A Study Note on Lie Group and Representation Theory and its Applications

November 17, 2024

Chapter 1

Introduction

1.1 About me and the note

I'm a Japanese older man who studies Lie group and representation theory as a hobby. Currently, I'm interested in the theory and its applications. For Lie group and representation theory, I'm interested in harmonic analysis and invariant metrics on homogeneous spaces. For the applications, I'm cuorious the application to statistics and mathematical programming and dynamic system. This note is the result of studying facts based on [1], [4]. I assume that the readers have knowledge of set theory[21], topological space[21], linear algebra[20], group theory and ring theory[22], calculus[18][19], complex analysis[16][17], the Lebesgue integral[15], differential manifold[23], Functional Analysis[12]. Except for this knowledge, I make efforts to make this note as self-contained as possible. However, I use some facts without proofs because I don't have enough time to provide proofs for them. For those facts, I give references to the proofs. Also, I introduce some facts without strict definitions and proofs for further study in the future. I mark the sections where all facts are of this kind with a star, $\star$. To be honest, I have never read the note again. Therefore, there may be many mistakes. I hope you understand.

1.2 My Motivation for Studying Lie Group and Representation Theory

In my mind, many mathematical problems can be clearly solved by ingenious tools that are not easily conceived by most people, myself included, such as expansions in terms of convenient functions, families of probability distributions that are easy to compute, handy metrics, and matrix decomposition formulas. For such expansion, the examples are Fourier series, expansion from spherical harminic functions, inverse formulas from Fourier transformation and wavelet transformation, and others. Exponential families are famous examples of families of probability distributions. Poincare metric is a good example of handy metrics. Jordan normal form is a popular example of decomposition formula of a matrix.

What is the mechanism behind these tools? Although I'm not a genius like Fourier or Poincare, when I encounter a new mathematical problem, is there a method I can use to invent or apply the appropriate tools?

When there is a group which is represented as the set of transformation the space in your problem, it is probably that you can get the tools by using group and representation theory. Therefore, I'm studying Lie group and representation theory.

1.3 Structure of Chapters in the Note

Chapter 2 is the priliminaries for the all chapters. Chapter 3 contains basic topics for Lie groups and Lie algebras. In Chapter 4, I introduces the theory of irreducible decomposition of unitary representation of general Lie group. In Chapter 5, I show the theory of irreducible decomposition of unitary representation of compact Lie group. Chapter 6 contains basic topics for homogeneous spaces where a Lie group is represented as the set of transformation the spaces. In Chapter 7, I show the classification theory of irreducible representation of compact Lie groups. In Chapter 8, I show the classification theory of irreducible representation of compact Lie groups. In the chapters after Chapter 9, I introduces the theories which apply Lie group and representation theory.

Contents

Part I

Lie Group and Representation Theory

Chapter 2

Preliminary

2.1 Linear algebra

2.1.1 Some facts without proof

For the proof, see [13].

Theorem 2.1.1 (Hahn Banach Theorem1)**.** *Let*

(S1) $(V, \{p_n\}_{n\in\mathbb{N}})$ *is a semi-normed space.*

(S2) $x, y \in V$ *such that* $x \neq y$.

Then there is real-valued continuous linear function f *such that* $f(x) \neq f(y)$.

2.1.2 Tensor Space

Clearly the following holds.

Proposition 2.1.2 (Tensor Space)**.** *Here are the settings and assumptions.*

(S1) K *denotes one of* $\mathbb{Q}, \mathbb{R}, \mathbb{C}$.

(S2) V, W *are* K*-vector spaces.*

(S3) By $V^\vee, W^\vee$ *denote by the set of all* K*-linear functionals of* V, W*, respectively.*

(S4) For $v \in V, w \in W$*, we set*

$$v \otimes w(f, g) := f(v)g(w) \ (f \in V^\vee, g \in W^\vee)$$

Then, for any $v \in V, w \in W$*,* $v \otimes w \in B(V, W)$*. We set*

$$V^\vee \otimes W^\vee := \langle\{v \otimes w | v \in V, w \in W\}\rangle$$

Proposition 2.1.3. *Here are the settings and assumptions.*

(S1) K *denotes one of* $\mathbb{Q}, \mathbb{R}, \mathbb{C}$.

(S2) V, W *are* K*-vector spaces.*

(S3) $w_1, ..., w_m \in W$ *are linear independent.*

(S4) $v_1, ..., v_m \in V \setminus \{0\}$.

Then, $\{v_i \otimes w_i\}_{i=1}^m$ *are linear independent.*

By Hahn-Banach Theorem,

Proof. there are $f_1, ..., f_m \in W^\vee$ such that $f_i(w_j) = \delta_{i,j}$ $(\forall i, j)$ and there are $g_1, ..., g_m \in W^\vee$ such that $g_i(v_i) \neq 0$ $(\forall i)$

Let us fix any $a_1, ...a_m \in K$ such that $\sum_{i=1}^m a_i v_i \otimes w_i = 0$. Since $0 = \sum_{i=1}^m a_i v_i \otimes w_i(g_j, f_j) = a_j$ $(\forall j)$, $\{v_i \otimes w_i\}_{i=1}^m$ are linear independent. □

2.1.3 Jordan Normal Form and Cayley-Hamilton Theorem

Proposition 2.1.4. *Let*

(S1) $m \in \mathbb{N} \cup [2, \infty)$

(S2) $f_1, ..., f_m \in \mathbb{C}[X] \setminus 0$.

(A1) $f_1, ..., f_m$ *don't have common divisor.*

then there are $h_1, ..., h_m \in \mathbb{C}[X]$ *such that*

$$\Sigma_{i=1}^{m} h_i f_i = 1 \tag{2.1.1}$$

Case when $m = 2$. When $\Sigma_{i=1}^{m} deg(f_i) = 0$, $deg(f_1) = deg(f_2) = 1$. In this case, the the claim in this Proposition holds. We assume the claim in this Proposition holds when $\Sigma_{i=1}^{m} deg(f_i) < K$. We can assume $deg(f_1) > 0$ There is $q, r \in \mathbb{C}[X]$ such that $f_1 = qf_2 + r$ and $deg(r) < deg(f_1)$ By the assumption of our mathematical induction, there are $h_1, h_2 \in \mathbb{C}[X]$ such that $h_1 r + h_2 f_2 = 1$. Because $r = qf_2 - f_1$, $-h_1 f_1 + (q + h_2) f_2 = 1$. □

Case when $m > 2$. We assume the claim in this Proposition holds when $m = K$. Let us set q is a maximum diviser of $f_1, ..., f_K$ and $g_i := \frac{f_i}{q_i}$ $(i = 1, 2, ..., K)$. Clearly, $g_1, ..., g_m$ don't have common divisor and f_{K+1} and q don't have common divisor. By the assumption of mathematical induction,
there are $h_1, ..., h_K, h_{K+1}, s \in \mathbb{C}[X]$ such that

$$\Sigma_{i=1}^{K} h_i g_i = 1 \tag{2.1.2}$$

and

$$sq + h_{K+1} f_{K+1} = 1 \tag{2.1.3}$$

Then $\Sigma_{i=1}^{K} h_i f_i = q$. Consequently,

$$\Sigma_{i=1}^{K} s h_i f_i + h_{K+1} f_{K+1} = 1 \tag{2.1.4}$$

□

Theorem 2.1.5. *Let*

(S1) $A \in M(n, \mathbb{C})$

then the followings hold.

(i) *There is* $P \in GL(n, \mathbb{C})$ *and* $\alpha_1, ..., \alpha_K \in \mathbb{C}$ *such that*

$$P^{-1}AP = \begin{pmatrix} J(\alpha_1) & O & ... & O \\ O & J(\alpha_2) & ... & O \\ ... & ... & ... & ... \\ O & & ... & J(\alpha_K) \end{pmatrix}$$

Here, for each i, *there are* $j_1, ..., j_{n_i}$ *such that*

$$J(\alpha_i) = \begin{pmatrix} J_1(\alpha_i) & O & ... & O \\ O & J_2(\alpha_i) & ... & O \\ ... & ... & ... & ... \\ O & & ... & J_{n_i}(\alpha_i) \end{pmatrix}$$

and $J_k(\alpha_i)$ *is a* j_k*-th square matrix*

$$J_k(\alpha_i) = \begin{pmatrix} \alpha_i & 1 & & & O \\ O & \alpha_i & 1 & & O \\ ... & ... & ... & ... & ... \\ O & & & \alpha_i & 1 \\ O & & & & \alpha_i \end{pmatrix}$$

We call $J_k(\alpha_i)$ *is a Jordan block.*

(ii) *If* $\alpha_i \neq \alpha_j$ *(for any* $i \neq j$*),* A *is diagonalizable.*

(iii) *For any* W_1 *and* W_2 *such that* W_1 *and* W_2 *are* $J_k(\alpha_i)$*-invariant subspaces and* $\mathbb{C}^\nu = W_1 \oplus W_2$, $W_1 = \mathbb{C}^\nu$ *or* $W_2 = \mathbb{C}^\nu$.

STEP1. Existence of the minimal polynomial of A. Because $E, A, A^2, ..., A^{n^2}$ are linearly dependent, there are $a_0, a_2, ..., a_n$ such that

$$\Sigma_{i=0}^{n^2} a_i A^i = 0 \tag{2.1.5}$$

So there is a $\varphi_A \in \mathbb{C}[X]$ such that

$$deg(\varphi_A) = min\{deg(\varphi)|\varphi \in \mathbb{C}[X] \ and \ \varphi(A) = 0\} \tag{2.1.6}$$

□

STEP2. Decomposition of $\mathbb{C}^n$ into generalized eigenspaces. By fundamental theorem of algebra, there are distinct $\alpha_1, ..., \alpha_K \in \mathbb{C}$

$$\varphi_A(x) = \Pi_{i=1}^K (x - \alpha_i)^{m_i} \tag{2.1.7}$$

We set $f_i \in \mathbb{C}[X]$ by $f_i(x) := \dfrac{\varphi_A(x)}{(x-\alpha_i)^{m_i}}$ $(i = 1, 2, ..., K)$. By Proposition(), then there are $h_1, ..., h_m \in \mathbb{C}[X]$ such that

$$\Sigma_{i=1}^K h_i(A) f_i(A) = E \tag{2.1.8}$$

We set $W_{i,j} := \{x \in \mathbb{C}^n | (A - \alpha_i E)^j x = 0\}$ and $W_i := W_{i,m_i}$ $(j = 1, 2, ..., m_i)$ For any $x \in \mathbb{C}^n$, $x = \Sigma_{i=1}^K h_i(A) f_i(A) x$. For each i, $h_i(A) f_i(A) x \in W_i$. So

$$\mathbb{C}^n = \Sigma_{i=1}^K W_i \tag{2.1.9}$$

□

STEP3. Showing $W_{i,k} \cap W_{j,l} = \{0\}$ $(i \neq j)$. We assume $k = l = 1$. Let us fix arbitary $x \in W_{i,1} \cap W_{j,1}$. Because $0 = Ax - Ax = \alpha_i x - \alpha_j x = (\alpha_i - \alpha_j)x$, $x = 0$. So $W_{i,1} \cap W_{j,1} = \{0\}$ $(i \neq j)$. Nextly we assume if $k + l \leq K$ then $W_{i,k} \cap W_{j,l} = \{0\}$ $(i \neq j)$. Let us fix arbitary i, j, k, l such that $i \neq j$. Let us fix arbitary $x_0 \in W_{i,k} \cap W_{j,l}$. We set $s : \mathbb{C}^n \ni x \mapsto [x] \in \mathbb{C}^n / W_{1,1}$. Because $AW_{1,1} \subset W_{1,1}$, $\tilde{A} : \mathbb{C}^n / W_{1,1} \ni [x] \mapsto [Ax] \in \mathbb{C}^n / W_{1,1}$ is well-definied and linear. We set $\tilde{W_{i,k}} := \tilde{A} s(W_{i,k})$ and $\tilde{W_{i,l}} := \tilde{A} s(W_{i,l})$ We can assume $k > 1$. Clearly $\tilde{W_{i,k}} \subset \{[x] \in \tilde{W_{i,k}} | (\tilde{A} - \alpha_i)^{k-1}[x] = 0\}$. So by the assumption of mathematical induction, $\tilde{W_{i,k}} \cap \tilde{W_{j,l}} = \{0\}$. This implies that $W_{i,k} \cap W_{j,l} \subset W_{i,1}$. Similarly, $W_{i,k} \cap W_{j,l} \subset W_{j,1}$. So $W_{i,k} \cap W_{j,l} \subset W_{i,1} \cap W_{j,1} = \{0\}$. □

STEP4. Showing $\Sigma_{i=1}^K W_i = \oplus_{i=1}^K W_i$. By STEP3, $\Sigma_{i=1}^2 W_i = \oplus_{i=1}^2 W_i$. We assume if $K \leq K_0$ then $\Sigma_{i=1}^K W_i = \oplus_{i=1}^K W_i$. We will show if $K = K_0 + 1$ then $\Sigma_{i=1}^K W_i = \oplus_{i=1}^K W_i$. By the assumption of mathematicalinduction,

$$\Sigma_{i=1}^{K_0} W_i / W_{K_0+1} = \oplus_{i=1}^K W_i / W_{K_0+1} \tag{2.1.10}$$

Let us fix arbitary $w_i \in W_i$ $(i = 1, 2, ..., K_0 + 1)$ such that $\Sigma_{i=1}^{K_0+1} w_i = 0$. By (11.16.21), $w_i \in W_i \cap W_{K_0+1}$ $(i = 1, ..., K_0)$. By STEP3, $w_i = 0$ $(i = 1, ..., K_0)$. So $w_K = 0$. □

STEP5. Constructing a basis of W_i. Let us fix i. There is $\nu \leq m_i$ such that

$$W_{i,\nu-1} \subsetneq W_{i,\nu} = W_i$$

If $\nu = 1$, then we take a basis of $W_{i,1} = W_i$. If $\nu > 1$, there are $w_1, ..., w_{r_\nu}$ such that $\{w_j + W_{i,v-1}\}_{j=1}^{r_\nu}$ is a basis of $W_{i,\nu}/W_{i,v-1}$. Clearly $A_i w_1, ..., A_i w_{r_\nu}$ are containd in $W_{i,\nu-1}$. Here,

$$A_i := A - \alpha_i E$$

We will show $\{A_i^j w_k\}_{k=1,...,r_\nu,\ j=0,...,\nu-1}$ are linear independent. Let us fix any $\{a_i^j w_k\}_{k=1,...,r_\nu,\ j=0,...,\nu-1} \subset \mathbb{C}$ such that

$$\sum_{k=1}^{r_\nu} \sum_{j=0}^{\nu-1} a_{k,j} A_i^j w_k = 0$$

Then

$$\sum_{k=1}^{r_\nu} a_{k,0} w_k = - \sum_{k=1}^{r_\nu} \sum_{j=1}^{\nu-1} a_{k,j} A_i^j w_k$$

Because the right side is contained in $W_{i,\nu-1}$ and $\{w_j + W_{i,v-1}\}_{j=1}^{r_\nu}$ are linear independent,

$$a_{k,0} = 0 \ (\forall k)$$

So,

$$\sum_{k=1}^{r_\nu}\sum_{j=0}^{\nu-1} a_{k,j+1}A_i^{j+1}w_k = 0$$

This implies that

$$\sum_{k=1}^{r_\nu} a_{k,1}w_k + \sum_{k=1}^{r_\nu}\sum_{j=1}^{\nu-1} a_{k,j+1}A_i^{j}w_k = \sum_{k=1}^{r_\nu}\sum_{j=0}^{\nu-1} a_{k,j+1}A_i^{j}w_k \in W_{i,\nu-1}$$

Because $\sum_{k=1}^{r_\nu}\sum_{j=1}^{\nu-1} a_{k,j+1}A_i^{j}w_k$ is contained in $W_{i,\nu-1}$,

$$\sum_{k=1}^{r_\nu} a_{k,1}w_k \in W_{i,\nu-1}$$

Because $\{w_j + W_{i,v-1}\}_{j=1}^{r_\nu}$ are linear independent,

$$a_{k,1} = 0 \ (\forall k)$$

Hereafter, by repeating this discussion,

$$a_{k,j} = 0 \ (\forall k, j)$$

We set

$$U_{i,k} := \left\langle \{A_i^j w_k\}_{j=0,\dots,\nu-1} \right\rangle$$

Clearly $U_{i,k}$ is A-invariant and the representation matrix of A respects to $\{A_i^j w_k\}_{j=0,\dots,\nu-1}$ is the Jordan block whose order is ν.

