$ *and any* $w \in W$ *such that* $\phi(w) > 0$

$$p(x|w) = v(x)exp(f(w) \cdot g(x))$$

Definition 3.7 (Conjugate prior distribution)**.** *The followings are settings and assumptions.*

(S1) $(Q, q, W, \Phi, \phi, p, v, g, f)$ *is in exponential family.*

(S2) $v \in \mathbb{R}^J$.

Then, we set

$$\varphi(u, v) := \varphi(u|v) := \frac{exp(v \cdot f(u))}{\int_W exp(v \cdot f(w))d\Phi(w)} \ (u \in W), z(v) := \int_W exp(v \cdot f(w))d\Phi(w)$$

We call $\varphi(\cdot_1|\cdot_2)$ *the conjugate prior distribution of the exponential family* $(Q, q, W, \Phi, \phi, p, v, g, f)$.

The following is clear.

Proposition 3.8 (Posterior Probability Distribution)**.** *The followings are settings and assumptions.*

(S1) (Q, q, W, Φ, ϕ, p) *is a probability model.*

(A2) q *is continuous.*

(S2) $X^n := \{X_i\}_{i=1}^n$ *is a sequence of* $\mathbb{R}^N$*-valued random variables such that* $X_i \sim Q$.

(A3) p *is continuous and for any* $x_1, ..., x_n \in q^{-1}((0, \infty))$ *there is* $w \in W$ *such that* $p(x_i, w) > 0$ $(\forall i \in \mathbb{N})$.

(A4) ϕ *is continuous and* $\phi > 0$ *in* W.

(S3) $\beta > 0$.

Then,

$$Z_n(\beta) := \int_W \phi(w)\Pi_{i=1}^n p(X_i|w)^\beta d\Phi(w) > 0$$

We set

$$r(w, X^n) := r(w|X^n) := \phi(w)\Pi_{i=1}^n p(X_i|w)^\beta \frac{1}{Z_n(\beta)} \ (w \in W)$$

We call $r(\cdot|X^n)$ *is the posterior distribution of* p*. And we call* β *an inverse temperature and* $Z_n(\beta)$ *the partition function, respectively.*

Proposition 3.9. *The followings are settings and assumptions.*

(S1) $(Q, q, W, \Phi, \phi, p, v, g, f)$ *is an exponential family.*

(A2) q *is continuous.*

(S2) $X^n := \{X_i\}_{i=1}^n$ *is a sequence of* $\mathbb{R}^N$*-valued random variables such that* $X_i \sim Q$.

(A3) p *is continuous and for any* $x_1, ..., x_n \in q^{-1}((0, \infty))$ *there is* $w \in W$ *such that* $p(x_i, w) > 0$ $(\forall i \in \mathbb{N})$.

(A4) ϕ *is continuous and* $\phi > 0$ *in* W.

(S3) $\beta > 0$ *is an inverse temperature.*

(S4) $v \in \mathbb{R}^J$.

(S5) $\hat{v} := v + \sum_{i=1}^n \beta g(X_i)$.

Then

(i) The partiation function is represented as below.

$$Z_n(\beta) = (\Pi_{i=1}^n v(X_i)^\beta)\frac{z(\hat{v})}{z(v)}$$

(ii) The posterior probability distribution is represented as below.

$$r(w|X^n) := \varphi(w|\hat{v})$$

Proof of (i).

$$
\begin{aligned}
& Z_n(\beta) := \int_W \phi(w)\Pi_{i=1}^n p(X_i|w)^\beta d\Phi(w) = \int_W \varphi(w|v)\Pi_{i=1}^n p(X_i|w)^\beta d\Phi(w) \\
= & \int_W \varphi(w|v)\Pi_{i=1}^n (v(X_i)\exp(f(w)\cdot g(X_i))))^\beta d\Phi(w) = \frac{1}{z(v)}\int_W \exp(v\cdot f(w))\Pi_{i=1}^n (v(X_i)\exp(f(w)\cdot g(X_i))))^\beta d\Phi(w) \\
= & \frac{1}{z(v)}\int_W \Pi_{i=1}^n v(X_i)^\beta \exp((v+\beta\sum_{i=1}^n g(X_i))\cdot f(w))d\Phi(w) = \frac{z(\hat{v})}{z(v)}\Pi_{i=1}^n v(X_i)^\beta
\end{aligned}
$$

□

Proof of (ii).

$$
\begin{aligned}
& r(w|X^n) := \phi(w)\Pi_{i=1}^n p(X_i|w)^\beta \frac{1}{Z_n(\beta)} = \varphi(w|v)\Pi_{i=1}^n p(X_i|w)^\beta \frac{z(v)}{z(\hat{v})\Pi_{i=1}^n v(X_i)^\beta} \\
= & \frac{\exp(v\cdot f(w))}{z(v)}(\Pi_{i=1}^n (v(X_i)\exp(f(w)\cdot g(X_i))))^\beta)\frac{z(v)}{z(\hat{v})\Pi_{i=1}^n v(X_i)^\beta} = \frac{\exp(\hat{v}\cdot f(w))}{z(\hat{v})} = \varphi(w|\hat{v})
\end{aligned}
$$

□

3.1.2 Probability Generating Function

Definition 3.10 (Probability Generating Function)**.** *Let*

(S1) $(\Omega = \mathbb{N}\cup 0, 2^\Omega, P)$ *is a probability space.*

then we set

$$G_P(z) := \sum_{i=0}^\infty P(i)z^i \ (z\in\mathbb{C}) \tag{3.1.1}$$

Proposition 3.11. *The followings hold.*

(i) Radius of convergence of $G_P(z)$ *is not less than 1.*

(ii) If $G_P = G_{P'}$ *then* $P = P'$*.*

(iii) If Y *is a random variable on any probability space such that* $Y \sim P$ *then* $G_P(z) = E(z^Y)$ *for any* $z \in D(0,1)$ *.*

(iii) If Y_1, Y_2 *is a random variable on any probability space such that* Y_1, Y_2 *are independent then* $G_{P_{Y_1+Y_2}} = G_{P_{Y_1}} G_{P_{Y_2}}$*.*

proof of (i). Because $0 \le P \le 1$, (i) holds. □

proof of (ii). By (i) and definition of G_P and G'_P, (ii) holds. □

proof of (iii). Let us fix any $z \in D(0,1)$. For any $N \in \mathbb{N}$,

$$
\begin{aligned}
E(z^Y) &= \sum_{i=0}^N \int_{\{Y=i\}} z^Y dQ + \int_{\{Y>N\}} z^Y dQ \\
&= \sum_{i=0}^N P(i)z^i + \int_{\{Y>N\}} z^Y dQ
\end{aligned} \tag{3.1.2}
$$

So

$$|E(z^Y) - \sum_{i=0}^N P(i)z^i| \le |\int_{Y>N} z^Y dQ| \le Q(\{Y>N\}) \tag{3.1.3}$$

Consequently (iii) holds. □

proof of (iv). It is enough to show (iv) by (iii). □

3.1.3 Bernoulli distribution

Definition 3.12 (Bernoulli distribution). *We call a probability distribution P on $\{0,1\}$ the Bernoulli distribution if for some $p \in [0,1]$ $P(\{1\}) = p$ and $P(\{0\}) = 1 - p$.*

Proposition 3.13 (Expectation and Variance of Bernoulli distribution). *Let us assumel a probability distribution P on $\{0,1\}$ is the Bernoulli distribution with $P(\{1\}) = p$.*

(i) $E(P) = p$

(ii) $V(P) = p(1-p)$,

(i). It is trivial. □

(i). $V(P) = \int_{\{0,1\}} x^2 dP - E(P)^2 = \int_{\{0,1\}} x dP - p^2 = p - p^2 = p(1-p)$ □

3.1.4 Binomial distribution

Definition 3.14 (Binomial distribution). *For some $p \in [0,1]$ and $n \in \mathbb{N}$ we call a probability distribution $B(n,p)$ on $\{0,1,...,n\}$ the Binomial distribution if $B(n,p)(\{i\}) = {}_nCip^i(1-p)^(n-i)$ $(i = 0,1,...,n)$.*

Clearly the following holds.

Proposition 3.15. *Let*

(S1) $(\Omega, \mathscr{F}, P)$ *is a probability space.*

(S2) $\{X_i\}_{i=1}^n$ *be independent random variables.*

(A1) The distribution of X_i is the Bernoulli distribution B with $B(\{1\}) = p$ $(\forall i)$.

then the distribution of $\Sigma_{i=1}^n X_i$ is $B(n,p)$.

By Proposition Proposition3.2 and Proposition3.1, Random variables like the one above exist.
$E(B(2,p)) = 1 \cdot {}_2C_1 p(1-p) + 2 \cdot {}_2C_2 p^2 = 2p + 0 \cdot p^2 = 2p$. $E_{B(2,p)}(x^2) = 2p + 2^2p^2 - 2p^2$. $E(B(3,p)) = 1 \cdot {}_3C_1 p(1-p)^2 + 2 \cdot {}_3C_2 p^2(1-p) + 3p^3 = 3p + 0 \cdot p^2 + 0 \cdot p^3 = 3p$. $E_{B(3,p)}(x^2) = 3p + 3^3p^2 - 3p^2 + 0 \circ p^3$. We can extend these fact to the following lemma and the following proposition.

Lemma 3.16.

(i) $\Sigma_{k=1}^l k {}_lC_k (-1)^k = 0$ $(\forall l \geq 2)$.

(ii) $\Sigma_{k=1}^l k^2 {}_lC_k (-1)^k = 0$ $(\forall l \geq 3)$.

(i). $L(x) := (1-x)^l = \Sigma_{k=1}^l {}_lC_k (-1)^k (-1)^k x^k$.
$L'(x) = l(1-x)^{l-1} = \Sigma_{k=1}^l k {}_lC_k (-1)^k (-1)^k x^{k-1}$.
So, if $l \geq 2$, then

$$\begin{aligned} 0 &= L'(1) \\ &= \Sigma_{k=1}^l k {}_lC_k (-1)^k (-1)^k \end{aligned} \tag{3.1.4}$$

□

(ii). $L(x) := (1-x)^l = \Sigma_{k=1}^l {}_lC_k (-1)^k (-1)^k x^k$.
$L''(x) = l(1-x)^{l-1} = \Sigma_{k=1}^l k(k-1) {}_lC_k (-1)^k (-1)^k x^{k-2}$.
So, if $l \geq 3$, then

$$\begin{aligned} 0 &= L''(1) \\ &= \Sigma_{k=1}^l k(k-1) {}_lC_k (-1)^k (-1)^k \\ &= \Sigma_{k=1}^l k^2 {}_lC_k (-1)^k (-1)^k - \Sigma_{k=1}^l k {}_lC_k (-1)^k (-1)^k \end{aligned} \tag{3.1.5}$$

By (i), $\Sigma_{k=1}^l k {}_lC_k (-1)^k (-1)^k = 0$. So $\Sigma_{k=1}^l k^2 {}_lC_k (-1)^k (-1)^k = 0$. □

Proposition 3.17 (Expectation and Variance of Binomial distribution).

(i) $E(B(n,p)) = np$

(ii) $V(B(n,p)) = np(1-p)$

proof1 of (i). Let us take $\{X_i\}_{\{}i = 1, 2., , , , n\}$ in Proposition3.15. $E(B(n,p)) = E(\Sigma_{i=1}^n X_i) = \Sigma_{i=1}^n E(X_i) = np$ □

proof1 of (ii). Let us take $\{X_i\}_{\{}i = 1, 2., , , , n\}$ in Proposition3.15. $V(B(n,p)) = \Sigma_{i=1}^n V(X_i) = np(1-p)$ □

proof2 of (i).

$$\begin{aligned}
E(B(n,p)) &= \sum_{k=1}^{n} k {}_nC_k p^k (1-p)^{n-k} \\
&= \sum_{k=1}^{l} k {}_nC_k p^k \sum_{i=0}^{n-k} {}_{n-k}C_i (-1)^i p^i \\
&= \sum_{l=1}^{n} \sum_{k=1,2\ldots,l,\ i=0,1,\ldots,n-k,\ k+i=l} k {}_nC_k p^k {}_{n-k}C_i (-1)^i p^i \\
&= \sum_{l=1}^{n} p^l \sum_{k=1,2\ldots,l,\ i=0,1,\ldots,n-k,\ k+i=l} k {}_nC_k {}_{n-k}C_i (-1)^i \\
&= \sum_{l=1}^{n} p^l \sum_{k=1}^{l} k {}_nC_k \cdot {}_{n-k}C_{l-k} (-1)^{l-k} \\
&= \sum_{l=1}^{n} (-1)^l p^l \sum_{k=1}^{l} k {}_nC_k \cdot {}_{n-k}C_{l-k} (-1)^k \\
&= \sum_{l=1}^{n} (-1)^l p^l \sum_{k=1}^{l} k \frac{{}_nP_l}{k!(l-k)!} (-1)^k \\
&= \sum_{l=1}^{n} (-1)^l p^l \sum_{k=1}^{l} k \frac{{}_nC_l \cdot l!}{k!(l-k)!} (-1)^k \\
&= \sum_{l=1}^{n} (-1)^l p^l {}_nC_l \sum_{k=1}^{l} k \frac{l!}{k!(l-k)!} (-1)^k \\
&= \sum_{l=1}^{n} (-1)^l p^l {}_nC_l \sum_{k=1}^{l} k {}_lC_k (-1)^k
\end{aligned} \tag{3.1.6}$$

By Lemma3.16, for any $l \geq 2$, $\sum_{k=1}^{l} k {}_lC_k (-1)^k = 0$. So $E(B(n,p)) = np$. □

proof2 of (ii). By the proof2 of (ii),

$$E_{B(n,p)}(x^2) = \sum_{l=1}^{n} (-1)^l p^l {}_nC_l \sum_{k=1}^{l} k^2 {}_lC_k (-1)^k \tag{3.1.7}$$

By Lemma3.16, for any $l \geq 3$, $\sum_{k=1}^{l} k^2 {}_lC_k (-1)^k = 0$.
So $E_{B(n,p)}(x^2) = \sum_{l=1}^{2} (-1)^l p^l {}_nC_l \sum_{k=1}^{l} k^2 {}_lC_k (-1)^k = np(1-p) + n^2p^2$. By (i), $V(B(n,p)) = E_{B(n,p)}(x^2) - E(B(n,p))^2 = np(1-p)$. □

3.1.5 Geometric distribution

Definition 3.18 (Geometric distribution)**.** *Let $p \in (0,1)$.*

$$P(k) := (1-p)^{k-1} p \ (k = 1, 2, \ldots) \tag{3.1.8}$$

We call P is Geometric distribution with p

Clearly P is a probability measure on $\{1, 2, \ldots, n, \ldots\}$.

Proposition 3.19. *Let P is Geometric distribution with p. Then*

$$G_P(z) = \frac{pz}{1-(1-p)z} \tag{3.1.9}$$

Proof.

$$\begin{aligned} G_P(z) &= \sum_{k=1}(1-p)^{k-1}pz^k \\ &= pz\sum_{k=1}(1-p)^{k-1}pz^{k-1} \\ &= pz\frac{1}{1-(1-p)z} \end{aligned} \tag{3.1.10}$$

□

Proposition 3.20. *Let P is Geometric distribution with p. Then*

$$E(P) = \frac{1}{p} \tag{3.1.11}$$

and

$$V(P) = \frac{1-p}{p^2} \tag{3.1.12}$$

proof1 of (3.1.11).

$$G'_P(z) = \frac{p(1-(1-p)z)+pz(1-p)}{(1-(1-p)z)^2}$$

So

$$\begin{aligned} E(P) &= G'_P(1) = \frac{p(1-(1-p)1)+p1(1-p)}{(1-(1-p)1)^2} \\ &= \frac{p^2+p-p^2)}{p^2} = \frac{1}{p} \end{aligned} \tag{3.1.13}$$

□

proof2 of (3.1.11).

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \tag{3.1.14}$$

By calculating the derivative,

$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^{k-1} \tag{3.1.15}$$

So

$$E(P) = p\sum_{k=1}^{\infty} k(1-(1-p))^{k-1} = p\frac{1}{(1-(1-p))^2} = \frac{1}{p} \tag{3.1.16}$$

□

proof of (3.1.12). By calculating the derivative of (3.1.17),

$$\frac{2}{(1-x)^3} = \sum_{k=2}^{\infty} k(k-1)x^{k-2} \tag{3.1.17}$$

So

$$\begin{aligned} E_P(x(x-1)) &= p\sum_{k=2}^{\infty} k(k-1)(1-p)^{k-1} \\ &= p(1-p)\sum_{k=2}^{\infty} k(k-1)(1-p)^{k-2} \\ &= p(1-p)\frac{2}{(p^3} = \frac{2(1-p)}{p^2} \end{aligned} \tag{3.1.18}$$

$$V(P) = E_P(x(x-1)) + E_P(x) - E_P(x)^2 = \frac{2(1-p)}{p^2} + \frac{p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2} \tag{3.1.19}$$

□

3.1.6 Negative binomial distribution

Definition 3.21 (Negative binomial distribution)**.** *We call a probability distribution P on $\{1, 2, ...\}$ the Negative binomial distribution if for some $p \in [0, 1]$ $P(\{k\}) = {}_{p_{r+k-2}}C_{r-1}(1-p)^{k-1}p^{r-1}$. We denote this distribution by $NB(r, p)$.*

Proposition 3.22.

$$G_{NB(r,p)}(z) = \frac{p^r z}{(1-(1-p)z)^r} \tag{3.1.20}$$

Proof. Because

$$\frac{1}{1-z} = \sum_{i=0}^{\infty} z^i \tag{3.1.21}$$

the following holds by $r-1$ times derivative.

$$\frac{(r-1)!}{(1-z)^r} = \sum_{i=r-1}^{\infty} i(i-1)...(i-r+2)z^i \tag{3.1.22}$$

□

Proposition 3.23. *Let $X_1, ..., X_r$ are independent random variables and for any i P_{X_i} is the geometric distribution. Then the distribution of $\sum_{i=1}^{r} X_i - (r-1)$ is $N(r, p)$.*

3.2 Descriptive statistics

3.2.1 Skewness

Definition 3.24 (Skewness)**.** *Let*

(S1) $\mu \in \mathscr{P}(\mathbb{R})$.

(A1) $\nu := E[\mu]$ *and* $\sigma^2 := V[\mu]$ *exist.*

Let us call $E[\frac{(x-\nu)^3}{\sigma^3}]$ *be the skewness of* μ.

Proposition 3.25. *Let*

(S1) f *is a probability density function on* $\mathbb{R}$.

(A1) $f(x) = f(-x)$ *a.e* $x > 0$.

(A2) $\int_{\mathbb{R}} |x|^i f(x)dx < \infty$ $(i = 1, 2)$.

(A3) $\int_{\mathbb{R}} xf(x)dx = 0$.

Then the skewness of the distribution from f is zero.

Proof. We denote S by the skewness of the distribution from f.

$$\begin{aligned} S &= \int_{\mathbb{R}} x^3 f(x)dx \\ &= \int_0^{\infty} x^3 f(x)dx + \int_{-\infty}^{0} x^3 f(x)dx \\ &= \int_0^{\infty} x^3 f(x)dx + \int_{\infty}^{0} (-y)^3 f(-y)(-1)dy \\ &= \int_0^{\infty} x^3 f(x)dx - \int_0^{\infty} y^3 f(y)dy \\ &= 0 \end{aligned} \tag{3.2.1}$$

□

Proposition 3.26. *Let*

(S1) f *is a probability density function on* $\mathbb{R}$.

(A1) $\int_{\mathbb{R}} |x|^i f(x)dx < \infty$ $(i = 1, 2, 3)$.

(S2) $d > 0$.

(A2) *For any* $\epsilon > 0$, *there is* $A, B, a, b \in \mathbb{R}$ *such that* $1 < A < B$ *and* $0 \leq a < b$ *and* $b \leq A$ *and* $(b-a) \leq (B-A)$ *and* $\frac{1}{b-a}\int_a^b xf(-x)dx \leq \frac{1}{B-A}\int_A^B xf(x)dx$ *and* $(A^2-1)\int_A^B xf(-x)dx - (b^2-1)\int_a^b xf(-x)dx \geq d$ *and* $|\int_0^\infty x^i f(x)dx - \int_A^B x^i f(x)dx| < \epsilon$ *and* $|\int_0^\infty x^i f(-x)dx - \int_a^b x^i f(x)dx| < \epsilon$ $(i = 1, 3)$.

(S3) *We denote the skewness of the distribution from* f *by* S.

Then $S \geq d$.