We set

$$V_i := \sum_{k=1}^{r_\nu} U_{i,k}$$

Because V_i is A-invariant,

$$\tilde{A} : W_i/V_i \ni w + V_i \mapsto Aw + V_i \in W_i/V_i$$

is well-defined and linear. Because $\{w_j + W_{i,\nu-1}\}_{j=1}^{r_\nu}$ is a spanning set of $W_{i,\nu}/W_{i,\nu-1}$, for any $w \in W_{i,\nu}$,

$$w + W_{i,\nu-1} \in V_i + W_{i,\nu-1}$$

So,

$$A_i^{\nu-1}w \in V_i$$

This implies

$$(\tilde{A}_i - \alpha_i E)^{\nu-1} = 0$$

By applying the above argument to $\tilde{A}_i$, $\tilde{A}_i$ is broken into Jordan blocks whose order is less than ν with respect to some basis. □

STEP6. Showing (ii). (i) implies (ii). □

STEP7. Showing (iii). We set $M := j_k$ and $J := J_k(\alpha_i)$ and $\alpha := \alpha_i$ and $J_\alpha := J - \alpha E$. There are $w_{1,1} \in W_1$ and $w_{1,2} \in W_2$ sucht that

$$e_1 = w_{1,1} + w_{1,2}$$

Because

$$J_\alpha e_1 = 0$$

and W_1 is J_α-invariant and W_2 is J_α-invariant,

$$J_\alpha w_{1,1} = J_\alpha w_{1,2} = 0$$

and the kernel of J_α is $\mathbb{C}e_1$, there are a_1 and a_2 such that

$$w_{1,1} = a_1 e_1, w_{1,2} = a_2 e_1$$

If $a_1 = 0$, then $a_2 = 1$ and $\mathbb{C}e_1 \subset W_2$. If $a_1 \neq 0$, then $\mathbb{C}e_1 \subset W_1$. By replacing W_1 by W_2, we can assume $\mathbb{C}e_1 \subset W_1$. There are $w_{2,1} \in W_1$ and $w_{2,2} \in W_2$ sucht that

$$e_2 = w_{2,1} + w_{2,2}$$

Because $J_\alpha e_2 = e_1$,

$$e_1 = J_\alpha w_{2,1} + J_\alpha w_{2,2}$$

Because $\mathbb{C}e_1 \subset W_1$,

$$J_\alpha w_{2,2} = 0$$

So,

$$w_{2,2} \in W_2 \cap \mathbb{C}e_1 = \{0\}$$

This implies $e_2 \in W_1$. By repeating this argument, $e_1, e_2, ..., e_M \subset W_1$. □

Proposition 2.1.6. *We succeed notations in Theorem4.3.14. Let us assume*

$$f_A(x) = \Pi_{i=1}^K (x - \alpha_i)^{n_i}$$

Then

$$dimW_i = n_i$$

Proof. By the proof of Theorem4.3.14, the order of $J(\alpha_i)$ is $dimW_i$. So,

$$f_A = \Pi_{i=1}^K f_{J(\alpha_i)} = \Pi_{i=1}^K (x - \alpha_i)^{\dim W_i}$$

So,

$$\dim W_i = n_i$$

□

By Theorem4.3.14, it is easy to show the following famous theorem.

Theorem 2.1.7 (Cayley-Hamilton theorem)**.** *Let*

(S1) $A \in M(n, \mathbb{C})$.

(S2) f_A *is the characteristic polynomial.*

then

$$f_A(A) = O \tag{2.1.11}$$

Proof. We will show this theorem by mathematical induction. If $n = 1$, then this theorem holds. Because $f_{P^{-1}AP} = f_A$ for any $P \in GL(n, \mathbb{C})$, by Theorem4.3.14, we can assume A is an upper triangle matrix.

$$\begin{aligned} & f_A(A) \\ = & \Pi_{i=1}^n (A - \alpha_i E) \\ = & (A - \alpha_1 E)\Pi_{i=2}^n (A - \alpha_i E) \\ = & \begin{pmatrix} 0 & x \\ 0 & X \end{pmatrix} \Pi_{i=2}^n (A - \alpha_i E) \end{aligned}$$

For any a_1, a_2, A and b_1, b_2, B there is c_1, c_2 such that

$$\begin{pmatrix} a_1 & a_2 \\ 0 & A \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ 0 & B \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ 0 & AB \end{pmatrix}$$

So, there are d_1, d_2 such that

$$\Pi_{i=2}^n (A - \alpha_i E) = \begin{pmatrix} d_1 & d_2 \\ 0 & O \end{pmatrix}$$

This implies

$$f_A(A) = \begin{pmatrix} 0 & x \\ 0 & X \end{pmatrix} \begin{pmatrix} d_1 & d_2 \\ 0 & O \end{pmatrix} = O$$

□

2.1.4 Kronecker Product

Definition 2.1.8 (Kronecker Product)**.** *Let K denotes one of $\mathbb{Q},\mathbb{R},\mathbb{C}$ and $A \in M(m,n,K)$ and $B \in M(p,q,K)$. Then*

$$A \otimes B = \{c_{i+k,j+l} := a_{i,j}b_{k,l}\}_{i,j,k,l\in\mathbb{N}} = \begin{pmatrix} a_{1,1}b_{1,1} & ... & a_{1,1}b_{1,q} & a_{1,2}b_{1,1} & ... & a_{1,2}b_{1,q} & ... & a_{1,n}b_{1,1} & ... & a_{1,n}b_{1,q} \\ a_{1,1}b_{2,1} & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ a_{1,1}b_{p,1} & ... & a_{1,1}b_{p,q} & a_{1,2}b_{p,1} & ... & a_{1,2}b_{p,q} & ... & a_{1,n}b_{1,q} & ... & a_{1,n}b_{p,q} \\ a_{2,1}b_{1,1} & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ a_{2,1}b_{p,1} & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ a_{m,1}b_{1,1} & ... & a_{m,1}b_{1,q} & a_{m,2}b_{1,1} & ... & a_{m,2}b_{1,1} & ... & a_{m,n}b_{1,1} & ... & a_{m,n}b_{p,q} \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ a_{m,1}b_{p,1} & ... & a_{m,1}b_{p,q} & a_{m,2}b_{p,1} & ... & a_{m,2}b_{p,q} & ... & a_{m,n}b_{p,1} & ... & a_{m,n}b_{p,q} \end{pmatrix}$$

We call $A \otimes B$ the Kronecker Product of A and B.

Proposition 2.1.9. *Here are the settings and assumptions.*

- *(S1)* $K = \mathbb{Q},\mathbb{R},\mathbb{C}$.
- *(S2)* $A \in M(m_1,m_2,K), B \in M(m_3,m_4,K), C \in M(n_1,n_2,K), D \in M(n_3,n_4,K)$.
- *(A1)* $m_2 = n_1, m_4 = n_3$.

Then

$$(A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$$

Proof. For any i_1,i_2,j_1,j_2,

$$(A \otimes B) \cdot (C \otimes D)_{(i_1,i_2),(j_1,j_2)} = \sum_{k_1,k_2} (A \otimes B)_{(i_1,i_2),(k_1,k_2)} (C \otimes D)_{(k_1,k_2),(j_1,j_2)} = \sum_{k_1,k_2} a_{i_1,k_1} b_{i_2,k_2} c_{k_1,j_1} b_{k_2,j_2}$$
$$= \sum_{k_1} a_{i_1,k_1} c_{k_1,j_1} \sum_{k_2} b_{i_2,k_2} b_{k_2,j_2} = (A \cdot C)_{i_1,j_1} (B \cdot D)_{i_2,j_2} = ((A \cdot C)_{\otimes} (B \cdot D))_{(i_1,i_2),(j_1,j_2)}$$

□

Proposition 2.1.10. *Here are the settings and assumptions.*

- *(S1)* $A \in M(m,\mathbb{C}), B \in M(n,\mathbb{C})$.
- *(S2)* $\lambda_1,...,\lambda_m$ *are the eigenvalues of* A.
- *(S3)* $\mu_1,...,\mu_n$ *are the eigenvalues of* B.
- *(A1)* $\lambda_i\mu_j$ $(i = 1,2,...,m, j = 1,2,...,n)$ *are distinct.*

Then

$$\lambda_i\mu_j \ (i = 1,2,...,m, j = 1,2,...,n)$$

are the all eigenvalues of $A \otimes B$.

Proof. Let x_i denote an eigenvector of A with respect to λ_i $(i = 1,2,...,m)$ and y_j denote an eigenvector of B with respect to μ_j $(j = 1,2,...,n)$. By Proposition2.1.9, the vector $x_i \otimes y_j$ is an eigenvector of $A \otimes B$ with respect to $\lambda_i\mu_j$ $i = 1,2,...,m, j = 1,2,...,n$. □

Proposition 2.1.11. *Here are the settings and assumptions.*

- *(S1)* $A \in M(m,\mathbb{C}), B \in M(n,\mathbb{C})$.

Then

$$det(A \times B) = det(A)det(B)$$

Proof. By applying triangulization of matrices, we can show that there are $\{A_i\}_{i=1}^{\infty} \subset M(m,\mathbb{C}), \{B_i\}_{i=1}^{\infty} \subset M(m,\mathbb{C})$ such that A_i, B_i satisfies the settings and the assumptions in Proposition2.1.10 for any i and

$$\lim_{i\to\infty} A_i = A, \lim_{i\to\infty} B_i = B$$

So,

$$\lim_{i\to\infty} det(A_i \otimes B_i) = det(A \otimes B), \lim_{i\to\infty} det(A_i) = det(A), \lim_{i\to\infty} det(B_i) = det(B)$$

By Proposition2.1.10,

$$det(A_i \otimes B_i) = det(A_i)det(B_i)\ (\forall i)$$

Consequently,

$$det(A \otimes B) = det(A)det(B)$$

□

2.2 Topological Space

2.2.1 The Case of General Metric Space

Definition 2.2.1 (Totally bouded metric space)**.** *Let*

(S1) (X, d) is a metric space.

(X, d) is totally bounded if for any $\epsilon > 0$ there are finite points $\{x_i\}_{i=1}^N$ such that $X = \cup_{i=1}^N B(x_i, \epsilon)$.

Proposition 2.2.2. *Let*

(S1) (X, d) is a metric space.

then the followings are equivalent.

(i) (X, d) is a totally bounded metric space.

(ii) For any sequence $\{x_i\}_{i=1}^\infty \subset X$ there is a subsequence $\{x_{\varphi(i)}\}_{i=1}^\infty$ which is a cauchy sequence.

(i) $\implies$ (ii). It is easy to show. □

(ii) $\implies$ (i). Let us assume (X, d) is not totally bounded. Then there is $\epsilon > 0$ such that for any finite subset $\{x_i\}_{i=1}^N$ $X \supsetneq \cup_{i=1}^{\varphi(n)} B(x_i, \epsilon)$.

Let us fix $x_1 \in X$. Because $X \supsetneq B(x_1, \epsilon)$. Let us fix $x_2 \in X \setminus \cup_{i=1}^1 B(x_i, \epsilon)$. By repeating the procedure in the same way below, there is $\{x_i\}_{i=1}^\infty$ such that $x_{n+1} \notin \cup_{i=1}^n B(x_i, \epsilon)\ (\forall n)$. Clearly $\{x_i\}_{i=1}^\infty$ does not contain subsequence which is a cauchy sequence. □

Proposition 2.2.3. *Let*

(S1) (X, d) is a totally bounded metric space.

(X, d) is separable.

Proof. For each $n \in \mathbb{N}$, $\{x_{n,i}\}_{i=1}^{\varphi(n)}$ such that $X = \cup_{i=1}^{\varphi(n)} B(x_{n,i}, \frac{1}{n})$. Clearly $\{x_{n,i} | n \in \mathbb{N},\ 1 \le i \le \varphi(n)\}$ is dense in X. □

Proposition 2.2.4. *Let*

(S1) (X, d) is a separable metric space.

(X, d) is secound countable.

Proof. Let us fix a countable dense set $\{x_n\}_{n=1}^\infty$ in X. Let us arbitary open covering $\{U_\lambda\}_{\lambda\in\Lambda}$.

We set $B := \{B(x_n, \frac{1}{m} | n \in \mathbb{N}$ and $m \in \mathbb{N}$ such that there is $B(x_n, \frac{1}{m} \subset U_\lambda$ for some $\lambda \in \Lambda\}$.

There is $\varphi : B \to \Lambda$ such that

$$b \subset U_{\varphi(b)}\ (\forall b \in B) \tag{2.2.1}$$

Clearly $\{U_{\varphi(b)} | b \ni B\}$ is countable.

Let us arbitary $x \in X$. There is $\lambda \in \Lambda$ such that $x \in U_\lambda$. There is $n \in \mathbb{N}$ such that $B(x, \frac{2}{n}) \subset U_\lambda$. There is m such that $d(x, x_m) < \frac{1}{n}$. We set $b := B(x_m, \frac{1}{n})$. Clearly $x \in b \subset U_\lambda$. So $x \in b \subset U_{\varphi(b)}$. Consequently, $X = \cup_{b\in B} U_{\varphi(b)}$ □

Proposition 2.2.5. *Let*

(S1) (X, d) is a metric space.

then the followings are equivalent.

(i) (X, d) is compact.

(ii) (X, d) is sequentially compact.

(iii) (X, d) is totally bounded and complete.

(i) $\implies$ (ii). It is easy to show. □

(ii) $\iff$ (iii). It is easy to show. □

(iii) and (ii) $\implies$ (i). We assume X is totally bounded and complete and X is not compact.

By Proposition2.2.4 and Proposition2.2.3, X is second countable.

So there is a open set covering $\{U_i\}_{i=1}^{\infty}$ such that for any finite subset $A \subset \mathbb{N}$ $X \supsetneq \cup_{i \in A} U_i$. Then $\{x_i\}_{i=1}^{\infty}$ such that $x_{n+1} \notin \cup_{i=1}^{n} U_i$. By (ii), there is a subsequence $\{x_{\varphi(i)}\}_{i=1}^{\infty}$ such that

$$\lim_{i \to \infty} x_{\varphi(i)} =: x \in X \tag{2.2.2}$$

exists.

There is n such that $x \in U_n$. There is $\epsilon > 0$ such that $B(x, \epsilon) \subset U_n$. By (2.2.2), there is $\varphi(m) > n$ such that $x_{\varphi(m)} \in B(x, \epsilon) \subset U_n$. Because $x_{\varphi(m)} \notin \cup_{i=1}^{\varphi(m)-1} \supset U_n$, $x_{\varphi(m)} \in U_n$ and $x_{\varphi(m)} \notin U_n$. It implies contradiction. □

Proposition 2.2.6. *Let*

(S1) (X, d) is a metric space.

(A1) $A \subset X$ is dense and totally bounded.

then (X, d) is totally bounded.

Proof. Let fix arbitary sequence $\{x_i\}_{i=1}^{\infty} \subset X$. By (A1), there is a sequcence $\{a_i\}_{i=1}^{\infty} \subset A$ such that $d(x_i, a_i) < \frac{1}{i}$ $(\forall i)$. By (A1) and Proposition2.2.2, there is a cauchy sequence $\{a_{\varphi}(i)\}_{i=1}^{\infty} \subset A$. Let fix arbitary $\epsilon > 0$. There is $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{\epsilon}{3}$ and $d(a_{\varphi(i)}, a_{\varphi(j)}) < \frac{\epsilon}{3}$ $\forall i > n_0$, $\forall j > n_0$. For any $i > n_0$ and any $j > n_0$

$$\begin{aligned} d(x_{\varphi(i)}, x_{\varphi(j)}) &\leq d(x_{\varphi(i)}, a_{\varphi(i)}) + d(a_{\varphi(i)}, a_{\varphi(j)}) + d(a_{\varphi(j)}, x_{\varphi(j)}) \\ &\leq \frac{1}{\varphi(i)} + \frac{\epsilon}{3} + \frac{1}{\varphi(j)} \\ &< \epsilon \end{aligned}$$

So $\{x_{\varphi}(i)\}_{i=1}^{\infty}$ is a cauchy sequence. Consequently X is totally bounded. □

Proposition 2.2.7. *Let us set $X := [0, 1]^{\mathbb{N}}$. Let us define $d : X \times X \to [0, \infty)$*

$$d(x, y) := \Sigma_{i=1}^{\infty} \frac{|y_i - x_i|}{2^i} \tag{2.2.3}$$

then (X, d) is a compact metric space.

Proof. Clearly (X, d) is a metric space. By Proposition2.2.5, it is enough to show X is sequential compact. Let us fix arbitary $\{x_i\}_{i=1}^{\infty} \subset X$. There is a subsequence $\{x_{\varphi(1,i)}\}_{i=1}^{\infty}$ and $y_1 \in [0, 1]$ such that $\lim_{i \to \infty} x_{\varphi(1,i),1} = y_1$. There is a subsequence of $\{x_{\varphi(1,i)}\}_{i=1}^{\infty}$ $\{x_{\varphi(2,i)}\}_{i=1}^{\infty}$ and $y_2 \in [0, 1]$ such that $\lim_{i \to \infty} x_{\varphi(2,i),i} = y_i$ $(i = 1, 2)$. By repeating the procedure in the same way below, we get $\varphi(1, i)\}_{n, i \in \mathbb{N}}$. We set $x_{\psi(i)} := x_{\varphi(i,i)}$ (for $i \in \mathbb{N}$) and $y := (y_1, y_2, ...)$. Clearly $\{x_{\psi(i)}\}_{i=1}^{\infty}$ converges to y. □

Proposition 2.2.8. *Let*

(S1) (X, d) is a separable metric space.

there is a metric $\tilde{d}$ such that (X, d) is homeomorphic to $(X, \tilde{d})$ and $(X, \tilde{d})$ is totally bounded.

Proof. $(X, min\{d, 1\})$ is a metric space and $(X, min\{d, 1\})$ is homeomorphic to (X, d). So we can assume (X, d) satisfies $0 \leq d \leq 1$.

Let us fix $\{x_i\}_{i=1}^{\infty} \subset X$ which is dense in X. We set $i : X \ni x \mapsto (d(x, x_i))_{i=1}^{\infty}[0, 1]^{\mathbb{N}}$. Clearly $i : X \to i(X)$ is homeomorphism. By Proposition2.2.5 and Proposition2.2.7, $i(X)$ is totally bounded. □

Proposition 2.2.9. *Let*

(S1) (X, d) is a separable metric space.

then there is a compact metric space $(\tilde{X}, \tilde{d})$ *and an homeomorphic mapping* $i : (X, d) \to i(X) \subset \tilde{X}$ *such that* $i(X)$ *is dense in* $\tilde{X}$

Proof. This proposition is proved by Proposition2.2.8 and Proposition2.2.6 and Proposition2.2.5 and Proposition11.5.8. □

Proposition 2.2.10. *Let*

(S1) (X, d) *is a metric space.*

(S2) $A \subset X$.

(S3) $r > 0$.

Then there is $f \in C_+(X)$ *such that* $0 \leq f \leq 1$ *on* X *and* $f|A \equiv 1$ *and* $supp(f) \subset \{x|d(x, A) \leq r\}$.

Proof. We set $f : \mathbb{R} \ni x \mapsto 1 - \frac{1}{r} min(r, d(x, A)) \in [0, 1]$. f satisfies the above condition. □

By Proposition, the following holds.

Proposition 2.2.11. *Let*

(S1) (X, d) *is a metric space.*

(A1) $A \subset X$ *and* $B \subset X$ *and* $d(A, B) > 0$.

then thre are $f \in C_+(X)$ *and* $g \in C_+(X)$ *such that* $0 \leq f \leq 1$ *on* X *and* $0 \leq g \leq 1$ *on* X *and* $f|A \equiv 1$ *and* $g|B \equiv 1$ *and* $d(\overline{supp(f)}, \overline{supp(g)}) > 0$.

2.2.2 The Case of Compact Metric Space

Proposition 2.2.12. *Let*

(S1) (X, d) *is a compact metric space.*

then $C(X)$ *is separable.*

Proof. By Proposition11.5.11, $C(X) \subset C_b(X)$. So it is enough to show $\{f \in C_+(X)|0 \leq f \leq 1 \text{ on } X\}$ is separable. By Proposition2.2.5, X is totally bounded. So for each $n \in \mathbb{N}$, there are $x_{n,1}, x_{n,2}, ..., x_{n,\varphi(n)}$ such that $X = \cup_{i=1}^{\varphi(n)} B(x_{n,i}, \frac{1}{n})$. By Proposition2.2.1, for each n and i and $m \in \mathbb{N}$ there is $f_{n,i,m} \in C_+(X)$ such that

$$f_{n,i,m}|B(x_{n,i}, \frac{1}{n}) \equiv 1 \tag{2.2.4}$$

and $supp(f_{n,i,m}) \subset B(x_{n,i}, \frac{1}{n} + \frac{1}{m})$ and

$$0 \leq f_{n,i,m} \leq 1 \tag{2.2.5}$$

on X.

We set $\Lambda := \{(n, i, m, q) \in \mathbb{N}^3 \times \mathbb{Q}|i \leq \varphi(n)\}$. For each λ which is a finite subset of Λ, $g_\lambda := max\{qf_{n,i,m}|(n, i, m, q) \in \lambda\}$. Then $B := \{g_\lambda|\lambda \text{ a finite subset of } \Lambda\}$ is a countable set.

We will show $\bar{B} = \{f \in C_+(X)|0 \leq f \leq 1 \text{ on } X\}$. Let us fix arbitary $f \in \{f \in C_+(X)|0 \leq f \leq 1 \text{ on } X\}$ and $\epsilon > 0$. By Proposition11.5.10, there is $N \in \mathbb{N}$ such that

$$|f(x) - f(y)| < \frac{\epsilon}{2} \tag{2.2.6}$$

(for any x, y such that $d(x, y) < \frac{1}{N}$). There are $q_i \in \mathbb{Q} \cup [0, 1]$ such that

$$|q_i - f(x_{2N,i})| < \frac{\epsilon}{2} \ (\forall i) \tag{2.2.7}$$

We set $g := max\{q_i f_{2N,i,2N}|i = 1, 2, ..., \varphi(2N)\}$. Clearly $g \in B$.

Let us fix arbitary $x \in X$. Because $X = \cup_{i=1}^{\varphi(2N)} B(x_{2N,i}, \frac{1}{2N})$, there is i such that $x \in B(x_{2N,i}, \frac{1}{2N})$.

By (2.2.4) and (2.2.6) and (2.2.7)

$$\begin{aligned} f(x) - \frac{\epsilon}{2} &< f(x_{2N,i} \\ &< q_i + \frac{\epsilon}{2} \\ &< q_i f_{2N,i,2N}(x) + \frac{\epsilon}{2} \\ &< g(x) + \frac{\epsilon}{2} \end{aligned} \tag{2.2.8}$$

So

$$f(x) - \epsilon < g(x) \tag{2.2.9}$$

There is j such that $g(x) = q_j f_{2N,j,2N}(x)$. By (2.2.5) and (2.2.6) and (2.2.7),

$$\begin{aligned} q_j f_{2N,j,2N}(x) &\leq q_j \\ &< f(x_{2N,j}) + \frac{\epsilon}{2} \\ &< f(x) + \epsilon \end{aligned} \tag{2.2.10}$$

So

$$|f(x) - g(x)| < \epsilon \tag{2.2.11}$$

Consequently, $\bar{B} = \{f \in C_+(X) | 0 \leq f \leq 1 \text{ on } X\}$

□

2.2.3 Baire Category Theorem

Theorem 2.2.13 (Baire Category Theorem I). *Let*

- *(S1) X is a complete metric space.*
- *(S2) $\{A_n\}_{n=1}^{\infty}$ is a sequence of closed sets of X such that $A_n \subset A_{n+1}$ $(\forall n \in \mathbb{N})$.*
- *(A1) $X = \cup_{n=1}^{\infty} A_n$.*

Then there is $n \in \mathbb{N}$ such that $A_n^{\circ} \neq \phi$.

Proof. Let us assume

$$A_n^{\circ} = \phi \ (\forall n \in \mathbb{N}) \tag{2.2.12}$$

Let us fix $x_0 \in A_1$. In this proof, for each $x \in X$ and $\epsilon > 0$ we denote $B(x, \epsilon) := \{y \in X | d(x, y) \leq \epsilon\}$. Since $x_0 \notin A_1^{\circ}$, $B(x_0, 1) \not\subset A_1$. Then there is $x_1 \in B(x_0, 1) \setminus A_1$. Because A_1^c is an open set, there is $\varphi(1) \in \mathbb{N} > 1$ such that $D(x_1, \frac{1}{\varphi(1)}) \subset A_1^c \cap B(x_0, 1)$. If you repeat this procedure in the same way below, there is $\varphi : \mathbb{N} \to \mathbb{N}$ and $\{x_n\}_{n=1}^{\infty} \subset X$ such that φ is narrow sense monotonically increasing and $D(x_n, \frac{1}{\varphi(n)}) \subset A_n^c \cap B(x_{n-1}, \frac{1}{\varphi(n-1)})$ $(\forall n \in \mathbb{N})$. Because clearly $\{x_n\}_{n=1}^{\infty}$ is a cauchy sequence, $x_{\infty} := \lim_{n \to \infty} x_n$ exists. By (A1), there is $n \in \mathbb{N}$ such that $x_{\infty} \in A_n$. Because $x_m \in D(n, \frac{1}{\varphi(n)}) \subset A_n^c$ $(\forall m \geq n)$, $x_{\infty} \in D(n, \frac{1}{\varphi(n)}) \subset A_n^c$. This is contradiction. □

Theorem 2.2.14 (Baire Category Theorem II). *Let*

- *(S1) X is a locally compact space.*
- *(S2) $\{A_n\}_{n=1}^{\infty}$ is a sequence of compact sets of X such that $A_n \subset A_{n+1}$ $(\forall n \in \mathbb{N})$.*
- *(A1) $X = \cup_{n=1}^{\infty} A_n$.*

Then there is $n \in \mathbb{N}$ such that $A_n^{\circ} \neq \phi$.

Proof. Let us assume

$$A_n^{\circ} = \phi \ (\forall n \in \mathbb{N}) \tag{2.2.13}$$

Since X is locally compact, there is a relative compact nonempty open subset V in X. Since $V \not\subset A_1^{\circ}$ and A_1 is compact and X is locally compact Hausdorff, there is a relative compact nonempty open subset $V_1 \in V \setminus A_1$. If you repeat this procedure in the same way below, there is a sequence of relative compact nonempty open subsets $\{V_k\}_{k \in \mathbb{N}}$ such that $V_{k+1} \subset V_k \subset \bar{V}$ $(\forall k \in \mathbb{N})$. Since $\bar{V}$ is compact, $x_{\infty} \in \cap_{k \in \mathbb{N}} \bar{V_k}$ exists. By (A1), there is $n \in \mathbb{N}$ such that $x_{\infty} \in A_n$. Because $V_m \subset A_n^c$ $(\forall m \geq n)$, $x_{\infty} \in \cap_{k \in \mathbb{N}} \bar{V_k} \subset A_n^c$. This is contradiction. □

2.2.4 Miscellaneouses

Proposition 2.2.15. *Let X and Y are topological space and $i : X \to Y$ is homeomorphism. And let $U \subset X$ ando $V := i(u)$. Then $i|U : U \to V$ is homeomorphisim.*

Proof. For any closed set in X A and any closed set in Y B, $i^{-1}(B \cap V) = i^{-1}(B) \cap V$ and $i(A \cap U) = i(A) \cap V$. So$i^{-1}(B \cap V)$ is closed set of X and $i(A \cap U)$ is closed set of Y. □

Proposition 2.2.16. *Let X is a topological space and $U \subset U' \subset X$. Let us assume the topology of U' is the relative topology respect to X. The relative topology of U respect to U' is equal to the relative topology of U respect to X.*

Proof. Because for any open set A in X $A \cap U = A \cap U' \cap U$, the Proposition holds. □

Proposition 2.2.17. *Let X be a Housdorff space and $C \subset X$ be a compact subset. Then C is a closed subset of X.*

Proof. Let us fix any $x \in X \setminus C$. For each $y \in C$, there are U_y and V_y such that U_y is an open neighborhood of x and V_y is an open neighborhood of y and $U_y \cap V_y = \phi$. Because C is compact, there are $V_{y_1}, ..., V_{y_m}$ such that $C \subset \cup_{i=1}^m V_{y_i}$. Because $\cap_{i=1}^m U_{y_i}$ is an open neighborhood of x and $\cap_{i=1}^m U_{y_i} \cap \cup_{i=1}^m V_{y_i} = \phi$, $x \notin \bar{C}$. Consequently, C is a closed subset. □

Definition 2.2.18 (Locally path-connected space)**.** *Here are the settings and assumptions.*

- *(S1) X is a topological space.*

We say X is locally path-connected if for any $U \in \mathcal{O}(X)$ and $x \in U$, there is V such that V is a path-connected open neighborhood of x and $V \subset U$.

The following clearly holds.