Proof.

$$\begin{aligned}
\int_0^\infty x^3 f(-x)dx &\leq \int_a^b x^3 f(-x)dx + \epsilon \\
&\leq \int_a^b x^3 f(-x)dx - \int_a^b xf(-x)dx + \int_a^b xf(-x)dx + \epsilon \\
&\leq \int_a^b (x^2-1)xf(-x)dx + \int_a^b xf(-x)dx + \epsilon \\
&\leq (b^2-1)\int_a^b xf(-x)dx + \int_0^\infty xf(-x)dx + 2\epsilon \\
&\leq (A^2-1)\int_A^B xf(-x)dx - d + \int_0^\infty xf(-x)dx + 2\epsilon \\
&\leq A^2\int_A^B xf(x)dx - d - \int_A^B xf(-x)dx + \int_0^\infty xf(x)dx + 2\epsilon \\
&\leq A^2\int_A^B xf(x)dx - d - \int_0^\infty xf(-x)dx + \int_0^\infty xf(x)dx + 3\epsilon \\
&\leq \int_A^B x^3 f(x)dx - d + 3\epsilon \\
&\leq \int_0^\infty x^3 f(x)dx - d + 4\epsilon \qquad (3.2.2)
\end{aligned}$$

So $S \geq d$. □

3.2.2 Kurtosis

Definition 3.27 (Kurtosis)**.** *Let*

(S1) $\mu \in \mathscr{P}(\mathbb{R})$.

(A1) $\nu := E[\mu]$ *and* $\sigma^2 := V[\mu]$ *exist.*

Let us call $E[\frac{(x-\nu)^4}{\sigma^4}] - 3$ *be the kurtosis of* μ *and denote it by* $Kurt(\mu)$.

Proposition 3.28. *The kurtosis of* $N(\mu, \sigma)$ *is 0.*

Proof. Let us denote by $C_\sigma := \frac{1}{\sigma\sqrt{2\pi}}$.

$$\begin{aligned}
E_{N(\mu,\sigma)}[(x-\mu)^4] &= C_\sigma \int_{-\infty}^\infty (x-\mu)^4 exp(-\frac{1}{2}(\frac{x-\mu}{\sigma})^2)dx \\
&= C_\sigma \int_{-\infty}^\infty (x-\mu)^4 exp(-\frac{1}{2}(\frac{x-\mu}{\sigma})^2)dx \\
&= C_\sigma \int_{-\infty}^\infty -\sigma^2(x-\mu)^3\{exp(-\frac{1}{2}(\frac{x-\mu}{\sigma})^2)\}'dx \\
&= 3C_\sigma \int_{-\infty}^\infty -\sigma^2(x-\mu)^2 exp(-\frac{1}{2}(\frac{x-\mu}{\sigma})^2)dx \\
&= 3\sigma^4 \qquad (3.2.3)
\end{aligned}$$

□

Proposition 3.29. *For $\tau > 0$ let us denote kurtosis of $h_\tau := \frac{1}{2\tau}\chi_{[-\tau,\tau]}$ by $k(h_\tau)$. Then $\lim_{\tau\to 0} k(h_\tau) = \infty$ and $\lim_{\tau\to\infty} k(h_\tau) = -3$.*

Proof. Because $E[xf] = 0$,

$$k(h_\tau) + 3 = \frac{E[x^4 h_\tau]}{(E[x^2 h_\tau])^2} \tag{3.2.4}$$

The followings hold.

$$E[x^4 h_\tau] = \frac{2}{5}\tau^5 \tag{3.2.5}$$

and

$$E[x^2 h_\tau] = \frac{2}{3}\tau^3 \tag{3.2.6}$$

So there is constant $C > 0$

$$k(h_\tau) + 3 \sim C\frac{\tau^5}{(\tau^3)^2} = C\frac{1}{\tau}\ (\tau \to 0 \text{ or } \tau \to \infty) \tag{3.2.7}$$

□

Proposition 3.30. *We set for $\epsilon > 0$ and $\delta > 0$*

$$f_{\epsilon,\delta}(x) = \begin{cases} \dfrac{1}{x^{(5+\delta)}} & \text{if } |x| > 1, \\ \\ \dfrac{1}{\epsilon}(\dfrac{1}{2} - \dfrac{1}{4+\delta}) & \text{if } |x| \le \epsilon \\ 0 & \text{otherwise} \end{cases} \tag{3.2.8}$$

Then $f_{\epsilon,\delta}$ is a probability density function. Let us denote the kurtosis of $f_{\epsilon,\delta}$ by $k(f_\delta)$. Then the followings hold.

(i) Then for any $\epsilon > 0$ $\lim_{\delta\to 0} k(f_{\epsilon,\delta}) = \infty$.

(ii) For any $\delta > 0$ $\lim_{\epsilon\to 0} k(f_{\epsilon,\delta}) = \infty$.

Proof. Because

$$\int_1^\infty \frac{1}{x^{(5+\delta)}}dx = \frac{1}{4+\delta} \tag{3.2.9}$$

$f_{\epsilon,\delta}$ is a probability density function.

Because $E[xf_{\epsilon,\delta}] = 0$,

$$k(f_{\epsilon,\delta}) + 3 = \frac{E[x^4 f_{\epsilon,\delta}]}{(E[x^2 f_{\epsilon,\delta}])^2} \tag{3.2.10}$$

The followings holds.

$$\begin{aligned} E[x^2 f_{\epsilon,\delta}] &= 2(\int_0^\epsilon x^2 f_{\epsilon,\delta}(x)dx + \int_1^\infty x^2 f_{\epsilon,\delta}(x)dx) \\ &= 2(\frac{\epsilon^3}{3}(\frac{1}{2} - \frac{1}{4+\delta}) + \int_1^\infty \frac{1}{x^{(3+\delta)}}dx) \\ &= 2(\frac{\epsilon^3}{3}(\frac{1}{2} - \frac{1}{4+\delta}) + \frac{1}{(2+\delta)})) \end{aligned} \tag{3.2.11}$$

$$\begin{aligned} E[x^4 f_{\epsilon,\delta}] &= 2(\int_0^\epsilon x^4 f_{\epsilon,\delta}(x)dx + \int_1^\infty x^4 f_{\epsilon,\delta}(x)dx) \\ &= 2(\frac{\epsilon^5}{3}(\frac{1}{2} - \frac{1}{4+\delta}) + \int_1^\infty \frac{1}{x^{(1+\delta)}}dx) \\ &= 2(\frac{\epsilon^5}{3}(\frac{1}{2} - \frac{1}{4+\delta}) + \frac{1}{\delta})) \end{aligned} \tag{3.2.12}$$

So, if we fix δ then there is constant $C > 0$

$$k(f_{\epsilon,\delta}) + 3 \sim C\frac{\epsilon^5}{(\epsilon^3)^2} = C\frac{1}{\epsilon}\ (\epsilon \to 0) \tag{3.2.13}$$

and if we fix ϵ then there is constant $C > 0$

$$k(f_{\epsilon,\delta}) + 3 \sim C\frac{1}{\delta}\ (\delta \to 0) \tag{3.2.14}$$

Then (i) and (ii) hold. □

3.3 Bayes's theorem

Theorem 3.31.

$$P(H_i|A) = \frac{P(H_i)P(A|H_i)}{\Sigma_{j=1}^{n} P(H_j)P(A|H_j)} \tag{3.3.1}$$

Proof. By the definition of conditional probability,

$$P(H_i|A) = \frac{P(H_i)P(A|H_i)}{P(A)} \tag{3.3.2}$$

and

$$P(A) = \Sigma_{j=1}^{n} P(A \cup H_j) = \Sigma_{j=1}^{n} P(H_j)P(A|H_j) \tag{3.3.3}$$

So, the above equation holds. □

3.4 Crude Monte Carlo method

Proposition 3.32. *Let*

- *(S1)* $(S := \{1, 2, ..., M\}, 2^{\Omega}, H)$ *is a probability space.*
- *(S2)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S3)* $\{X_n\}_{n=1}^{\infty}$ *is a sequence of independet random variables on* Ω *such that* $X_n(\Omega) \subset S$ *for any* $n \in \mathbb{N}$.
- *(A1)* $X_n \sim H$ *for any* $n \in \mathbb{N}$. $X_n \sim H)$ *means that* $P(\{X_n = i\}) = H(i)$
- *(S4)* g *is a function on* S.
- *(S5)* $\{Y_n\}_{n=1}^{\infty}$ *is a sequence of independet random variables on* Ω *such that* $Y_n(\Omega) \subset S$ *for any* $n \in \mathbb{N}$.
- *(A2)* $Y_n \sim C$ *for any* $n \in \mathbb{N}$. *Here,* C *is the counting measure of* S.

then

$$\operatorname*{plim}_{N\to\infty} \frac{\Sigma_{i=1}^{N} g(X_i)}{N} = \Sigma_{s\in S}\ g(s)H(\{s\}) = \#S \operatorname*{plim}_{N\to\infty} \frac{\Sigma_{i=1}^{N} g(Y_i)H(\{Y_i\})}{N} \tag{3.4.1}$$

STEP1. Showing (the left side)=(the middle side) . Clearly $\{g(X_n)\}_{n=1}^{\infty}$ is a sequence of independet random variables on Ω. By $(A1)$,

$$\int_{\Omega} g(X_n)dP = \Sigma_{s\in S}\ g(s)H(\{s\}) \tag{3.4.2}$$

and

$$\int_{\Omega} g(X_n)^2 dP = \Sigma_{s\in S}\ g^2(s)H(\{s\}) \tag{3.4.3}$$

So by weak law of large numbers (3.4.1) holds. □

STEP2. Showing (the right side)=(the middle side) . We set

$$G : S \ni s \mapsto g(s)H(\{s\})\#S \in \mathbb{R} \tag{3.4.4}$$

By applying the method of STEP1 to G and C,

$$\begin{aligned} \operatorname*{plim}_{N\to\infty} \frac{\Sigma_{i=1}^{N} g(Y_i)H(\{Y_i\})\#S}{N} &= \Sigma_{s\in S}\ g(s)H(\{s\})\#SC(\{s\}) \\ &= \Sigma_{s\in S}\ g(s)H(\{s\}) \end{aligned} \tag{3.4.5}$$

□

3.5 Chi-Squared Test for Categorical Data

Proposition 3.33. *Let*

(S1) $(\Omega, \mathscr{F}, P)$ *is a probability space.*

(S2) $\{X_i\}_{i=1}^{\infty}$ *is a sequence of* N*-dimensional vectors of random variables on* $(\Omega, \mathscr{F}, P)$.

(A1) $\{X_i\}_{i=1}^{\infty}$ *distribution converges to* $N(0, E_N)$.

then $\{|X_i|^2\}_{i=1}^{\infty}$ *distribution converges to* $\chi^2(N)$.

Proof. Let us fix arbitary $a > 0$.

Let λ be the N-dimensional Lebesugue's measure. By (A1) and $\lambda(\partial B(X, \sqrt{a})) = 0$,

$$\begin{aligned} \mu(\{|X_i|^2 \leq a\}) &= \mu(\{X_i \in \overline{B(X, \sqrt{a})}\}) \\ &\to N(0, E_N)(\overline{B(X, \sqrt{a})}) \ (i \to \infty) \end{aligned} \tag{3.5.1}$$

By the definition of chi-squared distribution with degree of free N,

$$N(0, E_N)(\overline{B(X, a)}) = \chi^2(N)([0, a]) \tag{3.5.2}$$

So $\{|X_i|^2\}_{i=1}^{\infty}$ distribution converges to $\chi^2(N)$. □

Theorem 3.34. *Let*

(S1) $(\Omega, \mathscr{F}, P)$ *is a probability space.*

(S2) $\{X_i\}_{i=1}^{\infty}$ *is a sequence of* K*-dimensional vectors of random variables on* $(\Omega, \mathscr{F}, P)$.

(S3) $\{\pi_k\}_{k=1}^{K} \subset (0, 1)$ *such that* $\Sigma_{k=1}^{K} \pi_k = 1$.

(A1) $P(\{X_{i,k} = 1\}) = 1 \ (\forall i, \forall k)$.

(A2) For any k, l *such that* $k \neq l$, $\{X_{i,k} = 1\} \cup \{X_{i,l} = 1\} = \phi \ (\forall i)$.

(S4) $O_{n,k} := \Sigma_{i=1}^{n} X_{i,k} \ (n \in \mathbb{N}, k \in \mathbb{N})$.

(S5) $E_{n,k} := n\pi_k \ (n \in \mathbb{N}, k \in \mathbb{N})$.

then

$$Q(n) := \Sigma_{k=1}^{K} \frac{(O_{n,k} - E_{n,k})^2}{n\pi_i} \tag{3.5.3}$$

distribution converges to $\chi^2(K - 1)$.

Proof. We set

$$Y_{n,k} := \sqrt{n}(\bar{X}_k - \pi_k) \ (n \in \mathbb{N}, k \in \mathbb{N}) \tag{3.5.4}$$

Then

$$Y_{n,K} := -\Sigma_{k=1}^{K-1} Y_{n,k} \ (\forall n) \tag{3.5.5}$$

and

$$O_{n,k} - E_{n,k} = \sqrt{n} Y_{n,k} \ (n \in \mathbb{N}, k \in \mathbb{N}) \tag{3.5.6}$$

$$Y_n := (Y_{n,1}, ..., Y_{n,K-1})^T \tag{3.5.7}$$

If we set $A := \{a_{i,j}\}_{i,j=1,\dots,K-1}$ by

$$a_{i,j} = \begin{cases} \frac{1}{\pi_i} + \frac{1}{\pi_K} & \text{if } (i = j), \\ \frac{1}{\pi_K} & \text{if } (i \neq j), \end{cases} \tag{3.5.8}$$

So

$$Q(n) = Y_n^T A Y_n \ (n \in \mathbb{N}) \tag{3.5.9}$$

and A is a nonnogetive definite symmetric matrix.

We set $(K-1)$-by-$(K-1)$ matrix $\Sigma := \{\sigma_{i,j}\}_{i,j=1,\dots,K-1}$ by $\sigma_{i,j} = cov(X_{1,i}, X_{1,j})$. Then

$$\sigma_{i,j} = \begin{cases} \pi_i(1 - \pi_i) & \text{if } (i = j), \\ -\pi_i \pi_j & \text{if } (i \neq j), \end{cases} \tag{3.5.10}$$

and

$$\sigma_{i,j} = cov(X_{n,i}, X_{n,j}) \ (\forall n, \forall i, \forall j) \tag{3.5.11}$$

By Proposition3.35, Σ is positive definite symmetric matrix.

By the central limit theomre(see [?]), $Y_n{}_{n=1}^{\infty}$ distribution converges to $N(0, \Sigma)$.

By Proposition3.33, $\{Q(n)\}_{n=1}^{\infty}$distribution converges to $\chi^2(K - 1)$.

□

Proposition 3.35. *Let A and B be matrixies in the proof oh Theorem3.34. Then* $A^{-1} = \Sigma$

Proof. For any $i \in \{1, 2, ..., K-1\}$

$$\begin{aligned}(A\Sigma)_{i,i} &= a_{i,i}\sigma_{i,i} + \Sigma_{k\neq i} a_{i,k}\sigma_{k,i} \\ &= (\frac{1}{\pi_i} + \frac{1}{\pi_K})\pi_i(1-\pi_i) + \Sigma_{k\neq i}\frac{1}{\pi_K}(-\pi_i\pi_j) \\ &= (1-\pi_i) + \pi_i\frac{(1-\pi_i) - \Sigma_{k\neq i}\pi_k}{\pi_K} \\ &= 1 \end{aligned} \tag{3.5.12}$$

For any $i \in \{1, 2, ..., K-1\}$ and any $j \in \{1, 2, ..., K-1\}$ such that $i \neq j$,

$$\begin{aligned}(A\Sigma)_{i,j} &= a_{i,i}\sigma_{i,j} + a_{i,j}\sigma_{j,j} + \Sigma_{k\neq i,j} a_{i,k}\sigma_{k,i} \\ &= (\frac{1}{\pi_i} + \frac{1}{\pi_K})(-\pi_i\pi_j) + \frac{1}{\pi_K}\pi_j(1-\pi_j) + \Sigma_{k\neq i,j}\frac{1}{\pi_K}(-\pi_k\pi_j) \\ &= (-\pi_j - \frac{\pi_j}{\pi_K}\pi_i) + (\frac{\pi_j}{\pi_K} - \frac{\pi_j}{\pi_K}\pi_j) - \frac{\pi_j}{\pi_K}\Sigma_{k\neq i,j}\pi_k \\ &= -\pi_j + \frac{\pi_j}{\pi_K} - \frac{\pi_j}{\pi_K}(1-\pi_K) \\ &= 0 \end{aligned} \tag{3.5.13}$$

□

3.6 Linear Regression

3.6.1 Preliminaries for Linear Regression

Throughout this section, we assume the following settings.

Setting 3.36 (Linear regression)**.** *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S2) Let* $X := \{X_{i,j}\}_{\{1\leq i\leq N, 1\leq j\leq K\}}$ *be a* (N, K) *matrix.*
- *(A1)* $X^T X$ *is a regular matrix of order* $(K+1)$*.*
- *(S3) Let* $\epsilon := \{\epsilon_i\}_{\{1\leq i\leq N\}}$ *be N random variables.*
- *(A2)* $\{\epsilon_i\}_{\{1\leq i\leq N\}} \overset{iid}{\sim} N(\mathbf{0}, \Sigma_{i=1}^N \sigma^2 E_N)$*, where* $\sigma > 0$*.*
- *(S4) Let* $\{\beta_i\}_{\{1\leq i\leq K\}}$ *be a real K-dimension vector.*
- *(S5) Let* $y := \{y_i\}_{\{1\leq i\leq N\}}$ *be N random variables which are defined by the following equation.*

$$y = X\beta + \epsilon \tag{3.6.1}$$

Remark 3.37. *By (A1),*

$$rank(X) = K \tag{3.6.2}$$

Definition 3.38 (Least squares estimate)**.** *Let*

$$\hat{\beta} := (X^T X)^{-1}(X^T y) \tag{3.6.3}$$

We call $\hat{\beta}$ *the least squares estimate of (3.6.1).*
And let

$$\hat{y} := X\hat{\beta} \tag{3.6.4}$$

We call $\hat{y}$ *the predicted values of (3.6.1).*
Lastly let

$$\hat{e} := y - \hat{y} \tag{3.6.5}$$

We call $\hat{e}$ *the residual of (3.6.1).*

Remark 3.39. $\hat{\beta}$ *is the point which minimize* $\mathbb{R}^K \ni z \mapsto |y - Xz|^2 \in [0,\infty)$*. And*

$$\hat{\beta} := \beta + (X^TX)^{-1}X^T\epsilon \tag{3.6.6}$$

and for each i $\hat{\beta}_i \sim N(\beta_i, \sigma^2\xi_i)$ *and* $\xi_i > 0$*, where* ξ_i *is* (i,i) *component of* $(X^TX)^{-1}$*.*

Definition 3.40 (Multivariate normal distribution)**.** *Let* X_i *be a random variable on* $(\Omega, \mathscr{F})$ $(i = 1, 2, ..., N)$*.* $\{X_i\}_{i=1}^N \sim N(\gamma, \Sigma)$ *if there is a natural number* l *and* (N, l) *matrix* A *and there are random variables* $\{\epsilon\}_{i=1}^l$ *on* $(\Omega, \mathscr{F})$ *such that* $\boldsymbol{\epsilon} := \{\epsilon\}_{i=1}^l$ *are i.i.d and* $\epsilon_i \sim N(0,1)$ $(\forall i)$ *and* $X = A\boldsymbol{\epsilon} + \gamma$ *and* $\Sigma = AA^T$*.*

3.6.2 Interval estimation of regression coefficients

Proposition 3.41.

$$\frac{|\hat{e}|^2}{\sigma^2} \sim \chi^2(N-K) \tag{3.6.7}$$

Proof. The following holds.

$$\hat{e} = (E_N - X(X^TX)^{-1}X^T)\epsilon \tag{3.6.8}$$

Let $A := (E_N - X(X^TX)^{-1}X^T)$ then A is symmetric and idempotent. So each eigenvalue of A is 0 or 1. And $tr(A) = N - tr(X(X^TX)^{-1}X^T) = N - tr((X^TX)^{-1}X^TX) = N - K$ so $rank(A) = N - K$. So by Proposition**??**, $\frac{|\hat{e}|^2}{\sigma^2} \sim \chi^2(N-K)$. □

Proposition 3.42. $\hat{\beta}$ *and* $\hat{e}$ *are independent.*

Proof. By (3.6.6) and (3.6.8), $cov(\hat{e}, \hat{\beta}) = 0$. So by Proposition3.42 $\hat{\beta}$ and $\hat{e}$ are independent. □

By Remark and Proposition3.41 and Proposition3.41 and Proposition3.42, the folloing Proposition holds.