Proposition 2.2.19. *Any topological manifold is locally path-connected.*

Definition 2.2.20 (Covering Space)**.** *Here are the settings and assumptions.*

- *(S1) E, B are path-connected and locally connected topological space.*
- *(S2) $p : E \to B$ is a surjective continuous map.*

We say (E, B, p) is a covering space if for any $b \in B$ there is U such that U is an open neighborhood of b and any connected component of $\pi^{-1}(U)$ V satisfies $\pi|V : V \to \pi(V)$ is a homeomorphism. We call E the total space, B the base space, p the projection.

Definition 2.2.21 (Finite covering Space)**.** *Here are the settings and assumptions.*

- *(S1) (E, B, p) is a covering space.*

We say (E, B, p) is a finite covering space if there is $m \in \mathbb{N}$ such that for any $b \in B$ $\#p^{-1}(b) = m$. We call m the covering degree of (E, B, p).

2.3 Calculus

2.3.1 Inverse function theorem

Lemma 2.3.1. *Let*

- *(S1) $I_b := (-b, b)^{n+1}$ and $J_b := (-b, b)^n$.*
- *(S2) $a \in I_b$.*
- *(A1) $f \in C^1(\bar{I}_b, \mathbb{R})$.*
- *(A2) $f(a) = 0$.*
- *(A3) $\alpha := inf_{x \in I_b} \frac{\partial f}{\partial x_1}(x)) > 0$.*

then $r \in C^1(J_b, \mathbb{R})$ such that

$$f(r(y), y) = 0 \ (\forall y \in J_b) \tag{2.3.1}$$

Proof. By (A3), for any $y \in J_b$ there exists only one $r(y) \in (-b, b)$ such that $f(r(y), y) = 0$.
Let us fix arbitary $y \in J_b$ and fix arbitary $i \in \{2, , ..., n\}$.
For $z \in \mathbb{R}$ such that $|z|$ is sufficient small,

$$\begin{aligned} 0 &= 0 - 0 \\ &= f(r(y + z\boldsymbol{e}_i), y + z\boldsymbol{e}_i) - f(r(y + z\boldsymbol{e}_i), y) \\ &\quad + f(r(y + z\boldsymbol{e}_i), y) - f(r(y), y) \\ &= \int_0^1 \frac{d}{dt} f(r(y + z\boldsymbol{e}_i), y + tz\boldsymbol{e}_i)dt \\ &\quad + \int_0^1 \frac{d}{dt} f(r(y) + t(r(y + z\boldsymbol{e}_i) - r(y)), y)dt \\ &= z\int_0^1 \frac{\partial f}{\partial x_i}(r(y + z\boldsymbol{e}_i), y + tz\boldsymbol{e}_i)dt \\ &\quad + (r(y + z\boldsymbol{e}_i) - r(y))\int_0^1 \frac{\partial f}{\partial x_1}(r(y) + t(r(y + z\boldsymbol{e}_i) - r(y)), y)dt \end{aligned} \tag{2.3.2}$$

By (2.3.2), $(r(y + z\boldsymbol{e}_i) - r(y)) \leq |z|\frac{1}{\alpha} sup_{\bar{I}_b}|\frac{\partial f}{\partial x_i}|$. So r is continuous on J_b.
By (A1) and (A3), $\int_0^1 \frac{\partial f}{\partial x_1}(r(y) + t(r(y + z\boldsymbol{e}_i) - r(y)), y)dt > 0$. So, by (2.3.2),

$$\frac{(r(y + z\boldsymbol{e}_i) - r(y))}{z} = \frac{(\int_0^1 \frac{\partial f}{\partial x_i}(r(y + z\boldsymbol{e}_i), y + tz\boldsymbol{e}_i)dt}{\int_0^1 \frac{\partial f}{\partial x_1}(r(y) + t(r(y + z\boldsymbol{e}_i) - r(y)), y)dt} \tag{2.3.3}$$

By (A1) and continuity of r and (2.3.3),

$$\lim_{z \to 0} \frac{(r(y + z\boldsymbol{e}_i) - r(y))}{z} = \frac{\partial f}{\partial x_i}(r(y), y)\frac{\partial f}{\partial x_1}(r(y), y) \tag{2.3.4}$$

Consequently $r \in C^1(J_b, \mathbb{R})$. □

Theorem 2.3.2 (Inverse function theorem)**.** *Let*

(S1) U is open set in $\mathbb{R}^n$.

(S2) $a \in U$.

(A1) $f \in C^1(U, \mathbb{R}^n)$.

(A2) $det(Jf(a)) > 0$ on U.

then there is $V \subset U$ such that V and $f(U)$ are open set and $f : V \to f(V)$ is injective and $f^{-1} \in C^1(f(V), V)$.

STEP1: case when $n = 1$. It is easy to show. □

STEP2-1: f is locally injective(case when $n > 1$). Let us fix arbitary $n_0 \in \mathbb{N}$. We assume the above theorem is true if $n \leq n_0$. Let us assume $n = n_0 + 1$. By (A2), for any $i \in \{1, 2, ..., n\}$ there is $u_i \in \mathbb{R}^n$ such that $Jf(a)u_i = e_i$. By setting for sufficient $b > 0$ $g : (-b, b)^n \ni \boldsymbol{t} \mapsto f(\sigma_{i=1}^n t_i v_i) \in \mathbb{R}^n$, We can assume $a = 0$ and $f(0) = 0$ and $[-c, c]^n \subset U$ for some $c > 0$. and

$$\frac{\partial f^i}{\partial x_i} > 0 \ on \ I_c := (-c, c)^n \ (\forall i > 0) \tag{2.3.5}$$

By (2.3.5), clearly f is injective on I_c.

□

STEP2-2: f is locally open map(case when $n > 1$). Next, we will show f is open map in I_c for sufficient small $c > 0$. And by Lemma2.3.1, there is $c' \in (0, c)$ and $r : C^1(J'_c, \mathbb{R})$ such that $f_1(r(y), y) = 0$ $(\forall y \in J_{c'})$. Here, $J'_c := (-c', c')^{n_0}$. By resetting c to be sufficient smaller, we can assum that $c = c'$.

We set $\tilde{f} = (f_2, ..., f_n)$. Let us set $g : J_c \ni y \mapsto \tilde{f}(r(y), y)\mathbb{R}^{n_0}$. $Jg(0) = \begin{pmatrix} \boldsymbol{e}_2^T \\ \boldsymbol{e}_3^T \\ ... \\ \boldsymbol{e}_n^T \end{pmatrix} \begin{pmatrix} dr \\ \boldsymbol{e}_1^T \\ ... \\ \boldsymbol{e}_{n_0}^T \end{pmatrix} = E_{n_0}$. By the assumption in mathematical induction, there is $c'' \in (0, c)$ such that $g(J_{c''})$ is an open set in $\mathbb{R}^{n_0}$ and g is injective on $J_{c''}$. By resetting c to be sufficient smaller, we can assum that $c = c''$.

Let us fix arbitary connected open interval $(x_1,x_2)\times J\subset I_c$. We set $I:=(x_1,x_2)$. We will show $f(I\times J)$ is open set. Let us fix arbitary point $f(x_0,y_0)\in f(I\times J)$. Because (2.3.5), For any $y\in J$, $f(x_1,y)<f(x_0,y)<f(x_2,y)$. Because $\bar{J}$ is compact, by (2.3.5), there is d_1 and d_2 such that

$$f_1(x_1,y)\le d_1<f_1(x_0,y)<d_2\le f_1(x_2,y)\ (\forall y\in J) \tag{2.3.6}$$

We set $W:=(d_1,d_2)\times g(J)$. Cleary $f(x,y)\in W$. Because g is open map, W is open set. We will show $W\subset f(I\times J)$. Let us fix arbitary $(u,g(y))\in I\times W$. Because $f_1(\cdot,y)$ is continuous and (2.3.6), by intermediate value theorem, there is $x\in I$ such that $f_1(x,y)=u$. So $f(x,y)=(f_1(x,y),g(y))=(u,g(y))$. This means $W\subset f(I\times J)$. Consequently, f is open map in I_c.

We replace f by $f|I_c$.

□

STEP2-3: For each i, $\frac{\partial f^{-1}}{\partial w_i}$ exists(case when $n>1$). Let us fix arbitary $(u_0,v_0)\in W$. Let us set $(x,y):=f^{-1}(u_0,v_0)$. By using an approach is same with one in STEP2-1, it is enough to show that for any i $\frac{\partial f^{-1}}{\partial w_i}(x_0,y_0)$ exists we can assume that $Jf(x_0,y_0)=E_n$.

Let us fix arbitary $i\in\{1,2,...,n\}$. Let us pick up $j\in(\{1,2,...,n\}\setminus\{i\}$. By swaping x_j by x_1 and swaping f_j by f_1, we can assume $j=1$. By using an approach is same with one in STEP2-2, there is $R\in C^1(J_c,\mathbb{R})$ such that

$$f_1(R(y),y)=u\ (\forall y\in J_c) \tag{2.3.7}$$

and $G:J_c\ni y\mapsto\tilde{f}(R(y),y)\in G(J_c)$ is injective and open map and class C^1 and G^{-1} is class C^1.

For any t such that $|t|$ is sufficient small,

$$f^{-1}((u,v)+t\boldsymbol{e}_i)=(R(G^{-1}((u,v)+t\boldsymbol{e}_i)),G^{-1}((u,v)+t\boldsymbol{e}_i)) \tag{2.3.8}$$

The right side of (2.3.8) is differentiable at $t=0$.

$\frac{\partial f^{-1}}{\partial w_i}(x_0,y_0)$ exists. □

STEP2-4 f^{-1} is class C^1(case when $n>1$). Lastly, we will show $f^{-1}\in C^1(W,I_c)$.

By STEP2-3,

$$ff^{-1}(w)=w\ (\forall w\in W) \tag{2.3.9}$$

So,

$$\frac{\partial f^{-1}}{\partial w_i}(w)=Jf(f^{-1}(w))^{-1}e_i\ (\forall i,\forall w\in W) \tag{2.3.10}$$

The right side of (2.3.10) is continuous with respect to w. So f^{-1} is class C^1. □

Remark 2.3.3. *There is a map which does not have global inverse map and has nonsingular Jacobi matrix at every point. $f:(0,\infty)\times\mathbb{R}\ni(r,\theta)\mapsto r\begin{pmatrix}r\cos(\theta)\\ r\sin(\theta)\end{pmatrix}\in\mathbb{R}^2\setminus\{0\}$ is a example of such maps.*

Remark 2.3.4. *[35] gives a sufficient condition for existence of global inverse map.*

The following proposition is easily proved by inverse mapping theorem.

Corollary 2.3.5. *(S1) U is open set in $\mathbb{R}^n$.*

(S2) $a\in U$.

(A1) $f\in C^1(U,\mathbb{R}^n)$.

(S3) V is open set in $\mathbb{R}^n$ such that $f(U)\subset V$.

(A2) $g\in C^1(V,\mathbb{R}^n)$.

(A3) $g\circ f=id_U$.

then there is $f(U)$ is open set.

2.3.2 Implicit function theorem

Theorem 2.3.6 (Implicit function theorem). *Let*

(S1) U *is open set in* $\mathbb{R}^{m+n}$.

(S2) $a := (a_1, a_2, ..., a_m)$ *and* $b := (b_1, b_2, ..., b_n)$ *and* $c := (a, b)$.

(S3) $c \in U$.

(A1) $f \in C^1(U, \mathbb{R}^m)$.

(A2) $f(c) = 0$.

(A3) $det(\{\frac{\partial f_i}{\partial x_j}(a)\}_{1 \leq i,j \leq m}) \neq 0$.

then there is an open subset in $\mathbb{R}^n$ V *and* $r \in C^1(V, \mathbb{R}^m)$ *such that* $b \in V$ *and* $r(b) = a$ *and*

$$(r(y), y) \subset U \text{ and } f(r(y), y) = 0 \ (\forall y \in V) \tag{2.3.11}$$

Proof. By resetting Bf for $B = \{\frac{\partial f_i}{\partial x_j}(a)\}^{-1}_{1 \leq i,j \leq m}$, we can assume.

Let us set $g : U \ni (x, y) \mapsto (f(x), y) \in \mathbb{R}^{m+n}$. Because $det(Jg(c)) \neq 0$, by inverse function theorem, there is an open neighborhood of c $U' := B(a, \epsilon) \times B(b, \epsilon) \subset U$ such that $g(U')$ is open subset and $g : U' \to g(U')$ is class C^1 homeomoriphisim.

We set $r : B(b, \epsilon) \ni g^{-1}(0, y) \in U'$. Clearly r satisfies the conditions in the above theorem. □

2.3.3 Method of Lagrange multiplier

Theorem 2.3.7 (Method of Lagrange multiplier). *Let*

(S1) U *is open set in* $\mathbb{R}^{m+n}$.

(S2) $a := (a_1, a_2, ..., a_m)$ *and* $b := (b_1, b_2, ..., b_n)$ *and* $c := (a, b)$.

(S3) $c \in U$.

(S4) $g \in C^1(U, \mathbb{R})$.

(A1) $f \in C^1(U, \mathbb{R}^m)$.

(A2) $f(c) = 0$.

(A3) $rank(Jf(c)) = m$.

(A4) a *is a maximum point of* g *in* U.

then there is $\lambda \in \mathbb{R}^m$ *such that*

$$dg(a) = \Sigma_{i=1}^m \lambda_i df_i(a) \tag{2.3.12}$$

Proof. By swapping variables, we can assume (A3) in Theorem2.3.6. We pick r in Theorem2.3.6. We define $(s_1, s_2, ..., s_n)$ by $(s_1, s_2, ..., s_n) := \begin{pmatrix} Jr(b) \\ E_n \end{pmatrix}$ Clearly $dim < s_1, s_2, ..., s_n >= n$. So $dim < s_1, s_2, ..., s_n >^{\perp} = m$ Because $f(r(\cdot), \cdot) \equiv 0$ in U, $< df_1, df_2, ..., df_m > \subset < s_1, s_2, ..., s_n >^{\perp}$. By (A3), $< df_1, df_2, ..., df_m > = < s_1, s_2, ..., s_n >^{\perp}$. Because $g(r(\cdot), \cdot)$ achives maximum at b, $dg \in < s_1, s_2, ..., s_n >^{\perp}$. Consequnently, there is $\lambda \in \mathbb{R}^m$ such that

$$dg(a) = \Sigma_{i=1}^m \lambda_i df_i(a) \tag{2.3.13}$$

□

2.4 Differential Manifold

Definition 2.4.1. *Let* M *is a n-dimensional manifold and* $p \in M$. *Let* f *and* g *be* C^∞ *function in some neighborhood of* p. *We define* $f \sim g$ *if* $f = g$ *in some neighborhood.*

Definition 2.4.2. *Let*

(S1) M *is a n-dimensional manifold.*

(S2) $p \in M$.

(S3) Denote $\mathfrak{F}(p)$ be the quotient set defined by equivalent relation in Definition.

We call $v : \mathfrak{F}(p)^*$ is a tangent vector if

$$v(fg) = v(f)g(p) + f(p)v(g) \ (\forall f, g \in \mathfrak{F}(p)) \tag{2.4.1}$$

We denote the set of all tangent vectors by $T(p)$.