Proposition 3.43. *For each* $i \in \mathbb{N} \cap [1, K]$*,*

$$\frac{(\hat{\beta}_i - \beta_i)\sqrt{(N-K)}}{|\hat{e}|\sqrt{\xi_i}} \sim t(N-K) \tag{3.6.9}$$

In the above equation, t_{N-K} *is the t-distribution whose degrees of freedom is* $N - K$ *and* ξ_i *is* (i,i) *component of* $(X^TX)^{-1}$*.*

The following is a remark.

Proposition 3.44.

$$E(\frac{|\hat{e}|^2\xi_i}{N-K}) = V(\hat{\beta}_i)\ (\forall i) \tag{3.6.10}$$

Proof. By Proposition3.41, $E(\dfrac{|\hat{e}|^2\xi_i}{N-K}) = \sigma^2\xi_i$. By Remark3.6.2, $V(\hat{\beta}_i) = \sigma^2\xi_i$ □

By the above remak, $\dfrac{|\hat{e}|\sqrt{\xi_i}}{\sqrt{N-K}}$ is denoted by $se(\hat{\beta}_i)$.

3.6.3 Decomposition of TSS

Proposition 3.45.

$$(\hat{y}, \hat{e}) = 0 \tag{3.6.11}$$

Proof. By (3.6.6),

$$X^T\hat{y} = X^TX\hat{\beta} = X^T(X\beta + \boldsymbol{\epsilon}) = X^Ty \tag{3.6.12}$$

So

$$\begin{aligned} (\hat{y}, \hat{e}) &= \beta^TX^T\hat{e} \\ &= \beta^TX^T(y - \hat{y}) \\ &= 0 \end{aligned}$$

□

Proposition 3.46. *Let*

(A1) There is a K-by-K matrix B such that the first column of XB is 1_N

then

$$\overline{\hat{y}} = \overline{y} \tag{3.6.13}$$

Proof. By (3.6.6),

$$X^T\hat{y} = X^TX\hat{\beta} = X^T(X\beta + \boldsymbol{\epsilon}) = X^Ty \tag{3.6.14}$$

So the following holds.

$$B^TX^T\hat{e} = 0 \tag{3.6.15}$$

The fitst component of the $B^TX^T\hat{e}$ is $\overline{\hat{y}} - \overline{y}$. So $\overline{\hat{y}} = \overline{y}$. □

Proposition 3.47. *Let*

(S1) $TSS := |y - \overline{y}1_n|^2$

(S2) $RSS := |\hat{y} - \overline{y}1_n|^2$

(S3) $ESS := |y - \hat{y}|^2$

(A1) (A1) in Proposition3.46

then

$$TSS = RSS + ESS \tag{3.6.16}$$

Proof. Because

$$TSS = y^T(E - \frac{1}{N}1_{N,N})y \tag{3.6.17}$$

and

$$RSS = y^T(X^T(X^TX)^{-1}X - \frac{1}{N}1_{N,N})y \tag{3.6.18}$$

and

$$ESS = y^T(E - X^T(X^TX)^{-1}X)y \tag{3.6.19}$$

$TSS = RSS + ESS$. □

3.6.4 Cochran's theorem

Proposition 3.48. *Let*

(S1) $m \in \mathbb{N}$ and A_i:N-by-N symmetric matrix $(i = 1, 2, ..., m)$

(A1) $E_N = \Sigma_{i=1}^m A_i$

(A2) $N = \Sigma_{i=1}^m rank(A_i)$

then

$$A_iA_j = \delta_{i,j}A_i \ (\forall i, \ \forall j) \tag{3.6.20}$$

where $\delta_{i,j}$ is a Kronecker delta.

Proof. Let $V_i := A_i\mathbb{R}^N$ and $n_i := rank(A_i)$ and $\{v_{i,j}\}_{1\le j\le n_i}$ be a basis of V_i $(i = 1, 2, ..., m)$. By (A1) and (A2), $\{v_{i,j}\}_{1\le i\le m, 1\le j\le n_i}$ is a basis of $\mathbb{R}^N$. and

$$\mathbb{R}^N = \bigoplus_{i=1}^m V_i \tag{3.6.21}$$

Let fix arbitary $i \in \{1, 2, ..., N\}$ and fix arbitary $x \in \mathbb{R}^N$. $A_ix = (\Sigma_{i=1}^m A_i)A_ix = (A_i)^2x + (\Sigma_{j\ne i}A_jA_ix$. By (3.6.21), $A_ix = {A_i}^2x$ and $A_jA_ix = 0$. □

By Proposition3.48 and Proposition**??** and Proposition, the following theorem holds.

Proposition 3.49 (Cochran's theorem)**.**
We take over (S1) and (A1) in Proposition3.48. And let

(S2) $(\Omega, \mathscr{F}, P)$ is a probability space.

(A1) $\boldsymbol{\epsilon} \sim N(0, E_N)$

(S3) $Q_i := \boldsymbol{\epsilon}^TA_i\boldsymbol{\epsilon}$ $(i = 1, 2, ..., m)$

then $Q_i \sim \chi^2(rankA_i)$ $(\forall i)$ and Q_i and Q_j are independent for all $(i, j) \in \{(i, j)|i \ne j\}$

3.6.5 Testing

Throughout this subsection, we assume

$$\beta = (\beta_0, 0, 0, ..., 0)^T \tag{3.6.22}$$

and

$$X = \begin{pmatrix} 1 & x_{1,1} & x_{1,2} & ... & x_{1,L} \\ 1 & x_{2,1} & x_{2,2} & ... & x_{2,L} \\ ... & ... & ... & ... & \\ 1 & x_{N,1} & x_{N,2} & ... & x_{N,L} \end{pmatrix} \tag{3.6.23}$$

Then

$$X\beta = \beta_0 1_{N,1} \tag{3.6.24}$$

So

$$\begin{aligned} \hat{y} &= X(X^TX)^{-1}X^Ty \\ &= X(X^TX)^{-1}X^T(X\beta + \epsilon) \\ &= \beta_0 1_{N,1} + X(X^TX)^{-1}X^T\epsilon \end{aligned} \tag{3.6.25}$$

And

$$\bar{y}1_{N,1} = \beta_0 \frac{1}{N} 1_{N,1} + 1_{N,N}\epsilon \tag{3.6.26}$$

Consequently,

$$RSS = \epsilon^T(X(X^TX)^{-1}X^T - \frac{1}{N}1_{N,1})\epsilon \tag{3.6.27}$$

Because $X(X^TX)^{-1}X^T$ is symmetric, $X(X^TX)^{-1}X^T$ and $\frac{1}{N}1_{N,1}$ are commutative.
And because $X(X^TX)^{-1}X^T$ is idempotent and symmetric, $(X(X^TX)^{-1}X^T - \frac{1}{N}1_{N,1})$ is idempotent and symmetric.
$rank(X(X^TX)^{-1}X^T - \frac{1}{N}1_{N,1}) = tr(X(X^TX)^{-1}X^T - \frac{1}{N}1_{N,1}) = L$
So by Proposition3.49, RSS and ESS are independent and $RSS \sim \chi^2(L)$ and $ESS \sim \chi^2(N-L-1)$.
So,

$$\frac{\frac{RSS}{L}}{\frac{ESS}{N-L-1}} \sim F(L, N-L-1) \tag{3.6.28}$$

3.6.6 Simple linear regression

Throughout this subsection, we set

$$T_x = \sum_{i=1}^{n} x_i,\ T_y = \sum_{i=1}^{n} y_i,\ T_{x,x} = \sum_{i=1}^{n} x_i^2,\ T_{x,y} = \sum_{i=1}^{n} x_i y_i \tag{3.6.29}$$

(1) Case1: there is intercept
Throughout this subsection, we assume

$$X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ ... & ... \\ 1 & x_n \end{pmatrix} \tag{3.6.30}$$

Then

$$\begin{aligned}
\hat{\beta} &= \begin{pmatrix} \hat{\alpha} \\ \hat{\gamma} \end{pmatrix} \\
&= (X^T X)^{-1} X^T y \\
&= (\begin{pmatrix} 1 & 1 & ... & 1 \\ x_1 & x_2 & ... & x_n \end{pmatrix} \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ ... & ... \\ 1 & x_n \end{pmatrix})^{-1} X^T y \\
&= \begin{pmatrix} n & T_x \\ T_x & T_{x,x} \end{pmatrix}^{-1} X^T y \\
&= \frac{1}{nT_{x,x} - T_x^2} \begin{pmatrix} T_{x,x} & -T_x \\ -T_x & n \end{pmatrix} \begin{pmatrix} T_y \\ T_{x,y} \end{pmatrix}
\end{aligned} \tag{3.6.31}$$

So

$$\begin{aligned}
\hat{\gamma} &= \frac{nT_{x,y} - T_x T_y}{nT_{x,x} - T_x^2} \\
&= \frac{T_{x,y} - \frac{1}{n} T_x T_y}{T_{x,x} - \frac{1}{n} T_x^2} \\
&= \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}
\end{aligned} \tag{3.6.32}$$

Consequently,

$$\hat{\gamma} = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} \tag{3.6.33}$$

(2) Case2: there is no intercept
Throughout this subsection, we assume

$$X = (x_1, x_2, ..., x_n)^T \tag{3.6.34}$$

Then

$$\hat{\beta} = \frac{T_{x,y}}{T_{x,x}} \tag{3.6.35}$$

3.6.7 Estimation about population mean

Throughout this section, we assume $X = 1_N$ is one and we define μ by $\beta = \mu 1_1$. The followings hold.

$$X^T X = N \tag{3.6.36}$$

$$Y := X(X^T X)^{-1} X^T = \frac{1}{N} 1_{N,N} \tag{3.6.37}$$

$$\hat{e} := y - \overline{y} 1_N \tag{3.6.38}$$

$$\frac{|\hat{e}|^2}{\sigma^2} \sim \chi^2(N-1) \tag{3.6.39}$$

$$\frac{(\mu - \overline{y})\sqrt{N(N-1)}}{|y - \overline{y}|} \sim t(N-1) \tag{3.6.40}$$

3.6.8 Estimation about difference between two population means

Throughout this section, we assume

$$X = \begin{pmatrix} 1_M & 0 \\ 0 & 1_N \end{pmatrix} \tag{3.6.41}$$

and

$$\beta = \begin{pmatrix} \mu_1 1_M \\ \mu_2 1_N \end{pmatrix} \tag{3.6.42}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} := y \tag{3.6.43}$$

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} := \epsilon \tag{3.6.44}$$

Then the followings hold.

$$X^T X = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} \tag{3.6.45}$$

$$Y := \begin{pmatrix} \frac{1}{M} 1_{M,M} & 0 \\ 0 & \frac{1}{N} 1_{N,N} \end{pmatrix} \tag{3.6.46}$$

$$\mu_1 = (\hat{y_1})_1 = \overline{y_1} + \overline{\epsilon_1} \tag{3.6.47}$$

$$\mu_2 = (\hat{y_2})_1 = \overline{y_2} + \overline{\epsilon_2} \tag{3.6.48}$$

So, by reproductive property of normal distribution,

$$\mu_1 - \mu_2 - (\overline{y_1} - \overline{y_2}) \sim N(0, (\frac{1}{M} + \frac{1}{N})\sigma^2) \tag{3.6.49}$$

And the following holds.

$$|\hat{e}|^2 = |y_1 - \mu_1 1_M|^2 + |y_2 - \mu_2 1_N|^2 \tag{3.6.50}$$

By Proposition3.42, $(\mu_1 - \mu_2 - (\overline{y_1} - \overline{y_2}))$ and $|y_1 - \mu_1 1_M|^2 + |y_2 - \mu_2 1_N|^2$ are independent. Consequently, the following holds.

$$\frac{(\mu_1 - \mu_2 - (\overline{y_1} - \overline{y_2}))\sqrt{M+N-2}}{\sqrt{(|y_1 - \mu_1 1_M|^2 + |y_2 - \mu_2 1_N|^2)(\frac{1}{M} + \frac{1}{N})}} \sim t(M+N-2) \tag{3.6.51}$$

3.6.9 One way analysis of variance

Throughout this section we set

$$y := (y_{1,1}, ..., y_{1,n_1}, y_{2,1}, ..., y_{2,n_2}, ..., y_{K,1}, ..., y_{K,n_K})^T \tag{3.6.52}$$

$$\beta := (\mu_1, \mu_2, ..., \mu_K)^T \tag{3.6.53}$$

$$\bar{y}_{i,\cdot} := \frac{\sum_{j=1}^{n_i} y_{i,j}}{n_i} \ (i = 1, 2, ..., K) \tag{3.6.54}$$

$$X := \begin{pmatrix} 1_{n_1} & O & O & O \\ 1_{n_2} & 1_{n_2} & O & O \\ ... & ... & ... & ... \\ 1_{n_K} & O & O & 1_{n_K} \end{pmatrix} \tag{3.6.55}$$

Then

$$Y := X(X^TX)^{-1}X^T := \begin{pmatrix} \frac{1}{n_1}1_{n_1,n_1} & O & O & O \\ O & \frac{1}{n_2}1_{n_2,n_2} & O & O \\ ... & ... & ... & ... \\ O & O & O & \frac{1}{n_K}1_{n_K,n_K} \end{pmatrix} \tag{3.6.56}$$

In this subsection, hereafter, we assume there is a real number μ such that

$$\beta = \mu 1_K \tag{3.6.57}$$

Then the followings holds.

$$TSS = \boldsymbol{\epsilon}^T(E_N - \frac{1}{N}1_{N,N})\boldsymbol{\epsilon} \tag{3.6.58}$$

$$ESS = \boldsymbol{\epsilon}^T(Y - \frac{1}{N}1_{N,N})\boldsymbol{\epsilon} \tag{3.6.59}$$

$$rank(Y - \frac{1}{N}1_{N,N}) = K - 1 \tag{3.6.60}$$

$$RSS = \boldsymbol{\epsilon}^T(E_N - Y)\boldsymbol{\epsilon} \tag{3.6.61}$$

$$rank(E_N - Y) = N - K \tag{3.6.62}$$

So, by Cohchran's theorem, ESS and RSS are independent, and $ESS \sim \chi^2(K-1)$ and $RSS \sim \chi^2(N-K)$. Consequently, the following theoem holds.

Theorem 3.50. *Under the setting(3.6.55) and the assumption(3.6.57)*

$$(ESS/(K-1))/(RSS/(N-K)) \sim F(K-1, N-K) \tag{3.6.63}$$

And the followings hold.

$$(X^TX)^{-1} = \begin{pmatrix} \frac{1}{n_1} & 0 & ... & 0 \\ 0 & \frac{1}{n_2} & ... & 0 \\ ... & ... & ... & ... \\ 0 & ... & 0 & \frac{1}{n_K} \end{pmatrix} \tag{3.6.64}$$

$$\hat{\beta} = (\bar{y}_{1,\cdot}, \bar{y}_{2,\cdot}, ..., \bar{y}_{K,\cdot})^T \tag{3.6.65}$$

So, by Proposition3.43, the following theoem holds.

Theorem 3.51. *Under the setting(3.6.55)*

$$(\bar{y}_{i,\cdot} - \mu_i)\sqrt{\frac{(N-K)n_i}{ESS}} \sim t(N-K) \tag{3.6.66}$$

3.7 Principal Component Analysis

3.8 Kernel Method

3.8.1 Motivation

Kernel Method is a method for effectively analyzing high dimensional data which does not fit statistical linear model.

Terminology 3.52 (Feature Space, Feature Map)**.** *The followings are settings.*

- *(S1)* Ω *be a set.*
- *(S2)* $\mathcal{H}$ *be a real inner product space.*
- *(S3)* $\Phi : \Omega \to H$.

We call Ω *a feature space and* Φ *a feature map, respectively.*

I imagine Ω to be a high dimensional data set like a subset of $\mathbb{R}^{10000}$. And I assume that for a given statistical problem like regression or principal component analysis or others, Ω does not fit statistical linear model like linear regression or linear principal component analysis or others. So, I hope $\Phi(\Omega)$ does fit the linear model. Since Ω is high dimensional, in many case H is also high dimensional. In general, that impose us highly costed calculation of the inner product. However, if we find $k : \Omega \times \Omega \to \mathbb{R}$ such that

$$(\Phi(X), \Phi(Y)) = k(X, Y)\ (\forall X, Y \in \Omega)$$

the inner product is easy to calculate. Here, k is called a kernel function and H is called a reproducing kernel Hilbert space. Kernel method is the method to solove a given problem using (H, k). In addition, such statistical problems are often reduced to an optimization problem in H. By the theory of the kernel method, it is shown that a solution of the optimization problem can be expressed as a linear combination of $\{\Phi(X_i)\}_{i=1}^n$.

$$\sum_{i=1}^{m} \alpha_i \Phi(X_i)$$

3.8.2 Positive Definite Kernel Function

Definition 3.53 (Real Valued Positive Definite Kernel Function). *The followings are settings.*

(S1) Ω be a set.

(S2) k be a real valued function on Ω.

We say k is a positive definite kernel function if for any $x_1, ..., x_m \in \Omega$ $\{k(x_i, x_j)\}_{i,j=1,2,...,m}$ is a positive semi-definite symmetric matrix.

Definition 3.54 (Complex Valued Positive Definite Kernel Function). *The followings are settings.*

(S1) Ω be a set.

(S2) k be a complex valued function on Ω.

We say k is a complex valued positive definite kernel function if for any $x_1, ..., x_m \in \Omega$ $\{k(x_i, x_j)\}_{i,j=1,2,...,m}$ is a positive semi-definite Hermitian matrix.

Example 3.55. *Let G be a topological group and ϕ be a positive definite function on G. Then*

$$k(x, y) := \phi(xy^{-1})\ (x, y \in G)$$

is a positive definite kernel function. For detail, see [14].

3.8.3 Reproducing Kernel Hilbert Space(RKHS)

Definition 3.56 (Reproducing Kernel Hilbert Space). *The followings are settings.*

(S1) Ω is a set.

(S3) H is a Hilbert space.

We say H is a real reproducing kernel Hilbert space over Ω if

$$H \subset Map(\Omega, \mathbb{R})$$

and for each $x \in \Omega$ there exists $k_x \in H$ such that

$$(u, k_x) = u(x)\ (\forall u \in H)$$

We call a function

$$k : \Omega^2 \ni (x, y) \mapsto k_x(y) \in \mathbb{R}$$

reproducing kernel.

Proposition 3.57. *The followings are settings.*

(S1) H is a real reproducing kernel Hilbert space over Ω.

Then the reproducing kernel is uniquely determined and is a positive definite kernel function.

Proof. See [14]. □

The following theorem shows that a positive definite kernel function identify a reproducing Hilbert space.

Theorem 3.58 (Moore-Aronszajn)**.** *The followings are settings.*

(S1) k *is a real positive definite kernel function over* Ω*.*

Then there is a reproducing kernel Hilbert space H *over* Ω *such that*

(i) k *is a reproducing kernel of* H*.*

(ii) For any $x \in \Omega$, $k(\cdot, x) \in H$*.*

(iii) $\{k(\cdot, x)\}_{x\in\Omega}$ *are dense in* H*.*

Proof. See [14]. □

Moore-Aronzjan Theorem also gives us a good feature map.

Proposition 3.59. *The followings are settings.*

(S1) Ω *is a feature space.*

(S2) k *is a real positive definite kernel function over* Ω*.*

(S3) H *is a reproducing kernel space with* k*.*

(S4) We define a feature map by

$$\Phi : \Omega \ni x \mapsto k(\cdot, x) \in H$$

Then

$$(\Phi(x), \Phi(y)) = k(x, y) \ (\forall x, y \in \Omega)$$

Proof. The proposition is clear from the definition of reproducing kernel space. □

The following theorem clarify a form of a solution of optimization problems in a reproducing Hilbert space.

Theorem 3.60 (Representer Theorem)**.** *The followings are settings.*

(S1) Ω *is a feature space.*

(S2) Λ *is a set.*

(S3) $\{(X_i, Y_i)\}_{i=1}^N \subset \Omega \times \Lambda$*.*

(S4) $\Psi : [0, \infty) \to \mathbb{R}$ *a strictly monotone increasing function.*

(S5) H *is a reproducing kernel Hilbert space.*

(S6) $L : H^N \mapsto \mathbb{R}$*.*

(S7) $h_1, ..., h_m \in H$*.*

Then the optimization problem

$$\min_{f\in H, c\in\mathbb{R}^m} F(f, c) := (L(\{f(X_i) + \sum_{\alpha=1}^{m} c_\alpha h_\alpha(X_i)\}_{i=1}^N) + \Psi(||f||))$$

has solutions in $\langle\{k_{X_i}\}_{i=1}^N\rangle$*.*

Proof. We set $H_0 := \langle\{k_{X_i}\}_{i=1}^N\rangle$. Let us fix any $f \in H$ and $c \in \mathbb{R}^m$. Then there are $f_0 \in H_0$ and $f_1 \in H_0^\perp$ (See [14].). From this, $||f_0||^2 \le ||f||^2$. So,

$$f(X_i) = (f, k_{X_i}) = (f_0, k_{X_i}) = f_0(X_i)$$

and

$$\Psi(||f_0||^2) \le \Psi(||f||^2)$$

This implies $F(f_0, c) \le F(f_0, c)$. □

3.8.4 Kernel Principal Components Analysis

Proposition 3.61. *The followings are settings and assumptions.*

(S1) Ω *is a feature space.*

(S2) H *is a reproducing kernel Hilbert space over* Ω *with the reproducing kernel* k*.*

(S3) $\Phi : \Omega \to H$ *is a feature map such that*

$$\Phi(x) = k_x \ (\forall x \in \Omega)$$

(S4) $\{X_i\}_{i=1}^N \subset \Omega$*.*

(S5) $\tilde{\Phi}(X_i) := \Phi(X_i) - \frac{1}{N}\sum_{j=1}^N \Phi(X_j)$*.*

(S6) *We call the optimization problem*

$$\max_{f \in H, ||f||=1} \frac{1}{N}\sum_{i=1}^N((f, \Phi(X_i)) - \frac{1}{N}\sum_{j=1}^N(f, \Phi(X_j))^2$$

problemA1.

(S7) *We set*

$$\tilde{K}_{i,j} := (\tilde{\Phi}(X_i), \tilde{\Phi}(X_j)) = k(X_i, X_j) - \frac{1}{N}\sum_{b=1}^N k(X_i, X_b) - \frac{1}{N}\sum_{a=1}^N k(X_a, X_j) + \frac{1}{N^2}\sum_{a,b=1}^N k(X_a, X_b) \ (i, j = 1, 2, ..., N)$$

We call $\tilde{K} := \{K_{i,j}\}_{i,j=1}^N$ *the centering gram matrix. Let* $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N$ *denote all eigenvalues of* $\tilde{K}$*. For each* i*, let* u_i *denote an unit eigenvector regarding to* λ_i*.*

(S8) *We call the optimization problem*

$$\max_{a \in \mathbb{R}^N, a^T\tilde{K}a=1} a^T\tilde{K}^2 a$$

problemB1.

Then the followings holds.

(i) *A solution of the problemA1 exists in* $\langle\{\tilde{\Phi}(X_i) | i = 1, 2, ..., N\}\rangle$*.*

(ii) *For any solution of the problemB1, denoted by* a*,* $\sum_{i=1}^m a_i\tilde{\Phi}$ *is a solution of problemA1.*

(iii) $f^1 := \sum_{i=1}^N \alpha_i^1\tilde{\Phi}(X_i)$*,* $\alpha_i^1 = \frac{1}{\sqrt{\lambda_1}}u^1$*. Then* f^1 *is a solution of problemA1.*

(iv) $(\tilde{\Phi}(X_i), f^1) = \sqrt{\lambda_i}u_1^1$ *for any* i*.*

(v) *We define the optimization problem*

$$\max_{f \in H, ||f||=1, f\perp\langle f_1\rangle} \frac{1}{N}\sum_{i=1}^N((f, \Phi(X_i)) - \frac{1}{N}\sum_{j=1}^N(f, \Phi(X_j))^2$$

and we call it problemA2. By the same way, we define problemA3,...,problemAN. And $f^p := \sum_{i=1}^N \alpha_i^1\tilde{\Phi}(X_i)$*,* $\alpha_i^p = \frac{1}{\sqrt{\lambda_p}}u^p$*. Then* f^p *is a solution of problemAp* $(p = 1, 2, ..., N)$*.*

(iv) $(\tilde{\Phi}(X_i), f^p) = \sqrt{\lambda_i}u_p^1$ *for any* i *and* p*.*

4 Mathematical Programming

4.1 MILP and Branch-and-Bound Method

Definition 4.1 (MILP:Mixed integer linear programming)**.** *Let*

> *(S1)* $A \in M(m, n, \mathbb{Q})$, $G \in M(m, p, \mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.
>
> *(S2)* $S := \{(x, y) \in (\mathbb{Z}_+)^n \times (\mathbb{R}_+)^p | g(x, y) := Ax + Gy \leq b\}$

We call the following problem a MILP.

$$\begin{aligned} &max\ f(x, y) := c^t x + h^t y \\ &subject\ to\ (x, y) \in S \end{aligned}$$

We succeed notations in Definition4.1. And we set

$$S^0 := \{(x, y) \in (\mathbb{R}_+)^n \times (\mathbb{R}_+)^p | Ax + Gy \leq b\}$$

Let us assume the MILP has a opmimal solution (x^*, y^*) and the optimal optimal value z^*. So $S^0 \neq \phi$. Let us fix $(x, y) \in S^0$.

Algorithm Branch-and-Bound Method

Input: $S^0 \neq \phi$
Step 1: Take a $(x^0, y^0) \in S^0$ and $(\underline{x}, \underline{y}, \underline{z}) \leftarrow (x_0, y_0, f(x^0, y^0))$ and $\mathcal{S} \leftarrow S_0$
Step 2: Take $j \in \{1, 2, ..., n\}$. $S_{00} := \{(x, y) \in S | x_j \leq \lfloor x_j^0 \rfloor\}$ and $S_{01} := \{(x, y) \in S | x_j \geq \lceil x_j^0 \rceil\}$ and
    MILP$_{00}$: $max f(S_{00})$ and MILP$_{01}$: $max f(S_{01})$.
    Delete S_0 from $\mathcal{S}$ and add S_{00} and S_{01} to $\mathcal{S}$.
Step 3: **for** $S_\alpha \in \mathcal{S}$ **do**
        Solve $LP_\alpha : max f(S_\alpha)$.
        if LP_α is not feasible **then**
            Delete S_α from $\mathcal{S}$.
        else
            We set (x^α, y^α) which is a optimal solution and z^α which is its optimal value.
            Delete S_α from $\mathcal{S}$.
            if $x^\alpha \in \mathbb{Z}_+^n$ **then**
                if $z^\alpha > \underline{z}$ **then**
                    $(\underline{x}, \underline{y}, \underline{z}) \leftarrow (x^\alpha, y^\alpha, f(x^\alpha, y^\alpha))$.
                end if
            else$z^\alpha > \underline{z}$
                Take $j \in \{1, 2, ..., n\}$. $S_{\alpha 0} := \{(x, y) \in S_\alpha | x_j \leq \lfloor x_j^\alpha \rfloor\}$ and $S_{\alpha 1} := \{(x, y) \in S_\alpha | x_j \geq \lceil x_j^\alpha \rceil\}$.
                Add $S_{\alpha 0}$ and $S_{\alpha 1}$ to $\mathcal{S}$.
            end if
        end if
    end for
Output: $(\underline{x}, \underline{y}, \underline{z})$.

4.2 Meyer's Fundamental Theorem

4.2.1 Main result

The propositions shown in this subsection will not be presented with proofs in this subsection, but will be presented with proofs in the subsections that follow.

Definition 4.2 (Polyhedron)**.** *Let* $A \in M(m, n, \mathbb{R}), b \in \mathbb{R}^m$. *We call*

$$P := \{x \in \mathbb{R}^n | Ax \leq b\}$$

a Polyhedron in $\mathbb{R}^n$ *or a* $\mathcal{H}$*-polyhedron. We call the right side* $\mathcal{H}$*-representation. If* $A \in M(m, n, \mathbb{Q}), b \in \mathbb{Q}^m$ *then* P *is a rational polyhedron.*

Definition 4.3 (Recession cone). *Let P be a nonempty polyhedron. We call*

$$rec(P) := \{r \in \mathbb{R}^n | x + \lambda r \in P, \forall x \in P, \forall \lambda \in \mathbb{R}_+\}$$

the recession cone of P.

Notation 4.4. *Let*

(S1) $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

We set

$$P(A,G,b) := \{(x,y) \in (\mathbb{R}_+)^n \times (\mathbb{R}_+)^p | g(x,y) := Ax + Gy \leq b\}$$

Definition 4.5 (Convex, Convex combination). *Let $A \subset \mathbb{R}^n$. We say A is convex if $\sum_{i=1}^n \lambda_i a_i \in A$ for $a_1, ..., a_n \in A$ and $\lambda_1, ..., \lambda_n \subset [0,1]$ such that $\sum_{i=1}^n \lambda_i = 1$. We call the sum*

$$\sum_{i=1}^n \lambda_i a_i$$

convex combination of $a_1, ..., a_n$.

Proposition 4.6. *Let*

(S1) $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

(S2) $S := \{(x,y) \in (\mathbb{Z}_+)^n \times (\mathbb{R}_+)^p | g(x,y) := Ax + Gy \leq b\}$

Then

(i)

$$sup\{c^t x + h^t y | (x,y) \in S\} = sup\{c^t x + h^t y | (x,y) \in conv(S)\}$$

Furthermore, there is $(x,y) \in S$ such that $c^t x + h^t y = sup\{c^t x + h^t y | (x,y) \in S\}$ $\iff$ there is $(x,y) \in conv(S)$ such that $c^t x + h^t y = sup\{c^t x + h^t y | (x,y) \in S\}$

(ii) $ex(conv(S)) \subset S$

Theorem 4.7 (Meyer(1974)[9] Fundamental Theorem). *Here are the settings and assumptions.*

(S1) $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $b \in \mathbb{Q}^m$.

(S2) $S := \{(x,y) \in P(A,G,b) | x \in (\mathbb{Z}_+)^n\}$.

Then there are $A' \in M(m,n,\mathbb{Q})$, $G' \in M(m,p,\mathbb{Q})$, $b' \in \mathbb{Q}^m$ such that

$$conv(S) = P(A', G', b')$$

By Proposition4.6 and Theorem4.7, MILP

$$\begin{aligned} &\max\ f(x,y) := c^t x + h^t y \\ &\text{subject to } (x,y) \in S \end{aligned}$$

is equal to a pure LP

$$\begin{aligned} &\max\ f(x,y) := c^t x + h^t y \\ &\text{subject to } (x,y) \in P(A', G', b') \end{aligned}$$

We set

$$\tilde{A} := \begin{pmatrix} A \\ A' \end{pmatrix}, \tilde{G} := \begin{pmatrix} G \\ G' \end{pmatrix}, \tilde{b} := \begin{pmatrix} b \\ b' \end{pmatrix},$$

Then clearly

$$S = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^p | (x,y) \in P(\tilde{A}, \tilde{G}, \tilde{b}), x \in \mathbb{Z}^n\}$$

and MILP

$$\begin{aligned} &\max\ f(x,y) := c^t x + h^t y \\ &\text{subject to } (x,y) \in S \end{aligned}$$

has a continuous ralaxation

$$\max f(x,y) := c^t x + h^t y$$
$$\text{subject to } (x,y) \in P(\tilde{A}, \tilde{G}, \tilde{b})$$

whose optimal value is equal to the one of the original MILP. And we can effectively find an optimal solution of this continuas ralaxation which is contained in S.

From the above discussion, the following can be shown.

Proposition 4.8. *Here are the settings and assumptions.*

(S1) $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

(S2) $S := \{(x,y) \in P(A,G,b) | x \in (\mathbb{Z}_+)^n\}$.

Then there is $M \in \mathbb{N}$ *and are* $\tilde{A} \in M(M,n,\mathbb{Q})$, $\tilde{G} \in M(M,p,\mathbb{Q})$, $\tilde{b} \in \mathbb{Q}^M$ *such that*

$$S = P(\tilde{A}, \tilde{G}, \tilde{b}) \cap \mathbb{Z}_+^n \times \mathbb{R}_+^p$$

and

$$conv(S) = P(\tilde{A}, \tilde{G}, \tilde{b})$$

4.2.2 Fourier elimination and Farkas Lemma

Definition 4.9 (Conic combination)**.** *Let* $v_1, ..., v_m \in \mathbb{R}^n$. *For every* $\lambda_1, ..., \lambda_m \geq 0$, *we call* $\sum_{i=1}^m \lambda_i v_i$ *a conic combination of* $v_1, ..., v_m$.

Theorem 4.10 (Fourier Elimination)**.** *Let*

(S1) $A \in M(m,n,\mathbb{R})$, $b \in \mathbb{R}^m$.

(S2) $I^+ := \{i | a_{i,n} > 0\}$, $I^- := \{i | a_{i,n} < 0\}$, $I^0 := \{i | a_{i,n} = 0\}$.

(S3) $a'_{i,k} := \dfrac{a_{i,k}}{|a_{i,n}|}$ $(i \in I^+ \cup I^-, k \in \{1,2,...,n-1\})$, $b'_i := \dfrac{b_i}{|a_{i,n}|}$ $(i \in I^+ \cup I^-)$.

(S4) $\tilde{A} := (A, b) \in M(m, n+1, \mathbb{R})$.

(S5) We set $\tilde{A}_{n-1} \in M(\#I^+ * \#I^- + \#I^0, n, \mathbb{R})$ *and* $b' \in \mathbb{R}^{(\#I^+ * \#I^- + \#I^0)}$ *by*

$$(kq\text{-th row of } \tilde{A}_{n-1}) = \frac{1}{|a_{k,n}|}(k\text{-th row of } \tilde{A}) + \frac{1}{|a_{q,n}|}(q\text{-th row of } \tilde{A}) \ (\forall k \in I^+, \forall q \in I^-)$$

and

$$((\#I^+ * \#I^- + j)\text{-th row of } \tilde{A}') = (j\text{-th row of } \tilde{A}) \ (j = 1,2,...,\#I^0)$$

(S6) $x^i := (x_1, ..., x_i)$ $(x \in \mathbb{R}^n)$

Then

(i) $Ax \leq b, x \in \mathbb{R}^n$ *is feasible if and only if*

$$\sum_{i=1}^{n-1} (a'_{k,i} + a'_{q,i}) x_i \leq b'_k + b'_q \ (\forall k \in I^+, \forall q \in I^-),$$
$$\sum_{i=1}^{n-1} a_{p,i} x_i \leq b_p \ (\forall p \in I^0)$$

(ii) If $A \in M(m,n,\mathbb{Q})$ *and* $b \in \mathbb{Q}^m$, *then* $a'_{k,i}, a'_{q,i}, b'_k, b'_q \in \mathbb{Q}$ $(\forall k \in I^+, \forall i \in \{1,2,...,n-1\}, \forall q \in I^-)$.

(iii) $\{x \in \mathbb{R}^n | Ax \leq b\} \neq \phi \iff \{x \in \mathbb{R}^{n+1} | \tilde{A}(x^t, -1)^t \leq 0\} \neq \phi \iff \{x \in \mathbb{R}^n | \tilde{A}_{n-1}((x^{n-1})^t, -1)^t \leq 0\} \neq \phi$.

(iv) For each $i \in \{0,1,...,n-1\}$, *there is* $m_i \in \mathbb{N}$ *and* $\tilde{A}_i \in M(m_i, i+1, \mathbb{R}$ *such that every row of* $\tilde{A}_i$ *is a conic combination of rows of* $\tilde{A}$ *and*

$$\{x \in \mathbb{R}^n | Ax \leq b\} \neq \phi \iff \{x \in \mathbb{R}^i | \tilde{A}_i((x^i)^t, -1)^t \leq 0\}$$

(v) If $\tilde{A} \in M(m, n+1, \mathbb{Q})$ *then* $\tilde{A}_i \in M(m_i, i+1, \mathbb{Q})$ $i \in \{0,1,...,n-1\}$.

(vi) $\{x \in \mathbb{R}^n | Ax \leq b\} \neq \phi \iff \tilde{A}_0 \leq 0.$

Proof of the 'only if' part in (i). Let us assume $x \in \mathbb{R}^n$ such that $Ax \leq b$. Then

$$\sum_{i=1}^{n-1} a'_{k,i} x_i + x_n \leq b'_k \ (\forall k \in I^+)$$

and

$$\sum_{i=1}^{n-1} a'_{q,i} x_i - x_n \leq b'_q \ (\forall q \in I^-)$$

So, by adding the left and right sides of these two inequalities, respectively, the following holds.

$$\sum_{i=1}^{n-1} (a'_{k,i} + a'_{q,i}) x_i \leq b'_k + b'_q \ (\forall k \in I^+, \forall q \in I^-),$$
$$\sum_{i=1}^{n-1} a_{p,i} x_i \leq b_p \ (\forall p \in I^0)$$

□

Proof of the 'if' part in (i). Let us assume

$$\sum_{i=1}^{n-1} (a'_{k,i} + a'_{q,i}) x_i \leq b'_k + b'_q \ (\forall k \in I^+, \forall q \in I^-),$$
$$\sum_{i=1}^{n-1} a_{p,i} x_i \leq b_p \ (\forall p \in I^0)$$

Then

$$\sum_{i=1}^{n-1} a'_{k,i} x_i - b'_k \leq -(\sum_{i=1}^{n-1} a'_{q,i} - b'_q) \ (\forall k \in I^+, \forall q \in I^-)$$

We set

$$x_n := \min\{-(\sum_{i=1}^{n-1} a'_{k,i} - b'_k) | k \in I^+\}$$

Then

$$x_n \geq \max\{(\sum_{i=1}^{n-1} a'_{q,i} - b'_q) | q \in I^-\}$$

So, $Ax \leq b$. □

Proof of (ii)-(iv). These are followed by (i). □

Theorem 4.11 (Farkas Lemma I)**.** *Let*

(S1) $A \in M(m, n, \mathbb{R})$, $b \in \mathbb{R}^m$.

Then

$$\{x \in \mathbb{R}^n | Ax \leq b\} = \phi \iff \{v \in \mathbb{R}^m | A^t v = 0, b^t v < 0, v \geq 0\} \neq \phi$$

Proof of 'only if' part. By Fourier elimination method (iv), there are $m_0 \in \mathbb{N}$ and $U \in M(m_0, n, \mathbb{R})$ such that $U \geq 0$ and $U\tilde{A} = (O_{m_i,n-1}, b^0)$ and $b^0 \not\geq 0$. Then there is $u \in \mathbb{R}^{m_0}$ such that $u^t b^0 < 0$. We set

$$v := (u^t U)^t$$

Then $v \geq 0$ and $Av = 0$ and $v^t b < 0$. □

Proof of 'if' part. Let us assume $\exists v \in \mathbb{R}^m$ such that $v^tA = 0$ and $v^tb < 0$ and $v \geq 0$. For any $x \in \mathbb{R}^n$, $v^tAx = 0$. So, $Ax \not\leq b$. □

Theorem 4.12 (Farkas Lemma II). *Let*

(S1) $A \in M(m, n, \mathbb{R})$, $b \in \mathbb{R}^m$.

Then

$$\{x \in \mathbb{R}^n | Ax = b, x \geq 0\} \neq \phi \iff \{u \in \mathbb{R}^m | A^tu \leq 0\} \subset \{u \in \mathbb{R}^m | u^tb \leq 0\}$$

Proof of ' $\Longrightarrow$ *'.* Let us fix $x \in \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$. Let us fix any $u \in \{u \in \mathbb{R}^m | A^tu \leq 0\}$. So, $b^tu \leq 0$. □

Proof of ' $\Longleftarrow$ *'.* Let us assume

$$\{x \in \mathbb{R}^n | Ax = b, x \geq 0\} = \phi$$

Then

$$\{x \in \mathbb{R}^n | Ax \leq b, -Ax \leq -b, x \geq 0\} = \{x \in \mathbb{R}^n | Bx \leq c\} = \phi$$

Here,

$$B := \begin{pmatrix} A \\ -A \\ -I_n \end{pmatrix}, c := \begin{pmatrix} b \\ -b \\ O_{n,1} \end{pmatrix}$$

and I_n is the n-th unit matrix. By Farkas Lemma I, there are $v \in \mathbb{R}^m_+$ and $v' \in \mathbb{R}^m_+$ and $w \in \mathbb{R}^n_+$ such that

$$B^t \begin{pmatrix} v \\ v' \\ w \end{pmatrix} = 0, \begin{pmatrix} v \\ v' \\ w \end{pmatrix}^t c < 0$$

This implies

$$A(-(v - v')) = -w, -(v - v')^tb > 0$$

We set $u := -(v - v')$. Then

$$u \in \{u \in \mathbb{R}^m | A^tu \leq 0\} \setminus \{u \in \mathbb{R}^m | u^tb \leq 0\}$$

□

4.2.3 Polyhedron and Minkowski Weyl Theorem

Definition 4.13 (Polytope). *We say $A \subset \mathbb{R}^n$ is a polytope if there are finite vectors $v_1, ..., v_m \in \mathbb{R}^n$ such that $A = conv(v_1, ..., v_m)$. We call $v_1, ..., v_m$ vertices of A. If $v_1, ..., v_m \in \mathbb{Q}^n$, we call A is a rational polytope.*

Definition 4.14 (Cone). *We say $C \subset \mathbb{R}^n$ is a cone if $0 \in C$ and for every $x \in C$ and $\lambda \in \mathbb{R}_+$ $\lambda x \in C$.*

By the definition of cone, the following holds.