Proposition 2.4.3. *Let us fix arbitary $f \in \mathfrak{F}(p)$ such that $df(p) = 0$. Then for any $v \in T(p)$, $v(f) = 0$*

Proof. Let us fix a local cordinate $\varphi := (x^1, ..., x^n)$ such that $\varphi(0) = p$. We set $\psi := (y_1, y_2, ..., y_n) := \varphi^{-1}$. By fundamental theorem of calculus, there are C^∞ functions in some neighborhood of 0 $\{\psi_i\}_{i=1}^n$ such that $f(x^1, ..., x^n) = \Sigma_{i=1}^n y_i \psi_i$ and $\psi_i(0) = 0$ $(\forall i)$. By (2.4.1), $v(f) = 0$. □

2.5 Functional Analysis

2.5.1 $L^1(\mathbb{R}^n)$

Proposition 2.5.1. *Let us fix $\epsilon > 0$. Then there is $j_\epsilon \in C_c(\mathbb{R}^n) \cup C_+(\mathbb{R}^n)$ such that*

(i) j_ϵ is a probability density function on $\mathbb{R}^n$.

(ii) $supp(j_\epsilon) \subset B(0, \epsilon)$.

The following proposition is easy to show.

Proposition 2.5.2. *Let*

(S1) j_ϵ is the function in Proposition2.5.1.

(S2) $f \in L^1(\mathbb{R}^n)$.

Then

(i) $j_\epsilon * f \in C^\infty(\mathbb{R}^n)$

(ii) $supp(j_\epsilon * f) \subset \{x \in \mathbb{R}^n | d(x, supp(j_\epsilon * f)) \leq \epsilon\}$

(iii) $||j_\epsilon * f||_1 \leq ||f||_1$

(iv) $\lim_{\epsilon \to 0} j_\epsilon * f = f$ in $L^1(\mathbb{R}^n)$.

(i) and (ii). It is easy to show. □

(iii) and (iv). It is enable to show by an approach which is similar to the approach in the proof of Proposition2.5.6. □

By (iv) of Proposition2.5.2 and Proposition11.5.12, the following holds.

Proposition 2.5.3. *$C_c^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$.*

Proposition 2.5.4. *Let*

(S1) $\{f_n\}_{n=1}^\infty \subset L^1(\mathbb{R}^n)$ and $f \in L^1(\mathbb{R}^n)$.

(A1) $\lim_{n\to\infty} f_n = f$ in $L^1(\mathbb{R}^n)$.

then $\lim_{n\to\infty} f_n = f$ (almost everyware pointwise convergence).

Proof. Let us fix arbitary $m \in \mathbb{N}$. We set

$$E_m := \{x \in \mathbb{R}^n | \varliminf_{n\to\infty} |f_n(x) - f(x)| \geq \frac{1}{m}\} \tag{2.5.1}$$

It is enough to show E_m is zero set.

$$\frac{1}{m}\mu(E_m) \quad \leq \quad ||f_n - f||_1 \to 0$$

□

2.5.2 Fourier Transform

Definition 2.5.5. *Let $\epsilon > 0$ and $n \in \mathbb{R}$.*

$$G_\epsilon(x) := \frac{1}{(2\pi\epsilon^2)^{\frac{n}{2}}} exp(-\frac{|x|^2}{2\epsilon^2}) \ (x \in \mathbb{R}^n) \tag{2.5.2}$$

Proposition 2.5.6. *The followings hold.*

(i) $G_\epsilon > 0$ *on* $\mathbb{R}^n$ $(\forall \epsilon > 0)$.

(ii) $\int_{\mathbb{R}^n} G_\epsilon dx = 1$.

(iii) *For any* $\delta > 0$, $\lim_{\epsilon \to 0} \int_{|x|>\delta} G_\epsilon dx = 0$.

(iv) *For any* $f \in L^1(\mathbb{R}^n)$, $||G_\epsilon * f||_1 \leq ||f||_1$.

(v) *For any* $f \in L^1(\mathbb{R}^n)$, $\lim_{\epsilon \to 0} G_\epsilon * f = f$ *in* $L^1(\mathbb{R}^n)$.

(vi) $\mathscr{F}^{-1}(\mathscr{F}(G_\epsilon)) = G_\epsilon$ $(\forall \epsilon > 0)$

(i) and (ii). Because G_ϵ is the probability denity function of $N(0, \epsilon E_n)$, (i) and (ii) hold. □

(iii). Because $\int_{|x| \leq \delta} G_\epsilon(x)dx = \int_{|x| \leq \frac{\delta}{\epsilon}} G_1(x)dx$, (iii) holds. □

(iv). By (i) and (ii),

$$\begin{aligned}
\int_{\mathbb{R}^n} |G_\epsilon * g(x)|dx &= \int_{\mathbb{R}^n} |\int_{\mathbb{R}^n} G_\epsilon(x-y)g(y)dy|dx \\
&\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_\epsilon(x-y)|g(y)|dydx \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_\epsilon(x-y)dx|g(y)|dy \\
&= \int_{\mathbb{R}^n} |g(y)|dy
\end{aligned}$$

□

(v). By (iv) and Proposition11.5.12, we can assume $f \in C_c(\mathbb{R}^n)$.
By Lebesugue's convergence theorem and (iv), it is enough to show $G_\epsilon * f$ pointwize converges to f.
Let us fix arbitary $\epsilon > 0$. Because f is uniform continuous on $\mathbb{R}^n$, $|f(x) - f(y)| < \frac{\epsilon}{2}$ (for any x, y such that $|x - y| < \delta$).
By (iii), there is $\tau_0 > 0$ such that $\int_{|x|>\delta} G_\tau dx < \frac{\epsilon}{2(2||f||_\infty + 1)}$ (for any $\tau < \tau_0$).
By (ii), for any $x \in \mathbb{R}^n$

$$\begin{aligned}
|G_\epsilon * f(x) - f(x)| &= |\int_{\mathbb{R}^n} G_\epsilon(y)(f(x-y) - f(x))dy| \\
&\leq \int_{|x|<\delta} G_\epsilon(y)|f(x-y) - f(x)|dy \\
&+ \int_{|x|\geq\delta} G_\epsilon(y)|f(x-y) - f(x)|dy \\
&\leq \frac{\epsilon}{2} + 2||f||_\infty \int_{|x|\geq\delta} G_\epsilon(y)dy \\
&\leq \epsilon
\end{aligned} \tag{2.5.3}$$

□

(vi). By Proposition11.5.2, (vi) holds. □

Proposition 2.5.7 (Inverse formula)**.** *For any* $f \in L^1(\mathbb{R}^n)$ *such that* $\mathscr{F}(f) \in L^1(\mathbb{R}^n)$,

$$f = \mathscr{F}^{-1}(\mathscr{F}(f)) \tag{2.5.4}$$

Proof. By (v) in Proposition2.5.6 and Proposition2.5.4, it is enough to show $G_\epsilon * f$ pointwize converges to $\mathscr{F}^{-1}(\mathscr{F}(f))$ on $\mathbb{R}^n$.

By (vi) in Proposition2.5.6 and Proposition11.5.2, for any $x \in \mathbb{R}^n$

$$\begin{aligned} G_\epsilon * f(x) &= \mathscr{F}^{-1}(\mathscr{F}(G_\epsilon)) * f(x) \\ &= \int_{\mathbb{R}^n} \mathscr{F}^{-1}(\mathscr{F}(G_\epsilon)(x-y))f(y)dy \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathscr{F}(G_\epsilon)(\xi)exp(i(x-y)\xi)d\xi f(y)dy \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathscr{F}(G_\epsilon)(\xi)exp(ix\xi)exp(-iy\xi)d\xi f(y)dy \\ &= \int_{\mathbb{R}^n} \mathscr{F}(G_\epsilon)(\xi)exp(ix\xi)\int_{\mathbb{R}^n} \frac{1}{(2\pi)^{\frac{n}{2}}} exp(-iy\xi)f(y)dyd\xi \\ &= \int_{\mathbb{R}^n} \mathscr{F}(G_\epsilon)(\xi)exp(ix\xi)\mathscr{F}(f)(\xi)d\xi \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} (2\pi)^{\frac{n}{2}}\mathscr{F}(G_\epsilon)(\xi)\mathscr{F}(f)(\xi)exp(ix\xi)d\xi \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} exp(-\frac{\epsilon^2}{2}|\xi|^2)\mathscr{F}(f)(\xi)exp(ix\xi)d\xi \end{aligned} \tag{2.5.5}$$

By Lebesuge's convergence theorem,

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} exp(-\frac{\epsilon^2}{2}|\xi|^2)\mathscr{F}(f)(\xi)exp(ix\xi)d\xi \to \mathscr{F}^{-1}(\mathscr{F}(f))(x) \tag{2.5.6}$$

□

Proposition 2.5.8 (Differential formula)**.** *Let*

(S1) $f \in C_c^\infty(\mathbb{R}^n)$.

(S2) $\boldsymbol{\alpha} \in \mathbb{Z}^n \cup [0,\infty)^n$.

(S3) $m := \Sigma_{i=1}^n \alpha_i$.

Then

(i) $D^{\boldsymbol{\alpha}} f \in C_c^\infty(\mathbb{R}^n)$ *and*

$$\mathscr{F}(f)(D^{\boldsymbol{\alpha}} f) = (i\xi)^{\boldsymbol{\alpha}}\mathscr{F}(f) \tag{2.5.7}$$

(ii) $\mathscr{F}(f) \in L^1(\mathbb{R}^n)$.

(i). It is enable to show by using integration by parts. □

(ii). It is enable to show by (i). □

2.5.3 Schwarz's Inequality

I will show a roundabout proof of Schwarz's Inequality. For a short proof, see([12]).

Theorem 2.5.9. *H is a inner product space. Then*

$$|(u,v)| \leq ||u||||v|| \ (\forall u \in H, \forall v \in H)$$

We can gave a proof of the above theorem without the following assumption

$$u \neq 0 \ then \ ||u|| \neq 0 \ (\forall u \in H) \tag{2.5.8}$$

Firstly, the above theorem is obviously true when $(u,v) = 0$. So we can assume $(u,v) \neq 0$. Let $\theta \in [0, 2\pi)$ be a argument of (u, v) and let $u' := exp(i(-\theta))u$. $exp(i(-\theta))(u,v)$ is a real positive number. So, if u', v satisfies Shuwartz' inequality, u, v satisfies the one since

$$\begin{aligned} |(u,v)| &= |exp(i(-\theta))(u,v)| \\ &= exp(i(-\theta))(u,v) \\ &= (u',v) \\ &\leq ||u'||||v|| \\ &= ||u||||v|| \end{aligned} \tag{2.5.9}$$

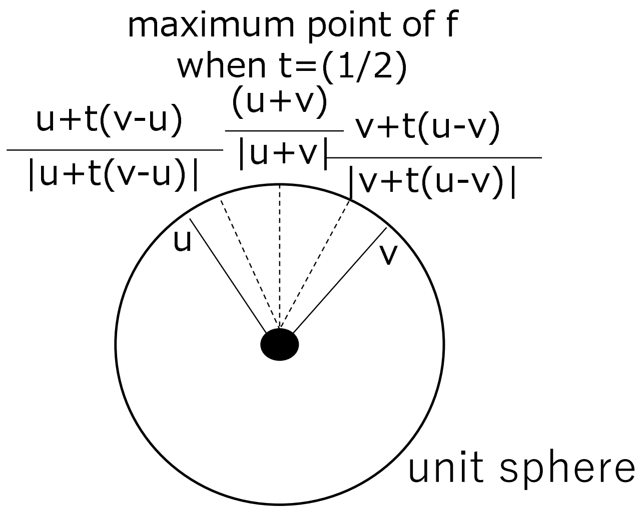

Figure 2.1: Image of poins on unit sphere

So we can assume

$$(u, v) \in \mathbb{R} \; and \; (u, v) > 0 \tag{2.5.10}$$

Hereafter, on these assumptions, we will show

- If $||u||||v|| \neq 0$, the above theorem is true
- $||u||||v|| \neq 0$

2.5.3.1 The theorem is true when u and v have not zero norm

In this case, we can assume u, v is in unit sphere. The above theorem claims that $(\frac{u}{||u||}, \frac{v}{||v||})$ reachs the maximum value 1 when $u = v$ on unit sphere $\{u \in H| \; ||u|| = 1\}$. By seeing Figure2.5.3.1 so we can guess

$$f : [0, 1] \ni t \mapsto \frac{(u + t(v - u), v + t(u - v))}{(||u + t(v - u)||)(||v + t(u - v)||)} \in \mathbb{R} \tag{2.5.11}$$

reachs the maximum value 1 at $t = \frac{1}{2}$.

The following is true

$$f(0) = (u, v) \tag{2.5.12}$$

.

It is enought to show the following proposition

Proposition 2.5.10.

$$f_p : [0, 1] \ni t \mapsto \frac{(u + t(v - u), v + t(u - v))}{((||u + t(v - u)|| + p)(||v + t(u - v)|| + p))} \in \mathbb{R} \tag{2.5.13}$$

reachs the maximum at $t = \frac{1}{2}$ *for all* $p > 0$.

Actually, if Proposition2.5.10 is true, then $f_p(0) \leq f_p(\frac{1}{2})$ $(\forall p > 0)$. So

$$\frac{(u, v)}{(1 + p)^2} \leq (\frac{\frac{1}{4}||u + v||^2}{(\frac{1}{2}||u + v|| + p)(\frac{1}{2}||u + v|| + p)}) \; (\forall p > 0) \tag{2.5.14}$$

Since $(u, v) > 0$, $||u + v|| > 0$. Reaching $p \to 0$, we get Shwartz Inequality

$$(u, v) \leq 1 = ||u||||v|| \tag{2.5.15}$$

So, hereafter, we will show Proposition2.5.10.

We define

$$a : \mathbb{R} \ni t \mapsto (u + t(v - u), v + t(u - v)) \in \mathbb{R} \tag{2.5.16}$$

So, we get

$$a(t) = -||u - v||^2 t^2 + ||u - v||^2 t + (u, v) \ (\forall t \in \mathbb{R}) \tag{2.5.17}$$

So a reachs the maximum at $t = \frac{1}{2}$. We notice a reachs the maximum at $t = \frac{1}{2}$ even if u, v is not in unit sphere.

$$||u + t(v - u)||^2 = ||v + t(u - v)||^2 = ||v - u||^2 t^2 + (2(u, v) - 2)t + 1 \tag{2.5.18}$$

and

$$||v - u||^2 = -(2(u, v) - 2) \tag{2.5.19}$$

So if we define

$$b : \mathbb{R} \ni t \mapsto ((||u + t(v - u)|| + p)(||v + t(u - v)|| + p)) \in \mathbb{R} \tag{2.5.20}$$

then b reachs the minimum at $t = \frac{1}{2}$. So the theorem is true when $||u||||v|| \neq 0$.

2.5.3.2 u and v have not zero norm

We will show $||u||||v|| \neq 0$. Firstly we assume $||u|| = ||v|| = 0$. Then

$$a(\frac{1}{2}) = \frac{1}{2}(u, v) \geq a(0) = (u, v) \tag{2.5.21}$$

So $(u, v) = 0$, this contradicts with (2.5.10). Secondly we assume $||u|| = 0,\ ||v|| \neq 0$. We can assume $||v|| = 1$. We define for positive number p

$$a_p : \mathbb{R} \ni t \mapsto (pu + t(v - pu), v + t(pu - v)) \in \mathbb{R} \tag{2.5.22}$$

Similarly to the above discussion a_p reachs the maximum at $t = \frac{1}{2}$.

$$0 \leq a_{\frac{1}{(u,v)}}(\frac{1}{2}) - a(0) = 1 - 2\frac{1}{(u, v)}(u, v) = -1 \tag{2.5.23}$$

This is a contradiction. So $||u||||v|| \neq 0$.