Proposition 4.15. *Any cone containing nonzero vector is not bounded.*

Definition 4.16 (Convex Cone). *We say $C \subset \mathbb{R}^n$ is a convex cone if C is cone and every conic combination of finite vectors of C is contained in C.*

Because every intersection of convex cones is also convex cone, the following holds.

Proposition 4.17 (Convex Cone generated by a set). *Let us assume A is any subset of $\mathbb{R}^n$. Then there is the minimum convex cone containing A. We denote this convex cone by $cone(A)$.*

Definition 4.18 (Polyhedral cone). *Let*

(S1) $A \in M(m, n, \mathbb{Q})$.

We call

$$P := \{x \in \mathbb{R}^n | Ax \leq 0\}$$

a Polyhedral cone.

Theorem 4.19 (Minkowski Weyl Theorem for cones). *Let*

(S1) $C \subset \mathbb{R}^n$.

Then C is a Polyhedral cone if and only if C is finite generated cone.

STEP1. Proof of 'if' part. Let us assume C is finite generated cone. Then there is $r_1, ..., r_k \in \mathbb{R}$ such that $C = cone(r_1, ..., r_k)$. We set $R = (r_1, ..., r_k)$.

By applying Fourier elimination method k times to the the following inequality

$$-\mu \leq 0, R\mu \leq x, -R\mu \leq -x$$

and Fourier elimination method (vi), there is $A \in M(m, n, \mathbb{R})$ such that the above inequality is equivalent to

$$Ax \leq 0$$

So, $C = \{x \in \mathbb{R}^n | Ax \leq 0\}$.

□

STEP2. Proof of 'only if' part. Let us assume C is a Polyhedral cone. So, there is $A \in M(m, n, \mathbb{R})$ such that $C = \{x \in \mathbb{R}^n | Ax \leq 0\}$. We set $C^* := \{y \in \mathbb{R}^n | \exists \nu \in \mathbb{R}^m_+ \text{ such that } A^t v = y\}$. Then

$$C^* = cone(a^1, ..., a^m)$$

Here, $a^i \in \mathbb{R}^n$ is the i-th row vector of A $(i = 1, 2, ..., m)$. By STEP1, there is $R \in M(n, k, \mathbb{R})$ such that

$$C^* = \{y \in \mathbb{R}^n | R^t y \leq 0\}$$

We denote the i-th column vector of R by r^i $(i = 1, 2, ..., k)$. We will show

$$C = cone(r_1, ..., r_k)$$

Let us fix any $x \in cone(r_1, ..., r_k)$. Then there are $\nu_1, ..., \nu_k \in \mathbb{R}_+$ such that $x = R\nu$. Because $a_i = A^t e_i$ $(i = 1, 2, ..., m)$, $a_i \in C^*$ $(i = 1, 2, ..., m)$. So, $AR \leq 0$. This implies $Ax = AR\nu \leq 0$. This means $x \in C$. We have shown $cone(r_1, ..., r_k) \subset C$.

Let us fix any $\bar{x} \in cone(r_1, ..., r_k)^c$. So, $\{\nu \in \mathbb{R}^k | R\nu = \bar{x}, \nu \geq 0\} = \phi$. By Farkas Lemma II, there is $y \in \mathbb{R}^n$ such that $R^t y \leq 0$ and $y^t \bar{x} > 0$. So, $y \in C^*$. Then there are $\nu \in \mathbb{R}^m_+$ such that $y = A^t \nu$. So, $\nu^t A\bar{x} > 0$. Because $\nu \in \mathbb{R}^m_+$, this implies $A\bar{x} \not\leq 0$. This means $\bar{x} \in C^c$. Consequently $C \subset cone(r_1, ..., r_k)$. □

Definition 4.20 (Minkowski sum)**.** *Let $A, B \subset \mathbb{R}^n$. We call*

$$A + B$$

the Minkowski sum of A and B.

Proposition 4.21. *Let*

(i) Minkowski sum of any two convex set is convex.

(ii) For any two subset $A, B \subset \mathbb{R}^n$,

$$conv(A + B) = conv(A) + conv(B)$$

Proof of (i). Let $A, B \subset \mathbb{R}^n$ be convex. For any $a_1, ..., a_m \in A$ and $b_1, ..., b_m \in B$ and $\lambda_1, ..., \lambda_m \subset [0, 1]$ such that $\sum_{i=1}^m \lambda_i = 1$,

$$\sum_{i=1}^m \lambda_i(a_i + b_i) = \sum_{i=1}^m \lambda_i a_i + \sum_{i=1}^m \lambda_i b_i \in A + B$$

So, $A + B$ is convex. □

Proof of (ii). By (i), $conv(A)+conv(B)$ is convex. And $A+B \subset conv(A)+conv(B)$. So, $conv(A+B) \subset conv(A)+conv(B)$. Let us fix any $a_1, .., a_k \in A$ and $b_a, ..., b_l \in B$ and $\lambda_1, ..., \lambda_k, \mu_1, ..., \mu_l \in [0, 1]$ such that $\sum_{i=1}^k \lambda_i = 1$ and $\sum_{i=1}^l \mu_i = 1$. Then

$$\sum_{i=1}^k \lambda_i a_i + \sum_{j=1}^l \mu_j b_j = \sum_{j=1}^l \mu_j (\sum_{i=1}^k \lambda_i a_i + b_j) = \sum_{j=1}^l \mu_j (\sum_{i=1}^k \lambda_i (a_i + b_j)) = \sum_{i,j} \lambda_i \mu_j (a_i + b_j) \in conv(A + B)$$

□

Theorem 4.22 (Minkowski-Weyl Theorem)**.** *A subset $P \subset \mathbb{R}^n$ is a Polyhedron if and only if there is a polytope Q a finite generated cone C such that*

$$P = Q + C$$

We call the right side $\mathcal{V}$-representation and call P a $\mathcal{V}$-polyhedron.

Proof of 'only if' part. Let us fix $A \in M(m, n, \mathbb{R})$ and $b \in \mathbb{R}^m$ such that $P = \{x \in \mathbb{R}^n | Ax \leq b\}$. We set

$$C_P := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} | Ax - yb \leq 0, y \leq 0\}$$

Then clearly

$$P = \{x \in \mathbb{R}^n | (x, 1) \in C_P\}$$

By Minkowski Weyl Theorem for cones, there are $r^1, r^2, ..., r^K \mathbb{R}^{n+1}$ such that

$$C_P := cone(r^1, r^2, ..., r^K)$$

Because C_P is a cone, we can assume $r^i_{n+1} = 0$ or 1 $(\forall i)$. So, there are $u_1, ..., u_k \in \mathbb{R}^n$ and $v_1, ..., v_l \in \mathbb{R}^n$ such that

$$C_P = cone(\begin{pmatrix} u_1 \\ 1 \end{pmatrix}, ... \begin{pmatrix} u_k \\ 1 \end{pmatrix}, \begin{pmatrix} v_1 \\ 0 \end{pmatrix}, ..., \begin{pmatrix} v_l \\ 0 \end{pmatrix})$$

So,

$$P = conv(u^1, ..., u^k) + cone(v^1, ..., v^l)$$

□

Proof of 'if' part. We assume we can get

$$P = conv(u^1, ..., u^k) + cone(v^1, ..., v^l)$$

Then

$$P = cone(\begin{pmatrix} u^1 \\ 1 \end{pmatrix}, ..., \begin{pmatrix} u^k \\ 1 \end{pmatrix}, \begin{pmatrix} v^1 \\ 0 \end{pmatrix}, ..., \begin{pmatrix} v^l \\ 0 \end{pmatrix}) \cap \mathbb{R}^n \times \{1\}$$

Because $cone(\begin{pmatrix} u^1 \\ 1 \end{pmatrix}, ..., \begin{pmatrix} u^k \\ 1 \end{pmatrix}, \begin{pmatrix} v^1 \\ 0 \end{pmatrix}, ..., \begin{pmatrix} v^l \\ 0 \end{pmatrix})$ is a Polyhedral cone, P is a Polyhedron. □

Proof of the last part. □

Proposition 4.23. *Let*

(i) Bounden Polyhedron is polytone.

(ii) If $A \in M(m, n, \mathbb{Q})$ and $b \in \mathbb{Q}^m$, then there are $v_1, ..., v_k \in \mathbb{Q}^n$ and $r_1, ..., r_l \in \mathbb{Z}^n$ such that

$$P := \{x \in \mathbb{R}^n | Ax \leq b\} = conv(v_1, ..., v_k) + cone(r_1, ..., r_l)$$

If P is bounded, P is a rational polytope.

(iii) $P \subset \mathbb{R}^n$ is a rational polyhedron if and only if P is a minkowski sum of a rational polytope and a convex cone generated by finite rational vectors.

Proof of (i). By Proposition4.15, (i) holds. □

Proof of (ii). By the proof of Theorem4.19, (ii) holds. □

Proof of (iii). By the proof of Theorem4.19, (iii) holds. □

4.2.4 Perfect formulation and Meyer's Foundamental theorem

Proposition 4.24. *Here are the settings and assumptions.*

(S1) $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

(S2) $S := \{(x,y) \in (\mathbb{Z}_+)^n \times (\mathbb{R}_+)^p | g(x,y) := Ax + Gy \leq b\}$

Then

(i)

$$sup\{c^t x + h^t y | (x,y) \in S\} = sup\{c^t x + h^t y | (x,y) \in conv(S)\}$$

Furthermore, there is $(x,y) \in S$ *such that* $c^t x + h^t y = sup\{c^t x + h^t y|(x,y) \in S\} \iff$ *there is* $(x,y) \in conv(S)$ *such that* $c^t x + h^t y = sup\{c^t x + h^t y|(x,y) \in S\}$

(ii) $ex(conv(S)) \subset S$

Proof of the first part of (i). Because $S \subset conv(S)$,

$$sup\{c^t x + h^t y | (x,y) \in S\} \leq sup\{c^t x + h^t y | (x,y) \in conv(S)\}$$

We can assume $z^* = sup\{c^t x + h^t y|(x,y) \in S\} < \infty$. Let us set $H := \{(x,y) \in \mathbb{R}^{n+p} | c^t x + h^t y \leq z^*\}$. Because H is convex and $S \subset H$, $conv(S) \subset H$. So,

$$sup\{c^t x + h^t y | (x,y) \in S\} \geq sup\{c^t x + h^t y | (x,y) \in conv(S)\}$$

□

Proof of the last part of (i). The part of $\implies$ is clear. We set $d := (c,h)$. Let us assume there is $\bar{z} = (\bar{x},\bar{y})$ such that $d^t\bar{z} = sup\{c^t x + h^t y|(x,y) \in conv(S)\}$. Then there are $\lambda_1, ..., \lambda_k > 0$ and $z_1, ..., z_k \in S$ such that $\bar{z} = \sum_{i=1}^k \lambda_i z_i$. Clearly $d^t z_i \leq d^t \bar{z}$ $(\forall i)$. Because $d^t\bar{z} = \sum_{i=1}^k d^t \lambda_i z_i$, there is i such that $d^t z_i \geq d^t \bar{z}$. So, $d^t z_i = sup\{c^t x + h^t y|(x,y) \in conv(S)\}$. □

Proof of (ii). Let us fix any $v \in ex(conv(S))$. Because $ex(conv(S)) \subset conv(S)$, there are $\lambda_1, ..., \lambda_m \in (0,1]$ and $v_1, ..., v_m \in S$ such that $v = \sum_{i=1} \lambda_i v^i$. We can assume $m > 1$. We set $v' := \sum_{i=2}^m \dfrac{\lambda_i}{1-\lambda_1} v^i$. Then $v' \in conv(S)$. Because $v = \lambda_1 v_1 + (1-\lambda_1)v'$ and $v \in ex(conv(S))$, $v = v_1 \in S$. □

Proposition 4.25. *Let* $r^1, ..., r^K \in \mathbb{R}^n$. *Then*

$$conv(\sum_{i=1}^K \mathbb{Z}_+ r^i) = cone(r_1, ..., r^K)$$

Proof. We will show this by Mathematical induction. If $K = 1$, then this proposition holds. Let us fix any $k \in \mathbb{N}$ and assume this proposition holds for every $K \leq k$.

We set $C := conv(\sum_{i=1}^{k+1} \mathbb{Z}_+ r^i)$. Clearly $C \subset cone(r^1, ..., r^{k+1})$. Let us fix $x \in cone(r^1, ..., r^{k+1})$. Then there are $\mu_1, ..., \mu_{k+1} \in \mathbb{R}_+$ such that $x = \sum_{i=1}^{k+1} \mu_i r^i$. We can assume $\mu_{k+1} > 0$. We set $\lambda := \dfrac{2\mu_{k+1}}{\lceil 2\mu_{k+1} \rceil}$. Because $0 \in C$, $2\mu_{k+1} r^{k+1} = (1-\lambda)0 + \lambda \lceil 2\mu_{k+1} \rceil r^{k+1} \in C$. By Mathematical induction assumption, $\sum_{i=1}^k 2\mu_i r^i \in C$. So,

$$\sum_{i=1}^{k+1} \mu_i r^i = \frac{1}{2}(2\mu_{k+1} r^{k+1} + \sum_{i=1}^k 2\mu_i r^i) \in C$$

So, $cone(r^1, ..., r^{k+1}) \subset C$. □

Theorem 4.26 (Meyer(1974)[9] Fundamental Theorem)**.** *Here are the settings and assumptions.*

(S1) $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

(S2) $S := \{(x,y) \in P(A,G,b) | x \in \mathbb{Z}^n\}$.

Then there are $A' \in M(m,n,\mathbb{Q})$, $G' \in M(m,p,\mathbb{Q})$, $b' \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$ *such that*

$$conv(S) = P(A', G', b')$$

STEP1. Decomposition of S. We can assume $S \neq \phi$. Then by Proposition4.23, there are $v^1, ..., v^t \subset \mathbb{Q}^{n+p}$ and $r^1, ..., r^q \subset \mathbb{Z}^{n+p}$ such that

$$P := P(A, G, b) = conv(v^1, ..., v^t) + cone(r^1, ..., r^q)$$

We set

$$T := \{\sum_{i=1}^{s} \lambda_i v^i + \sum_{j=1}^{q} \mu_j r^j | 0 \leq \lambda_i, \mu_j \leq 1 \ (\forall i, j), \sum_{i=1}^{s} \lambda_i = 1\} = conv(v^1, ..., v^t) + \sum_{j=1}^{q} [0, 1] r_j$$

Then T is bounded. There is $M \in \mathbb{N}$ and $D \in M(M, n+p, \mathbb{Q})$ such that

$$T = \{z \in \mathbb{R}^{n+p} | \exists \lambda \in \mathbb{R}^n_+, \exists \mu \in \mathbb{R}^p_+ \text{ s.t. } D \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \leq z, \sum_{i=1}^{s} \lambda_i \leq 1, -\sum_{i=1}^{s} \lambda_i \leq -1, \mu \leq 1\}$$

By Fourier elimination method, there are $C \in M(M, n, \mathbb{R})$ and $d \in \mathbb{Q}^n$ such that $T = \{x \in \mathbb{R}^n | Cx \leq d\}$. So, by Proposition4.23, T is a rational polytope.

Let

$$T_I := \{(x, y) \in \mathbb{Z}^n \times \mathbb{R}^p | (x, y) \in T\}, R_I := \{\sum_{j=1}^{q} \mu_j r^j | \mu_j \in \mathbb{Z}_+ \ (\forall j)\}$$

We will show

$$S = T_I + R_I$$

Because $T_I + R_I \subset T$ and i-th component of $T_I + R_I$ is integer for every $i \in \{1, 2, ..., s\}$, $T_I + R_I \subset S$.

Let us fix any $(x, y) \in \mathbb{Z}^n \times \mathbb{R}^p$ such that $(x, y) \in S$. Then there are $\lambda_1, ..., \lambda_s, \mu_1, ..., \mu_q \in [0, 1]$ such that $\sum_{i=1}^{s} \lambda_i = 1$ and

$$(x, y) = \sum_{i=1}^{s} \lambda_i v^i + \sum_{j=1}^{q} \mu_j r^j$$

We set

$$(x', y') := \sum_{i=1}^{s} \lambda_i v^i + \sum_{j=1}^{q} (\mu_j - \lfloor \mu_j \rfloor) r^j, r := \sum_{j=1}^{q} \lfloor \mu_j \rfloor r^j$$

Then $(x', y') \in T_I$ and $r \in R_I$. So, $(x, y) \in T_I + R_I$. Consequently, $S = T_I + R_I$. □

STEP2. Proof that $conv(S)$ is a rational polyhedron. By Proposition4.21 and STEP1,

$$conv(S) = conv(T_I) + conv(R_I)$$

Because $conv(R_I) = conv(r^1, ..., r^q)$, by Proposition4.25, $conv(R_I)$ is a rational polyhedral cone. So, it is enough to show

$$conv(T_I) \text{ is a rational polytope}$$

Since T is bounded, $X := \{x \in \mathbb{Z}^n | \exists y \in \mathbb{R}^p \text{ such that } (x, y) \in T_I\}$ is bounded and so is a finite set.

For each $x \in X$, we set $T_x := \{(x, y) | \exists y \in \mathbb{R}^p \text{ such that } (x, y) \in T_I\}$. For any $\bar{x} \in X$,

$$T_{\bar{x}} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p | x = \bar{x} \text{ and } (x, y) \in T\}$$

Because T is a rational polytope, $T_{\bar{x}}$ is a rational polytope. We denote th set of all vertices of $T_{\bar{x}}$ by $V_{\bar{x}}$ for any $\bar{x} \in X$. We set $V := \cup_{x \in X} V_x$. V is a finite set. We will show

$$conv(T_I) = conv(V)$$

Because $T_I = \cup_{x \in X} T_x = \cup_{x \in X} conv(V_x) \subset conv(V)$, $conv(T_I) \subset conv(V)$. Because $V = \cup_{x \in X} V_x \subset \cup_{x \in X} conv(V_x) = \cup_{x \in X} T_x = conv(T_I)$, $conv(V) \subset conv(T_I)$. So, $conv(T_I) = conv(V)$. Consequently, $conv(T_I)$ is a rational polytope. □

By the proof of Theorem4.24, the following holds.

Theorem 4.27. *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $G \in M(m, p, \mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

(S2) $S := \{(x, y) \in P(A, G, b) | x \in \mathbb{Z}^n\}$.

Then there are

$$a_1, ..., a_k \in P(A, G, b) \cap \mathbb{Z}^n \times \mathbb{Q}^p = S$$

and

$$r_1, ..., r_l \in \mathbb{Z}^{n+p}$$

such that

$$conv(S) = conv(a_1, ..., a_k) + cone(r_1, ..., r_l)$$

4.2.5 Sharp MILP Formulation

Definition 4.28 (MILP Formulation). *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $G \in M(m, p, \mathbb{Q})$, $B \in M(m, t, \mathbb{Q})$, $b \in \mathbb{Q}^m$.

(S2) $S \subset \mathbb{Q}^n$.

(S3) $T(A, G, B, b) := \{(x, y, z) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Z}^t | Ax + Gy + Bz \leq b\}$.

We say (A, G, B, b) is a MILP formulation for S if and only if S is equal to the image of

$$p_n : T(A, G, B, b) \ni (x, y, z) \mapsto x \in \mathbb{Q}^n$$

Clearly the following holds.

Proposition 4.29. *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $G \in M(m, p, \mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

(S2) $S := \{(x, y) \in P(A, G, b) | x \in (\mathbb{Z}_+)^n\}$.

(S3) We set

$$\tilde{A} := \begin{pmatrix} A \\ E_n \\ O_{p,n} \\ O_{n,n} \end{pmatrix}, \tilde{G} := \begin{pmatrix} G \\ O_{n,p} \\ -E_p \\ O_{n,p} \end{pmatrix}, \tilde{B} := \begin{pmatrix} B \\ -E_n \\ O_{p,n} \\ -E_n \end{pmatrix}, \tilde{b} := \begin{pmatrix} b \\ 0_n \\ 0_p \\ 0_n \end{pmatrix}$$

Then $(\tilde{A}, \tilde{G}, \tilde{B}, \tilde{b})$ is a MILP formultation for S.

Definition 4.30 (Sharp MILP Formulation). *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $G \in M(m, p, \mathbb{Q})$, $B \in M(m, t, \mathbb{Q})$, $b \in \mathbb{Q}^m$.

(S2) $S \subset \mathbb{Q}^n$.

(Aq) (A, G, B, b) *is a MILP formulation for* S.

We say (A, G, B, b) is sharp MILP formulation for S if and only if $conv(S)$ is equal to the image of

$$p_n : \tilde{T}(A, G, B, b) \ni (x, y, z) \mapsto x \in \mathbb{Q}^n$$

Here, $\tilde{T}(A, G, B, b)$ is a LP relaxation of $T(A, G, B, b)$.

Theorem 4.31. *Here are the settings and assumptions.*

(S1) $S \subset \mathbb{Q}^n$.

(A1) There are $A \in M(m, n, \mathbb{Q})$, $G \in M(m, p, \mathbb{Q})$, $B \in M(m, t, \mathbb{Q})$, $b \in \mathbb{Q}^m$ such that (A, G, B, b) is a MILP formulation for S.

Then there there are $M \in \mathbb{N}$ and $\tilde{A} \in M(M, n, \mathbb{Q})$, $\tilde{G} \in M(M, p, \mathbb{Q})$, $\tilde{B} \in M(M, t, \mathbb{Q})$, $\tilde{b} \in \mathbb{Q}^M$ such that $(\tilde{A}, \tilde{G}, \tilde{B}, \tilde{b})$ is a sharp MILP formulation for S.

Proof. We set

$$T_I := \{(x, y, z) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Z}^t | Ax + Gy + Bz \leq b\}$$

and $p_1 : T_I \ni (x, y, z) \mapsto x \in \mathbb{Q}^n$. Because (A, G, B, b) is a MILP formulation for S,

$$p_1(T_I) = S$$

By Theorem4.2.4, there are $M \in \mathbb{N}$ and $\tilde{A} \in M(M, n, \mathbb{Q})$, $\tilde{G} \in M(M, p, \mathbb{Q})$, $\tilde{B} \in M(M, t, \mathbb{Q})$, $\tilde{b} \in \mathbb{Q}^M$ such that

$$T_I = \{(x, y, z) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Z}^t | \tilde{A}x + \tilde{G}y + \tilde{B}z \leq \tilde{b}\}$$

$$conv(T_I) = \{(x, y, z) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Q}^t | \tilde{A}x + \tilde{G}y + \tilde{B}z \leq \tilde{b}\}$$

Because $conv(S) = conv(p_1(T_I)) = p_1(conv(T_I))$,

$$conv(S) = p_1(conv(T_I))$$

So, $(\tilde{A}, \tilde{G}, \tilde{B}, \tilde{b})$ is a sharp MILP formulation for S. □

4.2.6 Review

Meyer theorem states that the convex hull of the feasible region of MILP is a rational polyhedron. So, the feasibility and the optimal value of MILP are equivalent to the feasibility and the optimal value of some LP, respectively. By methods such as simplex method, we can find this LP solution in extreme points of feasible reasion. By Proposition4.24, this extreme point is a solution of original MILP problem.

I think there are the following three ideas that are important in the proof of Meyer theorem.

1. Fourier elimination method
2. Expressing the feasible region of MILP or LP in terms of the Minkowski sum of bounded and unbounded parts
3. Going back and forth between integer and continuous parts of a polyhedron

Fourier elimination method plays an important role throughout this section. Fourier elimination method is a method of solving linear inequalities

$$Ax \leq b \tag{4.2.1}$$

focusing on the sign of the coefficients of a certain variable and using only non-negative multipliers to eliminate the variable. (4.2.1) corresponds to another two linear inequalities. If there is a solution of (4.2.1), then there is $U \in M(m_0, n, \mathbb{R})$ such that $U \geq 0$ and $UA = 0$ and

$$0 \leq Ub \tag{4.2.2}$$

By focusing on row vectors of U, if there is no solutions of (4.2.1), then there is $u \in \mathbb{R}^n_+$ such that

$$A^t u = 0, u^t b < 0, u \geq 0 \tag{4.2.3}$$

Correspondance between (4.2.1) and (4.2.3) is stated by Farkas Lemma.

For idea2 on LP feasible reasion P, we state this idea as Minkowski Weyl Theorem.

$$P = conv(v^1, ..., v^s) + cone(r^1, ..., r^q) \tag{4.2.4}$$

By increasing the dimension of the solution space of the simultaneous inequalities by one as follows, Minkowski Weyl Theorem is boil down to the case in P is a polyhedral cone.

$$P = \tilde{P} \cap \mathbb{R}^n \times \{1\}, \tilde{P} := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} | (A, -b) \begin{pmatrix} x \\ y \end{pmatrix} \leq 0\} \tag{4.2.5}$$

By Fourier elimination method and Farkas Lemma, any polyhedral cone is equivalent to finite generated convex cone.

Meyer theorem is the following.

Theorem 4.32. *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $G \in M(m, p, \mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

(S2) $S := \{(x, y) \in P(A, G, b) | x \in \mathbb{Z}^n\}$.

Then $conv(S)$ *is a rational polyhedron.*

In the proof of Meyer theorem, we focus on Polyhedron $P := P(A, G, b)$ which is containing S. By Minkowski Weyl Theorem, we get

$$P = conv(v^1, ..., v^s) + cone(r^1, ..., r^q)$$

We focus a bounded part of P

$$T = conv(v^1, ..., v^s) + \sum_{j=1}^{q} [0, 1] r_j$$

We denote a integer part of T by T_I and denote a integer part of $cone(r^1, ..., r^q)$ by R_I. Then we get

$$S = T_I + R_I$$

So,

$$conv(S) = conv(T_I) + conv(R_I)$$

Because $conv(T_I)$ is a rational polytope and $conv(R_I)$ is a rational polyhedral cone, $conv(S)$ is a rational polyhedron.

4.3 MILP formulation

4.3.1 Minimal formulation

Definition 4.33 (Implied equations, redundant inequalities, and facet)**.** *Here are the settings and assumptions.*

(S1) $A \in M(m,n,\mathbb{Q})$, $b \in \mathbb{Q}^m$, $P := \{x \in \mathbb{Q}^n | Ax \le b\}$. a_i *is the i-th row vector of A..*

Then

(i) *We say* $F \subset P$ *is a face of* P *if and only if* $F = \{x | a_i^T x = b_i \ (\forall i \in L)\}$ *for some* $L \subset \{1, 2, ..., m\}$.

(ii) *We say* $F \subset P$ *is a proper face of* P *if and only if* F *is a face and* $F \ne \phi$ *and* $F \ne P$.

(iii) *We say* $F \subset P$ *is a facet of* P *if and only if* F *is a proper face and maximum with respect to inclusion.*

(iv) *We say* $a_i x \le b_i$ $(i \in L)$ *is implied equations of* P *if and only if* $a_i x \le b_i$ $(\forall i \in L)$ *for any* $x \in P$.

(v) *We say* $a_i x \le b_i$ $(i \in L)$ *is facet defining inequalities of* P *if and only if* $F := \{x | a_i x \le b_i \ (\forall i \in L)\}$ *is a facet of* P.

(vi) *We say* $a_i x \le b_i$ $(i \in L)$ *is redundant inequalities of* P *if and only if there is a subset* $I \subset \{1, 2, ..., m\}$ *such that* $P = \{x | a_i x \le b_i \ (\forall i \in I \setminus L)\}$.

(vii) *We say* $L \subset \{1, 2, ..., m\}$ *is a minimal formulation of* P *if and only if* $P = \{x | a_i x \le b_i \ (\forall i \in L)\}$ *and there is no* iL *such that* $a_i \le b$ *is a redundant inequality of* P.

4.3.2 Locally ideal formulation

Proposition 4.34 (Standard equity form for LP)**.** *Here are the settings and assumptions.*

(S1) $A \in M(m,n,\mathbb{Q})$, $b \in \mathbb{Q}^m$.

(S2) $S := \{x \in \mathbb{Q}^n | Ax \le b\}$.

(S3) *We set for* $x \in S$,

$$\Phi(x) := (y^+, y^-, z)$$

Here,

$$y_i^+ := \max\{x_i, 0\} \ (i = 1, 2, ..., n)$$
$$y_i^- := \max\{-x_i, 0\} \ (i = 1, 2, ..., n)$$
$$z_j := (a_j, x) - b_j \ (j = 1, 2, ..., m)$$

(S4) $\tilde{S} := \{(y^+, y^-, z) \in \mathbb{Q}_+^n | A(y^+ - y^-) + z \le b\}$.

Then Φ *is a bijective from* S *to* $\tilde{S}$. *We call* $\tilde{S}$ *the standard equity form of* S. *We call each* z_j *a slack variable.*

Definition 4.35 (Basic feasible solution for LP.)**.** *Here are the settings and assumptions.*

(S1) $A \in M(m,n,\mathbb{Q})$, $b \in \mathbb{Q}^m$.

Then

(i) *For* $x \in \mathbb{Q}^n$, *we say* $\bar{x}$ *is a basic solution of* $Ax = b$ *if and only if* $\{a_i | a_i$ *is the i-th column of* A *and* $\bar{x}_i > 0\}$ *are linear independent.*

(ii) *For* $x \in \mathbb{Q}_+^n$, *we say* $\bar{x}$ *is a basic feasible solution of*

$$Ax = b, x \ge 0$$

if and only if x *is a basic solution of* $Ax = b$.

Proposition 4.36. *Here are the settings and assumptions.*

(S1) $A \in M(m,n,\mathbb{Q})$, $b \in \mathbb{Q}^m$.

(S2) x *is a solution of* $Ax \le b, x \ge 0$.

(S3) $z = (z_1, .., z_m)$ *are nonzero slack variables for* $Ax + z = b, x, z \ge 0$.

(S4) $I := \{i \in \{1, 2, ..., m\} | a_i^T x = b_i\}$. *Here* a_i *is the i-th row vector of A.*

(S5) $J := \{j \in \{1, 2, ..., n\} | x_j \ne 0\}$.

Then (x, z) is a basic feasible solution iff $\{\{a_{i,j}\}_{i \in I}\}_{j \in J}$ are linear independent.

Proof. We set $I' := \{i \in \{1, 2, ..., m\} | a_i^T x < b_i\}$. (x, z) is a basic feasible solution iff $\{a^j\}_{j \in J} \cup \{e_i\}_{i \in I'}$ are linear independent. Here a^j is the j-th column of A. This is equivalent to $\{a^j - \sum_{i \in I'} a_{i,j} e_i\}_{j \in J} \cup \{e_i\}_{i \in I'}$ are linear independent. So, (x, z) is a basic feasible solution iff $\{\{a_{i,j}\}_{i \in I}\}_{j \in J}$ are linear independent. □

Definition 4.37 (Locally ideal)**.** *Here are the settings and assumptions.*

- *(S1)* $A \in M(m, n, \mathbb{Q})$, $G \in M(m, p, \mathbb{Q})$, $B \in M(m, t, \mathbb{Q})$, $b \in \mathbb{Q}^m$.
- *(S2)* $S \subset \mathbb{Q}^n$.
- *(S3)* $T(A, G, B, b) := \{(x, y, z) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Z}^t | Ax + Gy + Bz \leq b\}$.
- *(S4)* $\tilde{S} := \{w \in \mathbb{Q}^M | Cw = c, w \geq 0\}$ *is a standard equity form of* S *and* Φ *is the bijection from* S *to* $\tilde{S}$ *in Proposition4.34.*

We say (A, G, B, b) is a locally ideal MILP formulation for S if and only if $\tilde{S}$ has at most one basic feasible solution and for any basic feasible solution of $\tilde{S}$ w, $\Phi^{-1}(w) \in \mathbb{Q}^{n+p} \times \mathbb{Z}^t$.

We will show an example of MILP formulation which is not locally ideal but sharp.

Example 4.38. *Here are the settings and assumptions.*

- *(S1)* $S = \cup_{i=1}^n P_i$. $P_i := \{x \in \mathbb{Q}^n || x_i| \leq 1, x_j = 0 \ (j \neq i)\}$ $(i = 1, 2, ..., n)$.

Then

- *(i) The following is a MILP formulation for S.*

$$y_j - 1 \leq x_i \leq 1 - y_j \ (i = 1, 2, .., n, j \neq i), \tag{4.3.1}$$
$$y_i \geq 0, \ (i = 1, 2, .., n), \tag{4.3.2}$$
$$\sum_{i=1}^n y_i = 1 \tag{4.3.3}$$
$$y \in \mathbb{Z}^n$$

- *(ii)* $conv(S) = \{x \in \mathbb{Q}^n | \sum_{i=1}^n |x_i| \leq 1\}$
- *(iii) Equalities and Inequalities in (i) and the following is a sharp MILP formulation for S.*

$$\sum_{i=1}^n r_i x_i \leq 1 \ (r \in \{-1, 1\}^n) \tag{4.3.4}$$

- *(iv) If $n = 3$, the formulation in (iii) is not locally ideal.*
- *(v) The following is a sharp and locally ideal MILP formulation for S.*

$$-y_i \leq x_i \leq y_i \ (i = 1, 2, .., n), \tag{4.3.5}$$
$$y_i \geq 0, \ (i = 1, 2, .., n), \tag{4.3.6}$$
$$\sum_{i=1}^n y_i = 1 \tag{4.3.7}$$
$$y \in \mathbb{Z}^n$$

Proof of (i). It is clear. □

Proof of (ii). The part of $\subset$ is clear. Let us fix any x in the right side. We take $s \geq 1$ such that $\sum_{i=1}^n s|x_i| = 1$. Then

$$x = \sum_{i=1}^n r|x_i| \frac{sign(x_i)}{r} e_i$$

So, $x \in conv(S)$. □

Proof of (iii). We set $T := \{(x, y) \in \mathbb{Q}^n \times \mathbb{Q}^n | (x, y)$ satisfies equalities and inequalities of (i)$\}$. Clearly $p_1(T) \subset conv(S)$. Clearly T is convex. Because $P_i \times \{e_i\} \subset T$ $(\forall i)$, $S \subset p_1(T)$. So, $conv(S) \subset T$. □

Proof of (iv). Clearly $x_1 = x_2 = y_1 = y_2 = \dfrac{1}{2}, x_3 = y_3 = 0$ is a feasible solution. We will show this is a basic feasible solution. By Proposition4.36, it is enough to show the column vectors of

$$\begin{array}{r|cccc} & x_1 & x_2 & y_1 & y_2 \\ x_1 \leq 1 - y_1 & 1 & 0 & 0 & 1 \\ x_2 \leq 1 - y_2 & 0 & 1 & 1 & 0 \\ y_1 + y_2 = 1 & 0 & 0 & 1 & 1 \\ x_1 + x_2 = 1 & 1 & 1 & 0 & 0 \end{array}$$

are linear independent. Because this matrix is nonsingular, the column vectors of this matrix are linear independent. □

Proof of (v). By the same argument as the proof of (iii), we can show this formulation is sharp. For locally ideal property, it is enough to show for any basic feasible solution (x^+, x^-, y, z) there is $\#\{i | y_i \neq 0\} = 1$. Because $\sum_{i=1}^n y_i = 1$, $\#\{i | y_i \neq 0\} \geq 1$. For aiming contradiction, let us assume $\#\{i | y_i \neq 0\} > 1$. So, there are $i_1 \neq i_2$ such that $y_{i_1}, y_{i_2} > 0$. We can assume $i_1 =, i_2 = 2$. We will show in each case of the followings.

case1 $|x_1| < y_1$ or $|x_2| < y_2$.

case2 $|x_1| = y_1$ and $|x_2| = y_2$.

In case1, we can assume $|x_1| < y_1$. If $|x_2| < y_2$, then By Proposition4.36, the clumns vectors of the following matrix are linear independent.

$$\begin{array}{c|cc} & y_1 & y_2 \\ * & 0 & 0 \\ ... & ... & .... \\ * & 0 & 0 \\ \sum_i y_i = 1 & 1 & 1 \end{array}$$

This is contradiction. So, $|x_{i_2}| = y_{i_2}$. By Proposition4.36, the clumns vectors of the following matrix are linear independent.

$$\begin{array}{c|ccc} & y_1 & y_2 & x_2^* \\ * & 0 & 0 & 0 \\ ... & ... & ... & 0 \\ * & 0 & 0 & 0 \\ q_2 y_2 + r_2 x_2 \leq 0 & 0 & q_2 & r_2 \\ \sum_i y_i = 1 & 1 & 1 & 0 \end{array}$$

Here, $q_2 r_2 \neq 0$. So, the clumns vectors of the following matrix are linear independent.

$$\begin{array}{c|ccc} & y_1 & y_2 & x_2^* \\ * & 0 & 0 & 0 \\ ... & ... & ... & 0 \\ * & 0 & 0 & 0 \\ q_2 y_2 + r_2 x_2 \leq 0 & 0 & 0 & r_2 \\ \sum_i y_i = 1 & 1 & 0 & 0 \end{array}$$

This is contradiction.

In case2, By Proposition4.36, the clumns vectors of the following matrix are linear independent.

$$\begin{array}{c|cccc} & y_1 & y_2 & x_1^* & x_2^* \\ * & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... \\ * & 0 & 0 & 0 & 0 \\ q_1 y_1 + r_1 x_1 \leq 0 & q_1 & 0 & r_1 & 0 \\ q_2 y_2 + r_2 x_2 \leq 0 & 0 & q_2 & 0 & r_2 \\ \sum_i y_i = 1 & 1 & 1 & 0 & 0 \end{array}$$

Here, $q_1 r_1 q_2 r_2 \neq 0$. So, the clumns vectors of the following matrix are linear independent.

$$\begin{array}{c|cccc} & y_1 & y_2 & x_1^* & x_2^* \\ * & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... \\ * & 0 & 0 & 0 & 0 \\ q_1 y_1 + r_1 x_1 \leq 0 & 0 & 0 & r_1 & 0 \\ q_2 y_2 + r_2 x_2 \leq 0 & 0 & 0 & 0 & r_2 \\ \sum_i y_i = 1 & 1 & 1 & 0 & 0 \end{array}$$

This is contradiction.

Consequently, $\#\{i|y_i \neq 0\} \leq 1$. □

Memo 4.39. *We measured execution times in three formulations in Example4.38. Here are the settings.*

Version of SCIP: SCIP9.0.0.0

Target Machine: Ubuntu Desktop 22.04

Host Machine: Windows10

CPU:Inter Core i7-6700T@2.8GHz

DRAM: 32GB

Here are the problem.

$$maximize \sum_{i=1} c_i x_i, x \in S$$

For $n = 1000$*, the execution times are below.*

$$\text{Simple formulation(i)} : 119sec$$
$$\text{Locally ideal formulation(v)} : 0.028sec$$

For sharp formulation(iii) and $n = 17$*, the execution time is 229sec. See [10] for sample code.*

Proposition 4.40. *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $b \in \mathbb{Q}^m$.

(S2) $S := \{x \in \mathbb{Q}^{n_1} \times \mathbb{Q}^{n_2} | Ax \leq b\}$.

(S3) $T(A, G, B, b) := \{(x, y, z) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Z}^t | Ax + Gy + Bz \leq b\}$.

(S4) $\tilde{S} := \{w \in \mathbb{Q}^M | Cw = c, w \geq 0\}$ *is a standard equity form of* S *and* Φ *is the bijection from* S *to* $\tilde{S}$ *in Proposition4.34.*

Then

(i) For any $x \in ex(S)$, $\Phi(x)$ *is a basic feasible solution.*

(ii) For any $x \in S \setminus \{0\}$ *such that* $\Phi(x)$ *is a basic feasible solution,* $\Phi(x) \in ex(\tilde{S})$.

(iii) Let us assume $S \subset [0, \infty)^{n_1+n_2}$*. Then for any* $x \in S$ *such that* $\Phi(x)$ *is a basic feasible solution,* $x \in ex(S)$.