2.5.4 Projection

Proposition 2.5.11 (Bessel Inequality)**.** *Let*

(S1) V is a inner product space.

(S2) $\{v_i\}_{i=1}^N$ is a orthonormal system of V.

Then for any $u \in V$,

$$\sum_{i=1}^{N} |(u, v_i)|^2 \leq ||u||^2$$

Proof. By (S2),

$$0 \leq ||u - \sum_{i=1}^{N}(u, v_i)v_i||^2 = ||u||^2 - \sum_{i=1}^{N} |(u, v_i)|^2$$

This impliese the above inequality. □

Proposition 2.5.12. *Let*

(S1) V is a separable Hilbert space.

(S2) $\{v_i\}_{i=1}^\infty$ is a complete orthonormal system of V.

Then

(i) If $u \in V$ and $(u, v_i) = 0$ $(\forall i)$, then $u = 0$.

(ii) For any $u \in V$, $\sum_{i=1}^{\infty}(u, v_i)v_i$ converges and

$$u = \sum_{i=1}^{\infty}(u, v_i)v_i$$

(iii) *Any complete orthonormal system of* V *is countable.*

Proof of (i). We set $W := \sum_{i=1}^{\infty} \mathbb{C} v_i$. There is a sequcen $\{w_i\}_{i=1}^{\infty} \subset W$ such that $\lim_{i\to\infty} w_i = u$. So,

$$||u||^2 = \lim_{i\to\infty}(u, w_i) = 0$$

This implies $u = 0$. □

Proof of (ii). By bessel inequality, $\{\sum_{i=1}^{N}(u, v_i)v_i\}_{N\in\mathbb{N}}$ is a cauchy sequence in V. Because V is complete, $\sum_{i=1}^{\infty}(u, v_i)v_i$ converges. Because $(u - \sum_{i=1}^{\infty}(u, v_i)v_i, v_j) = 0$ $(\forall j)$, by (i), (ii) holds. □

Proof of (iii). Let us fix $\{w_\alpha\}_{\alpha\in\Lambda}$ which is any complete orthonormal system of V. For each $m, n \in \mathbb{N}$, there is a finite subset $\Lambda_{m,n} \subset \Lambda$ such that

$$d(v_m, \sum_{\alpha\in\Lambda_{m,n}} \mathbb{C} w_\alpha) < \frac{1}{n}$$

We set $\Lambda^* := \cup_{m,n}\Lambda_{m,n}$. Clearly Λ^* is at most countable and $\{w_\alpha\}_{\alpha\in\Lambda^*}$ is a complete orthonormal system of V. So, $\Lambda^* = \Lambda$. □

Proposition 2.5.13 (Projection Theorem)**.** *Let*

(S1) V *is a Hilbert space.*

(S2) W *is a closed subspace of* V*.*

then

$$V = W \oplus W^{\perp}$$

So, for each $v \in V$*, there is a unique* $w \in W$ *such that* $v - w \in W^{\perp}$*. We call* w *is the orthogonal projection of* v*. We set* $p_W : V \to W$ *by*

$$p_M : V \ni v \mapsto w \in W s.t. v - w \in W^{\perp}$$

We call p_W *is the orthogonal projection of* W*.*

Proof in general case. Let us fix any $v \in W$. We set

$$d := d(v, W)$$

Then there is $\{w_i\}_{i=1}^{\infty} \subset W$ such that

$$\lim_{n\to\infty} ||v - w_i|| = d$$

We will show $\{w_i\}_{i=1}^{\infty}$ is a cauchy sequence. For any $m, n \in \mathbb{N}$,

$$||w_m - w_n||^2 = ||v_m - w||^2 - 2Re(w_m - w, w_n - w) + ||w_n - w||^2$$

And

$$2Re(w_m - w, w_n - w) = ||(w_m - w) + (w_n - w)||^2 - ||w_m - w||^2 - ||w_n - w||^2$$

So,

$$||w_m - w_n||^2 + 4||\frac{w_m + w_n}{2} - w||^2 = 2||w_m - w||^2 + 2||w_n - w||^2$$

Because

$$||w_m - w_n||^2 + 4||\frac{w_m + w_n}{2} - w||^2 \geq ||w_m - w_n||^2 + 4d^2$$

$$||w_m - w_n||^2 \leq 2||w_m - w||^2 + 2||w_n - w||^2 - 4d^2$$

So, $\{w_i\}_{i=1}^{\infty}$ is a cauchy sequence. Because V is Hilbert space,

$$w := \lim_{n\to\infty} w_n$$

exists. Because W is closed, $w \in W$.

$$||v - w||^2 = ||v - w_n + w_n - w||^2 = ||v - w_n||^2 + 2Re(v - w_n, w_n - w) + ||w_n - w||^2$$

So,

$$||v - w||^2 = d^2$$

We set

$$u := v - w$$

Let us assume $u \notin W^{\perp}$. Then there is $w_0 \in W$ such that $(u, w_0) > 0$. So, for any $\delta > 0$

$$d^2 \leq ||u - \delta w_0||^2 = d^2 - 2\delta Re(u, w_0) + \delta^2 ||w_0||^2$$

This implies

$$2Re(u, w_0) \leq \delta ||w_0||^2$$

So, if we take $\delta < \dfrac{2Re(u, w_0)}{||w_0||^2}$, a contradiction arises. So $u \in W^{\perp}$. □

Proof in case W is separable. Because W is separable, by Gram-Schmit orthogonalization method, there a $\{w_i\}_{i=1}^{\infty}$ which is a complete orthonormal system of W. Let us fix any $u \in V$. By the same argument as the proof of Proposition2.5.12, $w := \sum_{i=1}^{\infty} (u, w_i) w_i$ converges. Because W is closed, $w \in W$. Clearly $u - w \perp W$. □

By the argument in the proof of Propoisition2.5.13, the following holds.

Proposition 2.5.14. *Let*

(S1) V is a pre Hilbert space.

(S2) W is a subspace of V.

(S3) $v \in V$.

(S4) $\{v_n\}_{n \in \mathbb{N}} \subset V$ such that

$$\lim_{n \to \infty} ||v - v_n|| = \inf_{u \in W} ||v - u||$$

then $\{v_n\}_{n \in \mathbb{N}}$ is a cauchy space.

Proposition 2.5.15. *Let*

(S1) V is a Hilbert space.

(S2) W is a closed subspace of V.

(A1) $p : V \to W$ is a surjective self adjoint linear operator such that $p^2 = p$.

then p is the orthogonal projection of W.

Proof. Let us set p_W the orthogonal projection of W. Let us fix any $v \in V$ and $w := p_W(v)$. Then, firstly, $p(v) - w \in W$ and there is $v' \in V$ such that $p(v') = w$.

$$p(v) - w = p(v) - p(v') = p(v) - p^2(v') = p(v) - p(w) = p(v - w)$$

Because $v - w \in W^{\perp}$, for any $w' \in W$,

$$(p(v) - w, w') = (p(v - w), w') = (v - w, p^* w') = (v - w, p(w')) = 0$$

So, $p(v) - w \in W^{\perp}$. These imply $p(v) = w$. □

By Proposition2.5.15, the following holds.

Proposition 2.5.16. *Let*

(S1) V is a Hilbert space.

(S2) $W_1, ..., W_m$ are closed subspace of V and $W_i \perp W_j$ $(\forall i \neq \forall j)$.

(A1) $p_i : V \to W_i$ is the orthogonal projection to W_i $(i = 1, 2, ..., m)$.

then

$$p := \sum_{i=1}^{m} p_i$$

is the orthogonal projection of $\oplus_{i=1}^{m} W_i$.

Proposition 2.5.17. *Let*

(S1) V is a Hilbert space.

(S2) $\{W_i\}_{i\in I}$ *is a family of closed subspaces of* V.

(A1) $W_i \perp W_j$ $(\forall i \neq \forall j)$.

(A2) $V = \oplus_{i\in I} W_i$.

(S3) We denote the orthogonal projection of W_i *by* p_i $(i \in I)$.

then for any $v \in V$

$$\inf\{||v - \sum_{j\in J} P_j v|| \; |J \subset I : finite\} = 0$$

Proof. Let us fix any $v \in V$ and $\epsilon > 0$. By (A2), there are $J \subset I$:finite and $\{v_i\}_{i\in J}$ such that $v_i \in W_i$ $(\forall i \in J)$ and $||v - \sum_{i\in J} v_i|| < \epsilon$. We set $p := \sum_{i\in J} P_i$. By Proposition2.5.16, p is the orthogonal projection of $\oplus_{i\in J} W_i$. By the proof of Projection theorem, $||v - p(v)|| \leq ||v - \sum_{i\in J} v_i||$. So, $||v - \sum_{j\in J} P_j v|| < \epsilon$. □

Proposition 2.5.18 (Riez representation theorem)**.** *Let*

(S1) V *is a Hilbert space.*

(S2) $f \in V^*$.

Then there is $u \in V$ *such that*

$$f(\cdot) = (\cdot, u)$$

Proof. We set $W := Ker(f)$. We can assume $f \neq 0$. Let us take $w_0 \in W^\perp \setminus \{0\}$. We can assume $f(w_0) = 1$. Let us fix $v \in V$ and $u := v - f(v)w_0$. Clearly $u \in W$, so $u \perp w_0$. This implies

$$(v, w_0) = f(v)||w_0||^2$$

□

2.5.5 Unit Sphere

Proposition 2.5.19. *Here are the settings and assumptions.*

(S1) V *is an inner product space.*

(A1) $\{v \in V| ||v|| = 1\}$ *is compact.*

Then $dimV < \infty$.

Proof. Let us assume $dimV = \infty$. Then there is a orthonormality $\{v_i\}_{i=1}^\infty \subset V$. Because there is no subsequence of $\{v_i\}_{i=1}^\infty$ which converges in V, $\{v \in V| ||v|| = 1\}$ is not compact. This is contradiction. □

Proposition 2.5.20. *Let*

(S1) V *is a Hilbert space.*

(S2) $\{v_i\}_{i=1}^\infty \subset \{v \in V| ||v|| = 1\}$.

Then there is subsequence $\{v_\varphi(i)\}_{i=1}^\infty$ *and* $v \in V$ *such that for any* $f \in V^*$

$$\lim_{i\to\infty} f(v_{\varphi(i)}) = f(v)$$

We denote this by

$$w - \lim_{i\to\infty} v_{\varphi(i)} = v$$

Proof. Because $(v_i, v_j) \in \mathbb{T}_1(\forall i, j)$ and $\mathbb{T}_1$ is compact, then there are subsequences $\{v_{\varphi_n(k)}\}_{k=1}^\infty$ $(n = 1, 2, ...)$ such that for each $n \in \mathbb{N}$ $\{v_{\varphi_n(k)}\}_{k=1}^\infty$ is a subsequence of $\{v_{\varphi_{n+1}(k)}\}_{k=1}^\infty$ and $\lim_{k\to\infty}(v_{\varphi_n(k)}, v_n)$ exists. We set

$$\psi(n) := \varphi_n(n) \; (n \in \mathbb{N})$$

Then for any $n \in \mathbb{N}$, $f(v_n) := (\lim_{k\to\infty}(v_n, v_{\psi(k)})$ exists. We set V_0 be the minimum sublinear space which contains $\{v_i\}_{i=1}^\infty$ and $V_1 := \bar{V_0}$. Let us fix any $w \in \bar{V_1}$. Then there is $\{w_i\}_{i=1}^\infty \subset V_0$ such that $\lim_{i\to\infty} w_i = w$. Let us fix any $\epsilon > 0$. Then there is $n_0 \in \mathbb{N}$ for any $m, n \geq n_0$ $||w_m - w_n|| \leq \epsilon$. $|f(w_m) - f(w_n)| = |f(w_m - w_n)| \leq ||w_m - w_n|| \leq \epsilon$. So, $f(w) := \lim_{n\to\infty} f(w_n)$ exists. Clearly $||f|| \leq 1$. So $f \in V_1^*$. By Riez representation theorem, there is $v \in V_1$ such that

$f = (\cdot, v)$. Let us fix any $u \in \bar{V_1}$ and $\epsilon > 0$. Then there is $u' \in V_0$ such that $||u - u'|| < \frac{\epsilon}{2}$. There is $n_0 \in \mathbb{N}$ such that for any $k \geq n_0$ $|(u', v_{\psi(k)}) - (u', v)| \leq \frac{\epsilon}{2}$. So $|(u, v_{\psi(k)}) - (u, v)| \leq \epsilon$. This means

$$\lim_{k\to\infty} (u, v_{\psi(k)}) = (u, v) \tag{2.5.24}$$

Let us fix any $g \in V^*$. Then $g|V_1 V_1^*$. By Riez representation theorem, there is $u_g \in V_1$ such that $g|V_1 = (\cdot, u_g)$. So,

$$\lim_{k\to\infty} g(v_{\psi(k)}) = g(v) \tag{2.5.25}$$

□

2.5.6 Miscellaneous

The following clearly holds.

Proposition 2.5.21. *Any finite linear subspace of a Hilbert space is closed.*

2.6 Topological group and representation

Definition 2.6.1 (Topological group)**.** *We call G is a topological group if G is a housdorff space and G is a group and $G \times G \ni (x, y) \mapsto xy \in G$ is continuous and $G \ni x \mapsto x^{-1} \in G$ is continuous.*

Proposition 2.6.2. *Let G is a topological group. Then the followings hold.*

(i) $i : G \ni x \mapsto x^{-1} \in G$ is isomorphism.

(ii) For any $g \in G$, $L_g : G \ni x \mapsto gx \in G$ is isomorphism.

(iii) For any $g \in G$, $R_g : G \ni x \mapsto xg \in G$ is isomorphism.

Proof of (i). For any open set U in G, $i(U) = i^{-1}(U)$. Because i is continuous, i is open map. So i is isomorhism. □

Proof of (ii). For any open set U in G, $L_g(U) = L_{(g^{-1})^{-1}}(U)$. Because $L_{g^{-1}}$ is continuous, L_g is open map. So L_g is isomorhism. □

Proof of (iii). It is possible to show (iii) by the approach which is similar to (ii). □

Proposition 2.6.3 (Semidirectproduct of groups)**.** *Let*

(i) G, H are groups.

(ii) $\sigma : G \to Aut(H)$ is a homomorphism of group.

(iii) We set

$$(g_1, h_1) \cdot (g_2, h_2) := (g_1 g_2, h_1 \sigma(g_1)(h_2)) \ (g_1, g_2 \in G, h_1, h_2 \in H)$$

Then $G \times H$ is a group with $\cdot$. We denote this group by $G \ltimes_\sigma H$.