Proof of (i). Let z is a slack variable for $Ax \leq b$. $I_n := \{i|x_i \neq 0\}$. $J_0 := \{j|z_j = 0\}$. If $J_0 = \phi$, then $Ax < b$. So, there is $x^1, x^2 \in \mathbb{Q}^n$ $t \in (0, 1)$ such that $Ax^1 Ax^2 < b$ and $x = tx^1 + (1-t)x^2$. This is contradiction. So, $J_0 \neq \phi$. If $x = 0$, $\Phi(x)$ is clearly basic feasible solution. So, we can assume $x \neq 0$. It is enough to show $\{\{a_{i,j}\}_{i \in I_n} | j \in J_0\}$ are linear independent. For aiming contradiction, let us assume $\{\{a_{i,j}\}_{i \in I_n} | j \in J_0\}$ are linear dependent. We set $k := \#I_n$ and

$$A' := \{a_{i,j}\}_{i \in I_n, j \in \{1,2,...,n\}}, A'' := \{a_{i,j}\}_{i \notin I_n, j \in \{1,2,...,n\}}, b' := \{b_i\}_{i \in I_n}, b'' := \{b_i\}_{i \notin I_n}$$

Then there is a $c \in \mathbb{Q}^n \setminus \{0\}$ such that

$$c_i = 0 \ (\forall i \notin I_n), x + sc \text{ is a solution of } A'x = b' \ (\forall s \in \mathbb{R})$$

Because $A''x < b''$, there is $s > 0$ such that $A''(x + sc) < b''$ and $A''(x - sc) < b''$. This means that $x \notin ex(S)$. This is contradiction. □

Proof of (ii). Let us fix any $x \in S \setminus \{0\}$ such that $\Phi(x)$ is a basic feasible solution. We can assume $|x_1|, .., |x_k| > 0, x_{k+1} = ... = x_n = 0$ and

$$a_{1,1}x_1 + ... + a_{1,k}x_k = b_1$$
$$...$$
$$a_{l,1}x_1 + ... + a_{l,k}x_k = b_l$$

and

$$rank \begin{pmatrix} a_{1,1} & ... & a_{1,k} \\ ... & ... & ... \\ a_{l,1} & ... & a_{l,k} \end{pmatrix} = k$$

So, the equation

$$a_{1,1}\bar{x}_1 + ... + a_{1,k}\bar{x}_k = b_1$$
$$...$$
$$a_{l,1}\bar{x}_1 + ... + a_{l,k}\bar{x}_k = b_l$$

has the unique solution.

For aiming contradiction, let us assume that $\Phi(x) \notin ex(S)$. Then there are $x^1 := (x^{1,+}, x^{1,-}, z^1)$ and $x^2 := (x^{2,+}, x^{2,-}, z^2)$ and $t \in (0,1)$ such that $x = tx^1 + (1-t)x^2$. So, $x_i^{j,+} = x_i^{j,-} = 0$ $(\forall i > k, j = 1,2)$ and $x_i^{j,+} = \delta_{+,sign(x_i)} x_i^{j,+}$ $(\forall i \leq k, j = 1,2)$ and $x_i^{j,-} = \delta_{+,sign(x_i)} x_i^{j,-}$ $(\forall i \leq k, j = 1,2)$. This implies $(x_1^{1,sign(x_1)}, ..., x_k^{1,sign(x_1)})$ and $(x_1^{2,sign(x_2)}, ..., x_k^{2,sign(x_2)})$ satisfies the equation

$$a_{1,1}\bar{x}_1 + ... + a_{1,k}\bar{x}_k = b_1$$
$$...$$
$$a_{l,1}\bar{x}_1 + ... + a_{l,k}\bar{x}_k = b_l$$

This is contradiction. □

Proof of (iii). (iii) is followed by the same argument of the proof of (ii). □

Definition 4.41 (Affine combination, Affine independent)**.**

(i) For $x_1, ..., x_m \in \mathbb{Q}^n$,

$$\sum_{i=1}^{m} \lambda_i x_i, \lambda_1, ..., \lambda_m \in \mathbb{Q}, \sum_{i=1}^{m} \lambda_i = 1$$

is called an affine combination of $x_1, ..., x_m$.

(ii) We say $x_1, ..., x_m \in \mathbb{Q}^n$ *are affinely independent if for any* $\lambda_1, ..., \lambda_m \in \mathbb{Q}$ *such that* $\sum_{i=1}^{m} \lambda_i = 0$ *and* $\sum_{i=1}^{m} \lambda_i x_i = 0$, $\lambda = 0$.

Definition 4.42 (Dimension)**.** *For* $S \subset \mathbb{R}^n$,

$$dim(S) := \max\{\#A | A \textit{ is a finite subset of } S \textit{ and } A \textit{ is affinely independent}\} - 1$$

Definition 4.43 (Pointed set)**.** *We say convex subset* $S \subset \mathbb{R}^n$ *is pointed if and only if* $ex(S) \neq \phi$.

Proposition 4.44. *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $b \in \mathbb{Q}^n$.

(S2) $P := \{x \in \mathbb{Q}_+^n | Ax \leq b\}$.

(A1) $P \neq \phi$.

Then P *is pointed.*

Proof. For $y \in P$, We set

$$N(y) := \#\{i | y_i \neq 0\}, M(y) := \#\{j | a_j^T y = b_j\}$$

Here, a_j is the j-th row vector of A. We set

$$K := \max\{N(y) | y \in P\}$$

If $K = n$, clearly $0 \in ex(P)$. So, we can assume $K < n$. We set

$$L := \max\{M(y) | y \in P, N(y) = K\}$$

Because $K < n$, $L > 0$. There is $x \in P$ such that $N(x) = K, M(x) = L$. We set $k := n - K$. We can assume $x_1, ..., x_k > 0, x_{k+1} = ... = x_n = 0$ and

$$A'x' = b'$$

Here, $A' := \{a_{i,j}\}_{i=1,...,L,j=1,2,...,n}$, $x' := (x_1, ..., x_k)$, $b' := (b_1, ..., b_L)^T$. For aiming contradiction, let us assume $rank(A') < n$. there is $r' \in \mathbb{Q}^k$ such that

$$A'(x' + tr) = b' \ (\forall t \in \mathbb{R})$$

So, there is $y \in P$ such that $N(y) > N(x)$ or $M(x) < M(y)$. This is contradiction. So, $rank(A') = n$. From this, $x \in ex(P)$. □

Definition 4.45 (Edge). *Let P be a nonempty polyhedron in $\mathbb{R}^n$. We call a face of P whose dimension is 1 an edge of P.*

Proposition 4.46. *Here are the settings and assumptions.*

- *(S1)* $a_1, ..., a_k \mathbb{Q}^n$.
- *(A1)* *For any i, $a_i \notin conv(a_1, ..., a_{i-1}, a_{i+1}, ..., a_k)$.*

Then

$$ex(conv(a_1, ..., a_k)) = \{a_1, ..., a_k\}$$

Proof. By Proposition4.24, it is enogh to show *supset* part. Let us assume there is i such that $a_i \notin ex(conv(a_1, ..., a_k))$. We can assume $i = k$. So there are $\lambda_1, ..., \lambda_k, \eta_1, ..., \eta_k, t \in (0, 1)$ such that $\sum_{i=1}^k \lambda_i = 1$ and $\sum_{i=1}^k \eta_i = 1$ and $a_k = t\sum_{i=1}^k \lambda_i a_i + (1-t)\sum_{i=1}^k \eta_i a_i$ and $\sum_{i=1}^k \lambda_i a_i \neq a_k$ and $\sum_{i=1}^k \eta_i a_i \neq a_k$. So, $t\lambda_k + (1-t)\eta_k < 1$. This implies

$$a_k = \sum_{i=1}^{k-1} \frac{t\lambda_i + (1-t)\eta_i}{1 - t\lambda_k - (1-t)\eta_k} a_i$$

So, $a_k \in conv(a_1, ..., a_{k-1})$. This is contradiction. □

Proposition 4.47. *Here are the settings and assumptions.*

- *(S1)* $P \subset \mathbb{R}^n$ *is a Poryhedron.*
- *(S2)* $a_1, ..., a_k, r_1, ..., r_l \mathbb{Q}^n$ *such that* $P = conv(a_1, ..., a_k) + cone(r_1, ..., r_l)$.
- *(A1)* *For any i, $a_i \notin conv(a_1, ..., a_{i-1}, a_{i+1}, ..., a_k)$.*

Then

$$ex(P) \subset \{a_1, ..., a_k\}$$

Proof. By Proposition4.24, it is enough to show $ex(P) \subset conv(a_1, ..., a_k)$. Let us fix any $x \in P \setminus conv(a_1, ..., a_k)$. There are $y \in conv(a_1, ..., a_k)$ and $z \in cone(r_1, ..., r_l) \setminus \{0\}$ such that $x = y + z$. Because $2z, 0 \in cone(r_1, ..., r_l)$, $y, y + 2z \in P$. So, $z = \frac{1}{2}(y + y + 2z)$. This means $x \notin ex(P)$. Consequently, $ex(P) \subset conv(a_1, ..., a_k)$. □

Proposition 4.48. *Here are the settings and assumptions.*

- *(S1)* $P \subset \mathbb{R}^n$ *is a Poryhedron.*
- *(A1)* P *is pointed.*

Then there are $a_1, ..., a_k, r_1, ...r_l \in \mathbb{Q}^n$

$$P = conv(a_1, ..., a_k) + cone(r_1, ..., r_l), ex(P) = \{a_1, ..., a_k\}, 0 \notin cone(r_1, ..., r_l)$$

Proof. By Minkowski-Weyl Theorem, Then there are $a_1, ..., a_k, r_1, ...r_l \in \mathbb{Q}^n$

$$P = conv(a_1, ..., a_k) + cone(r_1, ..., r_l)$$

For aiming contradiction, let us assume $0 \notin ex(cone(r_1, ..., r_l))$. The there are $z_1 \neq z_2 \in cone(r_1, ..., r_l)$ such that $0 = \frac{1}{2}(z_1 + z_2)$. For any i,

$$a_i = \frac{1}{2}((a_i + z_1) + (a_i + z_2))$$

This implies $a_i \notin ex(P)$. By Proposition4.47, $ex(P) = \phi$. This is contradiction. So, $0 \in ex(cone(r_1, ..., r_l))$.

By dropping some elements if necessary, we can assume that for each i

$$a_i \notin conv(a_1, ..., a_{i-1}, a_{i+1}, ..., a_k) + cone(r_1, ..., r_l)$$

For aiming contradiction, let us assume $a_i \notin ex(P)$. We can assume $i = k$. Then there are $y_1, y_2 \in conv(a_1, ..., a_k)$ and $z_1, z_2 \in cone(r_1, ..., r_l)$, $t \in (0,1)$ such that

$$a_k = t(y_1 + z_1) + (1-t)(y_2 + z_2), y_1 + z_1 \neq a_k, y_2 + z_2 \neq a_k$$

There are $\lambda_1, ..., \lambda_k, \eta_1, ..., \eta_k, t \in (0,1)$ such that $\sum_{i=1}^k \lambda_i = 1$ and $\sum_{i=1}^k \eta_i = 1$ and $y_1 = \sum_{i=1}^k \lambda_i a_i$ and $y_2 = \sum_{i=1}^k \eta_i a_i$. If $y_1 = y_2 = a_k$, $0 = tz_1 + (1-t)z_2$. This contradicts with $0 \notin ex(P)$. So, $y_1 \neq a_k$ or $y_2 \neq a_k$. This implies $t\lambda_k + (1-t)\eta_k < 1$. So,

$$a_k = \frac{1}{1 - t\lambda_k - (1-t)\eta_k}(\sum_{i=1}^{k}(t\lambda_i + (1-t)\eta_i)a_i + tz_1 + (1-t)z_2)$$

This means $a_k \in conv(a_1, ..., a_{k-1}) + cone(r_1, ..., r_l)$. This is contradiction. □

Proposition 4.49. *Here are the settings and assumptions.*

(S1) $S \subset \mathbb{Q}^n$.

(A1) (A, B, D, b) *is a locally ideal MILP formulation for* S.

(S2) We set

$$P := \{(x, u, y) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Q}^t | Ax + Bu + Dy \leq b\}, P_I := \{(x, u, y) \in P | y \in \mathbb{Z}^t\}$$
$$p : P \ni (x, u, y) \mapsto x \in \mathbb{Q}^n$$

(A2) P *is pointed.*

then (A, B, D, b) *is a sharp formulation for* S.

Proof. By Proposition4.48, there are $a_1, ..., a_k, r_1, ...r_l \in \mathbb{Q}^n$

$$P = conv(a_1, ..., a_k) + cone(r_1, ..., r_l), ex(P) = \{a_1, ..., a_k\}, 0 \notin cone(r_1, ..., r_l),$$
$$a_i = (\hat{x}^i, \hat{u}^i, \hat{y}^i)\ (i = 1, 2, ..., k), r_j = (\tilde{x}^j, \tilde{u}^j, \tilde{y}^j) \in \mathbb{Z}^n \times \mathbb{Z}^s \times \mathbb{Z}^t\ (j = 1, 2, ..., l)$$

By the assumption of locally idealness and Proposition4.40, $\hat{y}^i \in \mathbb{Z}^t$ $(\forall i)$.

Let us fix any $(x, u, y) \in P$. There are $\lambda_1, ..., \lambda_k, \eta_1, ..., \eta_k \in [0, 1) \in \mathbb{Q}$ such that $\sum_{i=1}^k \lambda_i = 1$ and $\sum_{i=1}^k \eta_i = 1$ and

$$(x, u, y) = \sum_{i=1}^{k} \lambda_i(\hat{x}^i, \hat{u}^i, \hat{y}^i) + \sum_{j=1}^{l} \eta_j(\tilde{x}^j, \tilde{u}^j, \tilde{y}^j)$$

We set

$$(x^1, u^1, y^1) := \sum_{i=1}^{k} \lambda_i(\hat{x}^i, \hat{u}^i, \hat{y}^i)$$

Because $y^1 \in \mathbb{Z}^t$ and (A, B, D, b) is a MILP formulation for S, $x^1 \in S$. Without loss of generality, we can assume $\lambda_1 > 0$. There is $\alpha \in \mathbb{Z} \cap (1, \infty)$ such that $\frac{\alpha}{\lambda_1} \sum_{j=1}^l \eta_j \in \mathbb{Z}^t$. We set

$$(x^2, u^2, y^2) := \sum_{i=1}^{k} \lambda_i(\hat{x}^i, \hat{u}^i, \hat{y}^i) + \alpha \sum_{j=1}^{l} \eta_j(\tilde{x}^j, \tilde{u}^j, \tilde{y}^j)$$

Then

$$(x^2, u^2, y^2) = \lambda_1((\hat{x}^1, \hat{u}^1, \hat{y}^1) + \frac{\alpha}{\lambda_1} \sum_{j=1}^{l} \eta_j(\tilde{x}^j, \tilde{u}^j, \tilde{y}^j)) + \sum_{i=2}^{k} \lambda_i(\hat{x}^i, \hat{u}^i, \hat{y}^i) \in conv(a_1, ..., a_k)$$

So, $x^2 \in conv(p(P))$. So, $x = (1 - \frac{1}{\alpha})x^1 + \frac{1}{\alpha}x^2 \in conv(p(P))$. Consequently, (A, B, D, b) is a sharp formulation. □

Proposition 4.50. *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $b \in \mathbb{Q}^m$.

(S2) $S := \{x \in \mathbb{Q}_+^{n_1} \times \mathbb{Z}_+^{n_2} | Ax \leq b\}$.

(S3) $\tilde{A} := \begin{pmatrix} A_1 & A_2 \\ O_{n_2,n_1} & E_{n_2} \end{pmatrix}$. *Here* $A_1 := (a^1, ..., a^{n_1})$ *and* $A_2 := (a^{n_1+1}, ..., a^n)$ *and each* a^i *is the i-th column vector of A.* $B := \begin{pmatrix} O_{m,n_2} \\ -E_{n_2} \end{pmatrix}$, $\tilde{b} := \begin{pmatrix} b \\ 0_{n_2} \end{pmatrix}$

(A1) $Q := \{x \in \mathbb{Q}_+^{n_1} \times \mathbb{Q}_+^{n_2} | Ax \leq b\}$ *has at least one basic feasible solution.*

Then $(\tilde{A}, B, \tilde{b})$ *is a locally ideal formulation for S iff* $(\tilde{A}, B, \tilde{b})$ *is a sharp formulation for S.*

Proof. Proposition4.44 and Propositon4.49, it is enough to show 'if' part. Let us assume $(\tilde{A}, B, \tilde{b})$ is a sharp formulation for S. Then

$$conv(S) = p(\{(x', x'', y) \in \mathbb{Q}^{n_1} \times \mathbb{Q}^{n_2} \times \mathbb{Q}^n | (x', x'') \in Q, x'' = y\}) = Q$$

By Theorem4.27, there are $\boldsymbol{a}_1, ..., \boldsymbol{a}_k \in S$ and $\boldsymbol{r}_1, ..., \boldsymbol{r}_l \subset \mathbb{Z}^n$ such that

$$Q = conv(\boldsymbol{a}_1, ..., \boldsymbol{a}_k) + cone(\boldsymbol{r}_1, ..., \boldsymbol{r}_l)$$

We can assume $\boldsymbol{a}_1, ..., \boldsymbol{a}_k$ are distinct. Let us fix any x which is a basic feasible solution of Q. By Proposition4.40, $x \in ex(Q)$. We will show $x \in S$. There are $\lambda_1, ..., \lambda_k, \eta_1, ..., \eta_l \in [0, 1]$ such that $\sum_{i=1}^k \lambda_i = \sum_{i=1}^l \eta_i = 1$ and

$$x = \sum_{i=1}^k \lambda_i \boldsymbol{a}_i + \sum_{j=1}^l \eta_j \boldsymbol{r}_j$$

For aiming contradiction, let us assume there is j such that $\eta_j > 0$. We can assume $j = 1$. Then We set

$$x^1 = \sum_{i=1}^k \lambda_i \boldsymbol{a}_i + \frac{1}{2}\eta_1 \boldsymbol{r}_1 + \sum_{j=2}^l \eta_j \boldsymbol{r}_j, x^2 = \sum_{i=1}^k \lambda_i \boldsymbol{a}_i + \frac{3}{2}\eta_1 \boldsymbol{r}_1 + \sum_{j=2}^l \eta_j \boldsymbol{r}_j$$

Then $x^1 \neq x^2$ and $x = \frac{1}{2}(x^1 + x^2)$. This contradicts with $x \in ex(Q)$. So,

$$x = \sum_{i=1}^k \lambda_i \boldsymbol{a}_i$$

For aiming contradiction, let us assume there is $i_1 \neq i_2$ such that $\lambda_{i_1}, \lambda_{i_2} > 0$. We can assume $i_1 = 1, i_2 = 2$. We set

$$y^1 = (\lambda_1 + \lambda_2)\boldsymbol{a}_1 + \sum_{i=3}^k \lambda_i \boldsymbol{a}_i, y^2 = (\lambda_1 + \lambda_2)\boldsymbol{a}_2 + \sum_{i=3}^k \lambda_i \boldsymbol{a}_i$$

Because $\boldsymbol{a}_1 \neq \boldsymbol{a}_2$, $y^1 \neq y^2$. And $x = \frac{\lambda_1}{\lambda_1 + \lambda_2} y^1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} y^2$. This contradicts with $x \in ex(Q)$. So, $x \in S$. □

4.4 Cutting Plane

Definition 4.51 (Valid Inequality)**.** *Let* $P \subset \mathbb{R}^n, c \in \mathbb{R}^n, \delta \in \mathbb{R}$. *We say the inequality* $c^T x \leq \delta$ *is invalid if* $c^T x \leq \delta$ *for any* $x \in P$.

4.5 Semidefinite Bounds

T.B.D

4.6 Reformulation and Relaxation

4.6.1 Lagrangian Relaxation

Definition 4.52 (Lagragian Relaxation)**.** *Here are the settings and assumptions.*

(S1) $c \in \mathbb{Q}^n$.

(S2) $b \in \mathbb{Q}^m$.

(S3) $A \in M(m, n, \mathbb{Q})$.

(S4) $p \in \mathbb{N}_{\leq n}$.

(S5) $F := \{x \in \mathbb{Z}^p \times \mathbb{Q}_{\geq 0}^(n-p) | Ax \leq b\}$.

(S6) $m_1 \in \mathbb{N}_{\leq n}, m_2 := m - m_1$.