Proof. Clearly $(1_G, 1_H)$ is the unit element of $G \ltimes_\sigma H$. Let us fix any $g_1, g_2, g_3 \in G$ and $h_1, h_2, h_3 \in H$.

$$\begin{aligned}
&(g_1, h_1) \cdot ((g_2, h_2) \cdot (g_3, h_3)) = (g_1, h_1) \cdot (g_2 g_3, h_2 \sigma(g_2)(h_3)) = (g_1 g_2 g_3, h_1 \sigma(g_1)(h_2 \sigma(g_2)(h_3))) \\
&= (g_1 g_2 g_3, h_1 \sigma(g_1)(h_2) \sigma(g_1)(\sigma(g_2)(h_3)))) = (g_1 g_2 g_3, h_1 \sigma(g_1)(h_2) \sigma(g_1 g_2)(h_3))) = (g_1 g_2, h_1 \sigma(g_1)(h_2))(g_3, h_3) \\
&= ((g_1, h_1) \cdot (g_2, h_2)) \cdot (g_3, h_3)
\end{aligned}$$

So, the associativity of $\cdot$ holds. For every $(g, h) \in G \ltimes_\sigma H$, $(g^{-1}, \sigma(g)(h)^{-1} h^{-1})$ is the inverse element of (g, h). Consequently, $G \ltimes_\sigma H$ is a group. □

Definition 2.6.4 (Representation of group)**.** *Let G be a group and V be a vector space on a field K. We call $\pi : G \to End_K(V)$ a representation of G if $\pi(1_G) = id_V$ and $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$ $(\forall g_1, g_2 \in G)$.*

Definition 2.6.5 (Continuous Representation of Group)**.** *Let G be a topological group and V be a Hilbert space on a field K. We call $\pi : G \to End_K(V)$ a continuous representation of G if (π, V) is a representation of G and $G \times V \ni (g, v) \mapsto \pi(g)v \in V$ is continuous.*

Definition 2.6.6 (Unitary Representation of Group). *Let G be a group and V be a Hilbert space on a field K. We call $\pi : G \to End_K(V)$ a unitary representation of G if (π, V) is a representation of G and $\pi(g)$ is a unitary operator for any $g \in G$.*

Definition 2.6.7 (Subrepresentation). *Let (π, V) be a continuous unitary representation of a topological group G and W be an invariant closed subspace of G. We call $(\pi|W, W)$ is a subrepresentation of π. We denote $\pi|W$ by π_1. We denote this by $\pi_1 < \pi$. And let (π_2, V_2) be a continuous unitary representation of a topological group G. We denote $\pi_2 \prec \pi$ if π_2 is isomorphic to a subrepresentation of G as continuous unitary representations.*

Proposition 2.6.8. *Let*

- *(S1) G is a topological group.*
- *(S2) (π, V) is a finite dimensional continuous representations of G.*

Then

$$G \ni g \mapsto \pi(g) \in GL(V)$$

is continuous.

Proof. Let us take $\{v_i\}_{i=1}^r$ such that $\{v_i\}_{i=1}^r$ is a orthonormal basis of V. For any $g_1, g_2 \in G$ and i, j

$$||(\pi(g_1)v_i, v_j) - (\pi(g_2)v_i, v_j)|| \leq ||\pi(g_1)v_i - \pi(g_2)v_i||$$

So, $(\pi(\cdot)v_i, v_j)$ is continuous. □

Proposition 2.6.9. *Let*

- *(S1) V is a vector space on $K := \mathbb{R}$ or $\mathbb{C}$.*
- *(S2) $A \in End_K(V)$.*
- *(S3) $A^*(f)(u) := f(Au)$ $(f \in V^*, u \in V)$.*

Then $A^* \in End_K(V^*)$.

Proof. For any $a, b \in K$ and $f, g \in V^*$ and $u \in V$,

$$A^*(af + bg)(u) = (af + bg)(Au) = af(Au) + bg(Au) = a(A^*f)(u) + b(A^*g)(u) = (a(A^*f) + b(A^*g))(u)$$

□

Proposition 2.6.10 (Contragredient representation). *Let*

- *(S1) G is a topological group.*
- *(S2) (π, V) is a representations of G.*

Then

- *(i) The following π^* is a homomorphism as groups.*

$$\pi^* : G \ni g \mapsto \pi(g^{-1})^* \in GL_{\mathbb{C}}(V)$$

We call π^* a the contragredient representation of π.

- *(ii) If (π, V) is a finite dimensional continuous representations of G, then π^* is continuous.*

Proof of (i). For any $g, h \in G$ and $f \in V^*$ and $u \in V$,

$$\pi^*(gh)f(u) = f(\pi(gh)^{-1}u) = f(\pi(h)^{-1}\pi(g)^{-1}u) = (\pi^*(h)f)(\pi(g)^{-1}u) = \pi^*(g)(\pi^*(h)f)(u)$$

□

Proof of (ii). Let us fix $\{v_1, .., v_m\}$ an orhonormal basis of V. We set $f_i := (\cdot, v_i)$ $(i = 1, 2, ..., m)$.

$$\pi(g)f(u) = f(\sum_{i=1}^{m} \pi(g^{-1}(u, v_i)v_i) = \sum_{i=1}^{m}(u, v_i)f(\pi(g^{-1})v_i) = \sum_{i=1}^{m} f(\pi(g^{-1})v_i)f_i(u)$$

So, π^* is continuous. □

Definition 2.6.11 (Intertwining operator, G-linear map.)**.** *Let*

(S1) G is a topological group.

(S2) (π_1, V_1) and (π_2, V_2) are representations of G.

We say $T : V_1 \to V_2$ is an intertwining operator or a G-linear map if T is a linear and

$$T \circ \pi_1 = \pi_2 \circ T$$

If π_1 and π_2 are continuous representations of G, we denote the set of all continous G-linear mapping from π_1 to π_2 by

$$Hom_G(V_1, V_2) \text{ or } Hom_G(\pi_1, \pi_2)$$

Definition 2.6.12 (Equivalent between two continuous representations of G)**.** *Let*

(S1) G is a topological group.

(S2) (π_1, V_1) and (π_2, V_2) are continuous representations of G.

We say π_1 and π_2 are equivalent if there is $T : V_1 \to V_2$ such that T is a bijective continuous G-linear and T^{-1} is a continuous G-linear.

Definition 2.6.13 (Equivalent between two unitary representations of G)**.** *Let*

(S1) G is a topological group.

(S2) (π_1, V_1) and (π_2, V_2) are unitary representations of G.

We say π_1 and π_2 are equivalent if there is $T : V_1 \to V_2$ such that T is a bijective unitary G-linear.

Definition 2.6.14 (G-linear map.)**.** *Let*

(S1) G is a topological group.

(S2) (π_1, V_1) and (π_2, V_2) are representations of G.

We say $T : V_1 \to V_2$ is an intertwining operator or a G-linear map if T is a linear and

$$T \circ \pi_1 = \pi_2 \circ T$$

The following is clear.

Proposition 2.6.15. *Let*

(S1) G is a topological group.

(S2) (π, V) is a continuous unitary representations of G.

(S2) W is a G-invariant subspace of V.

then $W^\perp$ is also a G-invariant subspace of V.

Definition 2.6.16 (Completely reducible)**.** *Let*

(S1) G is a topological group.

(S2) (π, V) is a continuous representations of G.

We say (π, V) is completely reducible if for any invariant subspace W_1 there is an invariant subspace W_2 such that $V = W_1 + W_2$.

Proposition 2.6.17. *Let*

(S1) G is a topological group.

(S2) (π, V) is a continuous unitary representations of G.

Then (π, V) is completely reducible.

Proof. Because of (S2), for any invarian subspace of W, $W^\perp$ is an invariant subspace. So, (π, V) is completely reducible. □

Proposition 2.6.18. *Let*

(S1) G is a topological group.

(S2) (π, V) is a continuous unitary representations of G.

(S2) W is a G-invariant closed subspace of V.

then the orthogonal projection of W, denoted by P_W, is G-linear.

By Proposition2.6.15, the following holds.

Definition 2.6.19 (Irreducible Representation)**.** *The followings are settings and assumptions.*

(S1) G is a topological group.

(S2) (π, V) is a continuous representation of G.

We say (π, V) is irreducible if there is no closed subspace of V $W \neq \{0\}$ such that $W \neq V$ and W is G invariant.

Proposition 2.6.20. *The followings are settings and assumptions.*

(S1) G is a topological group.

(S2) (π, V) is a finite dimensional irreducible continuous representation of G with a Hilbert space V.

(S3) $v \in V \setminus \{0\}$.

Then

$$\langle \pi(G)v \rangle = V$$

Proof. Since $\langle V \rangle$ is finite dimensional, $\langle \pi(G)v \rangle$ is finite dimensional. Since V is Hilbert space, finite dimensional subspce of V is closed. So, $\langle \pi(G)v \rangle$ is closed. Clearly $\langle \pi(G)v \rangle$ is G invariant. That impliese $\langle \pi(G)v \rangle = V$. □

Proposition 2.6.21. *Let*

(S1) G is a topological group.

(S2) (π, V) is a finite dimensional continuous unitary representations of G.

then (π, V) has an irreducible decomposition.

Proposition 2.6.22 (Shur Lemma)**.** *Let*

(S1) G is a compact Lie group.

(S2) (π_i, V_i) is a continous irreducible representation of G on $\mathbb{C}$ $(i = 1, 2)$.

(A1) Either V_1 or V_2 is finite dimensional.

(S2)

Then

$$Hom_G(V_1, V_2) = \begin{cases} \{0\} & (\pi_1 \not\simeq \pi_2) \\ \mathbb{C}T & (\pi_1 \simeq \pi_2) \end{cases}$$

Here T is an G-isomorphism from V_1 to V_2.

STEP1. Proof of $Hom_G(V_1, V_2) = \{0\}$ $(\pi_1 \not\simeq \pi_2)$. Let us assume $Hom_G(V_1, V_2) \neq \{0\}$. There is $f \in Hom_G(V_1, V_2) \setminus \{0\}$. Because $Ker(f)$ is closed G-invariant, $Ker(f) = \{0\}$. Because of (A1), $Im(f)$ is finite dimensional. By Proposition2.5.21, $Im(f)$ is closed G-invariant subspace of V_2. Becuase π_2 is irreducible, $Im(f) = V_2$. So, V_2 is finite dimensional and f is bijective. Then V_1 is finite dimensional. By Proposition2.5.21, $f^{-1} \in Hom_G(V_2, V_2)$. So, f is an G-isomorphism from V_1 to V_2. □

STEP2. Proof of $Hom_G(V_1, V_2) = \mathbb{C}T(\pi_1 \simeq \pi_2)$. Let us fix any $f \in Hom_G(V_1, V_2) \neq \{0\}$. By STEP1, f is an G-isomorphism from V_1 to V_2.

By (A1), V_1 and V_2 are finite dimensional. So, becuase $T \circ f$ has a eingenvalue λ, $Ker(T^{-1} \circ f - \lambda id) \neq \{0\}$. Because π_1 is irreducible, $Ker(T^{-1} \circ f - \lambda id) = V_1$. So, $f = \lambda T$. □

Proposition 2.6.23. *Let*

(S1) G is a commutative topological group.

(S2) (π, V) is a continous finite dimensional irreducible representation of G on $\mathbb{C}$.

then $\dim \pi = 1$.

Proof. Let us fix $v, w \in V \setminus \{0\}$. Because π is irreducible, $\pi(G)v = V$. So, there is $g \in G$ such that $\pi(g)v = w$. Because G is commutative, $A : V \ni u \mapsto \pi(g)u \in V$ is continuous G-linear and $ImA \neq \{0\}$. So, by Shur Lemma, there is $\lambda \in \mathbb{C}$ such that $A = \lambda id_V$. So, $w = \lambda v$. □

Definition 2.6.24 (Action of group.)**.** *Here are the settings and assumptions.*

(S1) G is a topological group.

(S2) X is a Housdorff space.

Then we say $\Phi \in C(G \times X, X)$ is an action of group G on X if

$$\Phi(e, x) = x \ (\forall x \in X)$$

and

$$\Phi(g_1, \Phi(g_2, x)) = \Phi(g_1 g_2, x) \ (\forall g_1, g_2 \in G, \forall x \in X)$$

Unless there are no confusion, let use denote $\Phi(g, x)$ by $g \cdot x$.

Definition 2.6.25 (Transitive action of group.)**.** *Here are the settings and assumptions.*

(S1) G is a topological group.

(S2) X is a Housdorff space.

(S3) G acts on X.

Then the action is transitive if for any $x \in X$ $G \cdot x = X$.

Definition 2.6.26 (Isotropy group.)**.** *Here are the settings and assumptions.*

(S1) G is a topological group.

(S2) X is a Housdorff space.

(S3) G acts on X.

(S4) $x \in X$.

We call

$$H := \{g \in G | g \cdot x = x\}$$

the isotropy group regarding x.

2.7 Homotopy and Fundamental group

Definition 2.7.1 (Path)**.** *Let*

(S1) X be a topological space.

We call each element of $C([0, 1], X)$ a path. For each $c \in C([0, 1], X)$, we call $c(0)$ the start point of c and $c(1)$ the end point of c. If $c(0) = c(1)$ then we call c a loop.

Definition 2.7.2 (Homotop of continuous maps)**.** *Let*

(S1) X, Y be a topological space.

(S2) $f, g \in C(X, Y)$.

We say f and g are homotop or homotopy equivalent if there is $\Phi \in C([0, 1] \times X, Y)$ such that $\Phi(0, \cdot) = f$ and $\Phi(1, \cdot) = g$. We call Φ a homotopy from f to g.

Definition 2.7.3 (Homotopy equivalent of continuous maps that have the same start point and end point)**.** *Let*

(S1) X be a topological space.

(S2) $x_0, x_1 \in X$.

(S3) $c_1, c_2 \in C([0, 1], X)$ such that $c_1(0) = c_2(0) = x_0$ and $c_1(1) = c_2(1) = x_1$.

We say c_1 and c_2 are homotop or homotopy equivalent preserving x_0 and x_1 if there is $\Phi \in C([0,1]\times[0,1]\times X)$ such that $\Phi(0,\cdot) = c_1$ and $\Phi(1,\cdot) = c_2$ and $\Phi(0,t) = x_0$ and $\Phi(1,t) = x_1$ for any $t \in [0,1]$. We call Φ a homotopy preserving x_0 and x_1. When there is no risk of misunderstanding, we simply refer to Φ as a homotopy.

Clearly, the following holds.

Proposition 2.7.4. *We succeed notations in Definition2.7.3. Homotop on $C(X,Y)$ is an equivalent relation on $C(X,Y)$.*

Definition 2.7.5 (Homotopy equivalent of topological spaces)**.** *Let*

(S1) X, Y be a topological space.

We say X and Y are homotopy equivalent if there is $\Phi \in C([0,1]\times X, Y)$ such that $\Phi(0,\cdot) = f$ and $\Phi(1,\cdot) = g$. We call Φ a homotopy.

Then, clearly, the followings hold.

Proposition 2.7.6 (Fundamental group)**.** *Let*

(S1) X be a topological space.

(S2) $x_0 \in X$.

(S3) Define

(i) Set

$$[([0,1],\partial I),(X,x_0)] := \{c \in C(I,X) | c(\partial I) \subset \{x_0\}\}$$

Here, $I := [0,1]$.

(ii) For each $c_1, c_2 \in [(I,\partial I),(X,x_0)]$,

$$c_1 \sim c_2$$

if there is a homotopy Φ from c_1 to c_2 such that $\Phi(t,\cdot) \in [(I,\partial I),(X,x_0)]$ $(\forall t \in I)$.