(S7) $A_1 := \begin{pmatrix} a_1 \\ ... \\ a_{m_1} \end{pmatrix}, A_2 := \begin{pmatrix} a_{m_1+1} \\ ... \\ a_m \end{pmatrix}$. *Here, a_i is the i-th row vector of A $(i = 1, 2, ..., n+p)$.*

(S8) $Q := \{x \in \mathbb{R}_+^n | A_2 x \leq b^2, x_j \in \mathbb{Z}\ (j = 1, 2, ..., p)\}$.

We call

$$\begin{gathered} z_I := \max_{x \in F} cx \\ \Longleftrightarrow \\ z_I := \max cx \\ Ax \leq b \\ x_j \in \mathbb{Z}\ (j = 1, 2, ..., p) \\ x_j \in \mathbb{Q}_{\geq 0}\ (j = 1, 2, ..., n) \end{gathered}$$

the original problem. And, for $\lambda \in \mathbb{R}_{>0}^{m_1}$, we call

$$\begin{gathered} z_{LR}(\lambda) := \max_{x \in Q}(cx + \lambda(b^1 - A_1 x)) \\ \Longleftrightarrow \\ z_I := \max(cx + \lambda(b^1 - A_1 x)) \\ A_2 x \leq b^2 \\ x_j \in \mathbb{Z}\ (j = 1, 2, ..., p) \\ x_j \in \mathbb{Q}_{\geq 0}\ (j = 1, 2, ..., n) \end{gathered}$$

$LR(\lambda)$, lagrangian relaxation.

Proposition 4.53. *We take over notations and settings in Definition4.65. Then*

$$z_{LR}(\lambda) \geq z_I\ (\forall \lambda \in \mathbb{R}_{>0}^{m_1})$$

Definition 4.54 (Unimodular Matrix)**.** *Let $A \in M(m, n, \mathbb{Z})$. We say A is unimodular if $rank(A) = m$ and for every m-th squared submatrix B*

$$\det B = 0, \pm 1$$

Definition 4.55 (Totally Unimodular Matrix)**.** *Let $A \in M(m, n, \mathbb{R})$. We say A is totally unimodular if for every squared submatrix B*

$$\det B = 0, \pm 1$$

Clearly the following holds.

Proposition 4.56. *For any totally unimodular matrix $A \in M(m, n, \mathbb{R})$, each $a_{i,j} = 0$ or 1.*

Theorem 4.57. *We take over notations and settings in Definition4.65. And*

(A1) $\{x | A_1 x \leq b^1, x \in conv(Q)\} \neq \phi$.

Then

$$z_{LD} = \max\{cx | A_1 x \leq b^1, x \in conv(Q)\}$$

Corollary 4.58. *We take over notations and settings and assumptions in Theorem4.57. Then*

$$z_I \leq z_{LD} \leq z_{LP}$$

Here, z_{LP} is an optimal solution of the continuous relaxation of the original problem.

Corollary 4.59. *We take over notations and settings and assumptions in Theorem4.57. And*

(A2) $conv(Q) = \{x \in \mathbb{R}^n_{\geq 0} | A_2 x \leq b^2\}$.

Then

$$z_{LD} = z_{LP}$$

Corollary 4.60. *We take over notations and settings and assumptions in Theorem4.57. And*

(A2) A_2 *is totally unimodular.*

(A3) b^2 *is integer.*

Then

$$z_{LD} = z_{LP}$$

4.6.2 Dantzig-Wolfe Reformulation

Definition 4.61 (Ray)**.** *Let* $a \in \mathbb{R}^n$*. We call* $[0, \infty)a$ *a Ray of* $\mathbb{R}^n$*.*

Definition 4.62 (Extreme Ray)**.** *Let* C *be a polyhedral cone. We call* $R \subset C$ *an extreme ray of* C *if* R *is an edge of* C*.*

Proposition 4.63 (Dantzig-Wolfe Reformulation)**.** *We take over notations and settings and assumptions in Theorem4.65. And*

(S9) $\{v^k\}_{k \in K}$ *is a finite subset of* $conv(Q)$*.*

(S10) We pick $\{v^k\}_{k \in K} \subset \mathbb{Q}^n$ *and* $\{u^h\}_{h \in H} \subset \mathbb{Q}^n$ *such that* $conv(Q) = conv(\{v^k\}_{k \in K}) + cone(\{u^h\}_{h \in H})$*. Remark that such* $\{v^k\}_{k \in K}$ *and* $\{u^h\}_{h \in H}$ *exist by Meyer's theorem.*

Then

(i) The problem

$$\max\{cx | A_1 x \leq b^1, x \in conv(Q)\}$$

is equal to the following problem.

$$\max(\sum_{k \in K}(cv^k)\lambda_k + \sum_{h \in H}(cr^h)\mu_h)$$
$$\sum_{k \in K}(A_1 v^k)\lambda_k + \sum_{h \in H}(A_1 r^h)\mu_h \leq b^1$$
$$\sum_{k \in K}\lambda_k = 1$$
$$\lambda \in \mathbb{R}^{\#K}_{\geq 0}, \mu \in \mathbb{R}^{\#H}_{\geq 0}$$

The formulation is called Dantzig-Wolfe relaxation ot the original proglem.

(ii) The original problem is equal to the following problem. The formulation is called the Dantzi-Wolfe reformulation of the original problem.

$$\max(\sum_{k \in K}(cv^k)\lambda_k + \sum_{h \in H}(cr^h)\mu_h)$$
$$\sum_{k \in K}(A_1 v^k)\lambda_k + \sum_{h \in H}(A_1 r^h)\mu_h \leq b^1$$
$$\sum_{k \in K}\lambda_k = 1$$
$$\sum_{k \in K}(v^k)\lambda_k + \sum_{h \in H}(r^h)\mu_h \in \mathbb{Z}^n$$
$$\lambda \in \mathbb{R}^{\#K}_{\geq 0}, \mu \in \mathbb{R}^{\#H}_{\geq 0}$$

4.6.3 Column Generation

Example 4.64 (Dantzig-Wolfe Reformulation and Column Generation)**.** *We take over notations and settings and assumptions in Proposition4.63.*

(i) $K' \subset K, H' \subset H$.

(ii) We call the following problem master problem.

$$\max(\sum_{k \in K'} (cv^k)\lambda_k + \sum_{h \in H'} (cr^h)\mu_h)$$
$$\sum_{k \in K'} (A_1 v^k)\lambda_k + \sum_{h \in H'} (A_1 r^h)\mu_h \leq b^1$$
$$\sum_{k \in K'} \lambda_k = 1$$
$$\sum_{k \in K'} (v^k)\lambda_k + \sum_{h \in H} (r^h)\mu_h \in \mathbb{Z}^n$$
$$\lambda \in \mathbb{R}_{\geq 0}^{\# K'}, \mu \in \mathbb{R}_{\geq 0}^{\# H'}$$

Then

(i) The master problem is unbounded, the Dantzig-Wolfe relaxation is also unbounded.

(ii) The dual of the master problem is the following.

$$\min(\pi b^1 + \sigma)$$
$$\pi(A_1 v^k) + \sigma \geq cv^k, k \in K'$$
$$\pi(A_1 r^h) \geq cr^h, h \in H'$$
$$\pi \in \mathbb{R}_{\geq 0}^m, \sigma \in \mathbb{R}$$

(iii) Let us assume the master problem has an optimal solution, and $(\bar{\pi}, \bar{\sigma})$ *is an solution of the dual problem. We set*

$$\bar{c}_k := cv^k - \pi(A_1 v^k) - \bar{\sigma} \ (k \in K),$$
$$\bar{c}_h := cr^h - \pi(A_1 r^h) \ (h \in H)$$

If $\bar{c}_k \leq 0$ $(\forall k \in K)$ *and* $\bar{c}_h \leq 0$ $(\forall h \in H)$*, then* $(\bar{\pi}, \bar{\sigma})$ *is an optimal solution of the Dantzig-Wolfe relaxation.*

(iv) We take over notations and settings and assumptions in (iii). We call the following problem the pricing problem.

$$\zeta := -\bar{\sigma} + \max_{x \in Q}(c - \bar{\pi} A_1)x$$

Then

(a) ζ *is unbounded if and only if there is* $h \in H$ *such that* $\bar{c_h} > 0$.

(b) ζ *is bounded and* $\zeta > 0$ *if and only if there is* $k \in K$ *such that* $\bar{c_k} > 0$.

(c) ζ *is bounded and* $\zeta \leq 0$ *if and only if there is* $k \in K$ *such that* $\bar{c_h} \leq 0, \bar{c_k} \leq 0$ $(\forall h \in H, \forall k \in K)$.

4.6.4 Benders Decomposition

Theorem 4.65 (Benders Theorem)**.** *Here are the settings and assumptions.*

(S1) $c \in \mathbb{Q}^n$.

(S2) $h \in \mathbb{Q}^p$.

(S3) $A \in M(m, n, \mathbb{Q})$.

(S4) $G \in M(m, p, \mathbb{Q})$.

(S5) $F := \{(x, y) \in \mathbb{Z}_{\geq 0}^n \times \mathbb{Q}_{\geq 0}^p | Ax + Fy \leq b\}$.

(S6) We call

$$\max_{(x,y)\in F}\{cx+hy\}$$
$$\iff$$
$$\begin{gathered}\max\{cx+hy\}\\ Ax+Gy\le b\\ x\in\mathbb{Z}^n_{\ge 0}\\ y\in\mathbb{Q}^p_{\ge 0}\end{gathered}$$

the original problem.

(S7) $Q:=\{u\in\mathbb{R}^m_{\ge 0}|u^TG\ge h\}$.

(S8) We pick $\{u^k\}_{k\in K}\subset\mathbb{Q}^m$ *such that* $conv(Q)=conv(\{u^k\}_{k\in K})$. *Remark that such* $\{u^k\}_{k\in K}$ *exist by Meyer's theorem.*

(S9) $C:=\{u\in\mathbb{R}^m_{\ge 0}|u^TG\ge 0\}$.

(S10) We pick $\{r^J\}_{j\in J}\subset\mathbb{Q}^m$ *and* $\{r^j\}_{j\in J}\subset\mathbb{Q}^m$ *such that* $C=cone(\{r^j\}_{j\in J})$. *Remark that such* $\{r^j\}_{j\in J}$ *and* $\{r^j\}_{j\in J}$ *exist by Meyer's theorem.*

Then the original problem is equal to the following problem.

$$\begin{gathered}\max\{\eta+cx\}\\ \eta\le u^k(b-Ax)\ (\forall k\in K)\\ (r^j)^T(b-Ax)\ge 0\ (\forall j\in J)\\ x\in\mathbb{Z}^n_{\ge 0}\\ \eta\in\mathbb{R}\end{gathered}$$

5 Event graph analysis

5.1 Max-plus algebra

Definition 5.1 (Semi-ring)**.** *Here are the settings.*

(S1) R *is a set.*

(S2) $\oplus, \otimes$ *are binomial operators on* R*.*

We say $(R, \oplus, \otimes)$ *is a semi ring if*

(i) *For any* $x, y, z \in R$,

$$(x \oplus y) \oplus z = x \oplus (y \oplus z)$$
$$(x \otimes y) \otimes z = x \otimes (y \otimes z)$$

(ii) *For any* $x, y, z \in R$,

$$x \oplus y = y \oplus x$$

(iii) *For any* $x, y, z \in R$,

$$x \otimes (y \oplus z) = x \otimes y \oplus x \otimes z$$

(iv) R *has the unit element* ϵ *with respect to* $\oplus$*.*

(v) R *has the unit element* e *with respect to* $\oplus$*.*

(vi) $\epsilon \otimes x = x \otimes \epsilon = \epsilon$*.*

We say R *is commutative if* $\otimes$ *is commutative. We say* R *is idempotent if* $\otimes$ *is idempotent.*

Definition 5.2 ($\mathbb{R}_{max}$)**.** *Here are the settings.*

(S1) $\mathbb{R}_{max} := \mathbb{R} \cup \{-\infty\}$*. We set* $\epsilon := -\infty$ *and* $e := 0$*.*

(S2) *For* $x, y \in \mathbb{R}_{max}$

$$x \oplus y := \max\{x, y\}$$
$$x \otimes y := x + y$$

We call $\mathcal{R}_{max} := (\mathbb{R}_{max}, \oplus, \otimes)$ *the max-plus algebra.*

Clearly the following holds.

Proposition 5.3. $\mathcal{R}_{max}$ *is a commutative and idempotent semi ring.*

5.2 Petri net and Event graph

Definition 5.4 (Petri net, place, transition)**.** *Here are the settings.*

(S1) $(\mathcal{N}, \mathcal{A})$ *is a directed graph.*

We say $(\mathcal{N}, \mathcal{A})$ *is a petri net if there is* $(\mathcal{P}, \mathcal{Q})$ *which is a pair of disjoint subsets of* $\mathcal{N}$ *satisfying the following two conditions.*

(i) $\mathcal{N} = \mathcal{P} \cup \mathcal{Q}, \mathcal{P} \cap \mathcal{Q} = \phi$*.*

(ii) $\mathcal{A} \subset \mathcal{P} \times \mathcal{Q} \cup \mathcal{Q} \times \mathcal{P}$*.*

We denote this petri net by $(\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A})$*.*

We call each element of $\mathcal{P}$ *a place and call each element of* $\mathcal{Q}$ *a transistion. Let us fix* $p \in \mathcal{P}$ *and* $q \in \mathcal{Q}$*. We say* p *is the input place of the transition* q *and* q *is the output place of the transition* p *if* $(p, q) \in \mathcal{A}$*. We say* p *is the output place of the transition* q *and* q *is the input place of the transition* p *if* $(p, q) \in \mathcal{A}$*.*

We denote the set of all input place of q *by* $\pi(q)$ *and denote the set of all input transition of* p *by* $\pi(p)$*.*

We denote the set of all output place of q *by* $\sigma(q)$ *and denote the set of all output transition of* p *by* $\sigma(p)$*.*

Definition 5.5 (Event graph)**.** *Here are the settings.*

(S1) $(\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A})$ *is a petri net.*

We say this petri net is an event graph if for each $p \in \mathcal{P}$ there is the unique $q_1 \in \mathcal{Q}$ such that $(p, q_1) \in \mathcal{A}$ and there is the unique $q_2 \in \mathcal{Q}$ such that $(q_2, p) \in \mathcal{A}$.

Definition 5.6 (Enability and Firing in petri net)**.** *Here are the settings.*

- *(S1)* $(\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A})$ *is a petri net.*
- *(S2)* $w : \mathcal{A} \to \mathbb{N}_{\geq 1}$*. We call* $w(a)$ *is the weight of* $a \in \mathcal{A}$*.*
- *(S3)* $M_1 : \mathcal{P} \to \mathbb{Z}_{\geq 0}$*. For each* $p \in \mathcal{P}$*, we say* p *is marked with* $M_1(p)$ *tokens.*
- *(S4)* $q \in \mathcal{Q}$*.*

Then

- *(i) We say* q *is enable if each input place* p *of* q *is marked with at least* $w(p, q)$ *tokens.*
- *(ii) Let us assume* q *is enable. We set for each* $p \in \mathcal{P}$

$$M_1(p) := M_0(p) + \chi_{\sigma(q)}(p)w(q, p) - \chi_{\pi(q)}(p)w(p, q)$$

We call M_1 *the firing of* M_0 *with respect to* q*.*

Definition 5.7 (Liveness, Autonomous, Time event graph)**.** *Here are the settings.*

- *(S1)* $G := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_0)$ *is an event graph with weight and token.*

Then

- *(i) We say* G *is liveness if for any cycle* c *of* G *there is* $p \in \mathcal{P}$ *whose output transition is enable.*
- *(ii) For each* $q \in \mathcal{Q}$*,* q *is a supplier transition if* $\pi(q) = \phi$*.*
- *(iii) We say* G *is autonomous if* G *is no supplier transitions.*
- *(iv) Let* $\tau : \mathcal{P} \to \mathbb{Z}_{\geq 0}$ *and* $\gamma : \mathcal{A} \cap \mathcal{P} \times \mathcal{Q} \to \mathbb{Z}_{\geq 0}$ *such that*

$$\gamma(p, q) \leq \tau(p)$$

Then (G, τ, γ) *with time event graph.*

Definition 5.8 (Enability and Firing in Time event graph)**.** *Here are the settings.*

- *(S1)* $G := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_0, \tau, \gamma_0)$ *is a time event graph.*
- *(A1) For any* $q_1, q_2 \in \mathcal{Q}$*, there is at most one* $p \in \mathcal{P}$ *such that* $(q, p), (p, q) \in \mathcal{A}$*.*
- *(A2)* $w = 1$ *on* $\mathcal{A}$*.*
- *(S2)* $q \in \mathcal{Q}$*.*

Then

- *(i) We say* q *is enable if each input place* p *of* q *is marked with at least* $w(p, q)$ *tokens and* $\tau(p) \leq \gamma(p, q)$*. We denote the all enable transitions by* $E(G)$*.*
- *(ii) Let us assume* q *is enable. We set for each* $p \in \mathcal{P}$

$$M_1(p) := M_0(p) + \chi_{\sigma(q)}(p)w(p, q) - \chi_{\pi(q)}(p)w(p, q), \gamma_1(p) := 0$$

We call (M_1, γ_1) *the firing of* (M_0, γ_0) *with respect to* q*.*

Clearly the following holds.

Proposition 5.9. *Here are the settings.*

- *(S1)* $G_0 := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_0, \tau, \gamma_0)$ *is a time event graph.*
- *(A1) For any* $q_1, q_2 \in \mathcal{Q}$*, there is at most one* $p \in \mathcal{P}$ *such that* $(q, p), (p, q) \in \mathcal{A}$*.*
- *(A2)* $w = 1$ *on* $\mathcal{A}$*.*
- *(S3) We set*

$$M_1(p) := M_0(p) + \chi_{E(G_0)}(q_1) - \chi_{E(G_0)}(q_2)$$

Here $q_1 \in \pi(p)$ *and* $q_2 \in \sigma(p)$*. And*

$$\gamma_1(p, q) := \begin{cases} \gamma_0(p, q) + 1 & M_0(p) > 0 \text{ and } q \text{ is not enable} \\ 0 & \text{otherwise} \end{cases}$$

(S4) We set $G_1 := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_1, \tau, \gamma_1)$.

Then G_1 *is a time event graph.*

Definition 5.10 (Firing time)**.** *Here are the settings.*

(S1) $G_0 := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_0, \tau, \gamma_0)$ *is a time event graph.*

(A1) For any $q_1, q_2 \in \mathcal{Q}$, *there is at most one* $p \in \mathcal{P}$ *such that* $(q,p), (p,q) \in \mathcal{A}$.

(A2) $w = 1$ *on* $\mathcal{A}$.

(S3) We define $\{G_t\}_{t=0}^{\infty}$ *inductively by the procedure defined in Proposition5.9.*

Then

$$x_q(k) := \{t_0 \in \mathbb{Z}_{\geq 0} | k = \#\{t \leq t_0 | q \in E(G_t)\}\} \ (q \in \mathcal{Q}, k \in \mathbb{N}_{\geq 1})$$

We call $x_q(k)$ *the* k*-th firing time of* q. *We set*

$$x(k) := (x_{q_1}(k), ..., x_{q_{\#\mathcal{Q}}})^T \ (k \in \mathbb{N}_{\geq 1})$$

Definition 5.11 (System Matrix)**.** *Here are the settings.*

(S1) $\{G_t := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_t, \tau, \gamma_t)\}_{t \in \mathbb{Z}_{\geq 0}}$ *is a sequence of time event graphs by the procedure defined in Proposition5.9.*

(S2) $\{x(k)\}_{k=1}^{\infty}$ *is the sequence by Definition5.10.*

(S3) We denote the maximum number of tokens at any one place in $\{G_t\}_{t \in \mathbb{Z}_{\geq 0}}$ *by* M.

Then for each $m \in \{0, 1, ..., M\}$

$$[A_m]_{j,l} := \begin{cases} a_{j,l} & p_{j,l} \text{ exists and } p_{j,l} \text{ has } m \text{ tokens in } G_0 \\ \epsilon & \text{otherwise} \end{cases} \quad (j, l = 1, 2, ..., \#\mathcal{Q})$$

Here $p_{j,l}$ *is the place such that* $(q_j, p_{j,l}), (p_{j,l}, q_l) \in \mathcal{A}$.

Proposition 5.12. *We succeed notations in Definition5.11. And let us assume any* G_t *is autonomous. Then*

$$x(k) = A_0 \otimes x(k) \oplus A_1 \otimes x(k-1) \oplus ... \oplus A_M \otimes x(k-M) \ (k = M+1, M+2, ...)$$

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