(iii) For each $c_1, c_2 \in [(I,\partial I),(X,x_0)]$,

$$c_2 \cdot c_1(t) = \begin{cases} c_1(2t) & (0 \leq t < \frac{1}{2}) \\ c_2(2t-1) & (\frac{1}{2} \leq t \leq 1) \end{cases}$$

(iii) Set

$$\pi_1(X,x_0) := [(I,\partial I),(X,x_0)]/\sim$$

(iv) For each $[c_1],[c_2] \in \pi_1(X,x_0)$

$$[c_2]\cdot[c_1] = [c_2 \cdot c_1]$$

Then $\sim$ is a equivalent relation on $[(I,\partial I),(X,x_0)]$ and $\cdot$ on $\pi_1(X,x_0)$ is well-defined and $\pi_1(X,x_0)$ is a group with respect to $\cdot$. We call $\pi_1(X,x_0)$ the fundamental group of X with base point x_0. If X is path-connected and $\pi_1(X,x_0) = \{e\}$, we say X is simply connected.

Proposition 2.7.7 (n-th Homotopy group)**.** *Let*

(S1) X be a topological space.

(S2) $x_0 \in X$.

(S3) $n \in \mathbb{N}$.

(S4) Define

(i) Set

$$[(I^n,\partial I^n),(X,x_0)] := \{c \in C(I^n,X) | c(\partial I^n) \subset \{x_0\}\}$$

Here, $I^n := [0,1]^n$.

(ii) For each $c_1, c_2 \in [(I^n,\partial I),(X,x_0)]$,

$$c_1 \sim c_2$$

if there is a homotopy Φ from c_1 to c_2 such that $\Phi(t,\cdot) \in [(I^n,\partial I),(X,x_0)]$ $(\forall t \in I)$.

(iii) For each $c_1, c_2 \in [(I := [0,1], \partial I), (X, x_0)]$,

$$c_2 \cdot c_1(t) = \begin{cases} c_1(2t_1, t_2, ..., t_n) & (0 \leq t_1 < \frac{1}{2}) \\ c_2(2t_1 - 1, t_2, ..., t_n) & (\frac{1}{2} \leq t_1 \leq 1) \end{cases}$$

(iii) Set

$$\pi_n(X, x_0) := [(I^n, \partial I^n), (X, x_0)]/ \sim$$

(iv) For each $[c_1], [c_2] \in \pi_n(X, x_0)$

$$[c_2] \cdot [c_1] = [c_2 \cdot c_1]$$

Then $\sim$ *is a equivalent relation on* $[(I^n, \partial I), (X, x_0)]$ *and* $\cdot$ *on* $\pi_n(X, x_0)$ *is well-defined and* $\pi_n(X, x_0)$ *is a group with respect to* $\cdot$*. We call* $\pi_n(X, x_0)$ *the* n*-th homotopy group of* X *with base point* x_0*.*

Chapter 3

Lie Group and Lie Algebra

3.1 Lie Group

Definition 3.1.1 (Locally isomorphism between two topological groups)**.** *Let G and H are topological groups. We say G and H are locally isomorphic if there is $U \subset G$ and $V \subset H$ and isomoriphism $i : U \to G$ such that U is a neighborhood of 1_G and V is a neighborhood of 1_H and the followings hold.*

(i) For any $x, y \in U$ such that $xy \in U$, $i(xy) = i(x)i(y)$.

(ii) For any $x, y \in U$, $xy \in U \iff i(x)i(y) \in V$.

Example 3.1.2. *$\mathbb{R}$ and $\mathbb{T}$ are locally isomorphic.*

Definition 3.1.3 (Lie subgroup of $GL(n, \mathbb{C})$)**.** *We say G is a Lie subgroup of $GL(n, \mathbb{C})$ if the followings hold.*

(i) G is a subgroup of $GL(n, \mathbb{C})$

(ii) G is a topological group

(iii) There is a neighborhood of e in G V such that

(iii-1) The topology of V is relative topology of $GL(n, \mathbb{C})$

(iii-2) There is a neighborhood of e in $GL(n, \mathbb{C})$ U such that if $x_j \in V$ $(j \in \mathbb{N})$ and $x_j \to x \in U$ then $x \in V$.

(iii-3) G has at most countable connected components.

Proposition 3.1.4. *Let*

(S1) G is a subgroup of $GL(n, \mathbb{C})$.

(A1) G is a topological group.

(A2) G has at most countable connected components.

Then the followings are hold.

(i) G is a Lie subgroup of $GL(n, \mathbb{C})$

(ii) There is V which is a neighborhood of 1_G and is a closed subset of $GL(n, \mathbb{C})$ and the topology of V is relative topology of $GL(n, \mathbb{C})$

Proof of that (ii) $\implies$ (i). We set $U := G$. V and U satisfies the condition (iii) in Definition3.1.3. □

Proof of that (i) $\implies$ (ii). By the condition (iii-1) in Definition3.1.3, there is W such that W is an open subset of $GL(n, \mathbb{C})$ and $V^\circ = V \cap W$. Clearly W is an open neighborhood of $1_{GL(n,\mathbb{C})}$. There is W_0 such that W_0 is an open subset of $GL(n, \mathbb{C})$ and $1_G \in W_0 \subset \bar{W_0} \subset U \cap W$. We set $V' := \bar{W_0} \cap V$. By the condition (iii-1) in Definition3.1.3, there is Z such that Z is an open subset of G and $V \cap W_0 = V \cap Z$. So $V' = \bar{W_0} \cap V$ is a neighborhood of 1_G in G. Because $\bar{W_0} \subset U$, by the condition (iii-2) in Definition3.1.3, V' is closed subset of $GL(n, \mathbb{C})$. □

Proposition 3.1.5. *Let*

(S1) G is a Lie subgroup of $GL(n, \mathbb{C})$.

Then, for any W which is a neighborhood of 1_G in G, there is V' such that V' is a closed subset of $GL(n, \mathbb{C})$ and V' is a neighborhood of 1_G.

Proof. There is $\epsilon > 0$ such that $B(1_G, 4\epsilon) \cap V \subset W \cap V$. Because $V \subset G$,

$$\overline{B(1_G, 2\epsilon)} \cap V \subset B(1_G, 4\epsilon) \cap V \subset W$$

Clearly $\overline{B(1_G, 2\epsilon)} \cap V$ is a closed subset of $GL(n, \mathbb{C})$.

There is Z such that Z is an open subset of G and $1 \in Z$ and $Z \subset V$. By Proposition2.2.15, $Z \cap B(1_G, \epsilon)$ is an open subset of Z. So, there is open subset of G O such that $Z \cap B(1_G, \epsilon) = Z \cap O$. So $Z \cap O$ is an open subset of G and $1 \in Z \cap O \subset \overline{B(1_G, 2\epsilon)} \cap V$. So, $\overline{B(1_G, 2\epsilon)} \cap V$ is a neighborhood of 1_G. By Proposition2.2.15, The topology of $\overline{B(1_G, 2\epsilon)} \cap V$ is the relative topology to $GL(n, \mathbb{C})$. □

Example 3.1.6. *Let λ be a irrelational number. Let $G := exp(i2\pi\lambda\mathbb{Z}) \subset GL(1, \mathbb{C})$. Le us assume G is a topological group respects to the discrete topology. $V := \{1\}$ is a neighborhood of 1 on G and V is a closed subset of $GL(1, \mathbb{C})$. So, G is a Lie subgroup of $GL(1, \mathbb{C})$. Because $\mathbb{T}$ is compact, there is subsequence $\{exp(i2\pi\lambda\varphi(m))\}_{m=1}^{\infty}$ and $x \in \mathbb{T}$ such that*

$$\lim_{m\to\infty} exp(i2\pi\lambda\varphi(m)) = x$$

Because λ is irrelational, $x \notin G$. So, G is not closed subset of $GL(1, \mathbb{C})$.

Definition 3.1.7 (Linear Lie group of $GL(n, \mathbb{C})$)**.** *We call $G \in GL(n, \mathbb{C})$ is a Linear Lie group of $GL(n, \mathbb{C})$ if G is closed subgroup of $GL(n, \mathbb{C})$*

Proposition 3.1.8. *If $G \in GL(n, \mathbb{C})$ is a Linear Lie group of $GL(n, \mathbb{C})$ then G is a Lie subgroup of $GL(n, \mathbb{C})$*

Proof. Clearly G satisfies Definition3.1.3. Because $GL(n, \mathbb{C})$ satisfies the second countable axiom, G satisfies the second countable axiom. So G has at most countable connected components. □

Definition 3.1.9 (General Lie group)**.** *We say G is a Lie group if G is a topological group such that there is a Lie subgroup of $GL(n, \mathbb{C})$ which is locally isomorphic to G.*

Proposition 3.1.10. *Let*

- *(S1) G_1 is a Lie group which is isomorphic to a Lie subgroup G_2 of $GL(n, \mathbb{C})$.*
- *(S2) V which is a neighborhood of 1_{G_2} in G_2 and U which is a neighborhood of 1_{G_1} in G_1 and isomorphism $i : U \to V$ satisfying the conditions in Definition3.1.1..*
- *(S3) $U' \subset U$ and $V' := i(U')$.*

Then $i|U'$ satisfying the conditions in Definition3.1.1.

Proof of condition(i). It is trivial. □

Proof of condition(ii). Let us fix any $x, y \in U'$. Let us assume $xy \in U'$. Then by condition(i), $i(x)i(y) = i(xy) \in V'$. Let us assume $i(x)i(y) \in U'$. Then $xy \in U$. $i(xy) = i(x)i(y) \in U'$. So $xy \in V'$. □

Proposition 3.1.11. *Let*

- *(S1) G_1 is a Lie group which is isomorphic to a Lie subgroup G_2 of $GL(n, \mathbb{C})$.*

Then there is $V := G \cap B(1_{G_2}, \epsilon)$ for some $\epsilon > 0$ which is a compact neighborhood of 1_{G_2} in G_2 and U which is a compact neighborhood of 1_{G_1} in G_1 and isomorphism $\tau : U \to V$ satisfying the conditions in Definition3.1.1.

Proof. Let us fix U and V and $\tau : U \to V$ such that U is a neighborhood of 1_{G_1} and V is a neighborhood of 1_{G_2} and $\tau : U \to V$ is isomorphism satisfying the conditions in Definition3.1.1. There is a open set B_1 of $GL(n, \mathbb{C})$ such that $V^{\circ} = G_2 \cap B_1$. There is $\epsilon > 0$ such that $B(1_{G_2}, 2\epsilon) \subset B_1$. We set $V_2 := \overline{B(1_{G_2}, \epsilon)} \cap G_2$ and $U_1 := \tau^{-1}(V_2)$. Because $\tau^{-1}(G \cap B(1_{G_2}, \epsilon)$ is open set in the relative topology with G and subset of U_1, U_1 is the neighborhood of 1_{G_1}. We set $\eta := \tau^{-1}$. Because $G_2 \cap \overline{B(1_{G_2}, \epsilon)} \subset G_2 \cap B_1 \subset V$, $V_2 = V \cap \overline{B(1_{G_2}, \epsilon)}$. So V_2 is a closed subset of V and U_1 is a closed subset of U.

By Proposition2.2.16 and Propositon2.2.15, $\tau|U_1$ is homeomorphism. So U_1 is compact. Also, by Proposition3.4.1, $\tau|U_1$ satisfies conditions in Definition3.1.1. □

In this note, unless otherwise stated, U and V are assumed to be the neighborhoods obtained in Proposition3.1.11.

Proposition 3.1.12. *Let*

- *(S1) G_1 is a Lie group which is isomorphic to a Lie subgroup G_2 of $GL(n, \mathbb{C})$.*

(S2) V *which is a neighborhood of* 1_{G_2} *in* G_2 *and* U *which is a neighborhood of* 1_{G_1} *in* G_1 *and isomorphism* $i : U \to V$ *satisfying the conditions in Definition3.1.1..*

Then $j := i^{-1}$ *satisfying the conditions in Definition3.1.1.*

Proof of condition(i). Let us fix any $z, w \in V$. Let us assume $zw \in V$. Then $i(j(z))i(j(w)) \in V$. So $j(z)j(w) \in U$. By condition(i), $i(j(z)j(w)) = i(j(z))i(j(w)) = zw$. So $j(z)j(w) = j(zw)$. □

Proof of condition(ii). Let us fix any $z, w \in V$. Let us assume $zw \in V$. By the proof of condition(i), $j(z)j(w) \in U$.
Inversely, let us assume $j(z)j(w) \in U$. Then by condition(ii), $zw = i(j(z))i(j(w) \in V$. □

3.2 Matrix exponential

Definition 3.2.1 (Operator Norm)**.** *For* $X \in M(n, \mathbb{C})$,

$$||X||_{op} := ||X|| := \sup_{||v||=1,\ v \in \mathbb{C}^n} |Xv|$$

Definition 3.2.2. *For* $X \in M(n, \mathbb{C})$,

$$||X||_\infty := \sup\{|x_{i,j}||i, j \in \{1, 2, ..., n\}\}$$

Proposition 3.2.3. *For* $X \in M(n, \mathbb{C})$,

$$||X||_\infty \le ||X||_{op} \le \sqrt{n}||X||_\infty$$

Proof of $||X||_\infty \le ||X||_{op}$*.* For any $i, j \in \{1, 2, ..., n\}$, $|x_{i,j}| \le |Xe_j| \le ||X||$. □

Proof of $||X||_{op} \le \sqrt{n}||X||_\infty$*.* We set $x_i := (x_{i,j})_{j=1}^n$ for each i. For any $u \in \mathbb{C}^n$ such that $|u| = 1$, by Schwartz's inequality,

$$|Xu| \le |((x_1|u), ..., (x_n|u))| \le \sqrt{n} \sup_{i=1,2,...,n} |x_i| \le \sqrt{n}||X||_\infty$$

□

Proposition3.2.3 implies the following.

Proposition 3.2.4. $M(n, \mathbb{C})$ *is banach space with the operator norm.*

Proposition 3.2.5. *Let*

(S1) $X \in M(n, \mathbb{C})$

Then for any eigenvalue λ *of* X

$$|\lambda| \le ||X||$$

Proposition 3.2.6. *Let*

(S1) $M := \{X \in M(n, \mathbb{C})|\ X \text{ is diagonalizable }\}$

Then M *is dense in* $M(n, \mathbb{C})$

Proof. Because M is triangularisable(See 2), there is $P \in GL(n, \mathbb{C})$ such that

$$P^{-1}MP := \begin{pmatrix} \alpha_1 & & * \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}$$

We set for each $0 \le s << 1$

$$E(s) := \begin{pmatrix} s & & 0 \\ & \ddots & \\ 0 & & s^n \end{pmatrix}$$

Because $P^{-1}MP + E(s)$ has not a duplicate eigenvalue, so $P^{-1}MP + E(s)$ is diagonalizable. So $M(s) := M + PE(s)P^{-1}$ is diagonalizable. $\lim_{s \to 0} M(s) = M$. □

Proposition 3.2.7. *(S1) $X \in M(n, \mathbb{C})$*

(S2) f is a power series whose radius of convergence is not less than $R > 0$.

then

(i) If $||X|| < R$ then $f(X)$ exists.

(ii) $f(X)$ is a horomoriphic function for each variable $x_{i,j}$.

Proof of (i). We set $f(x) =: \sum_{i=1}^{\infty} c_i X^i$. By the definition of the radius of convergence,

$$\sum_{i=1}^{\infty} |c_i| ||X||^i < \infty$$

This implies that $\{\sum_{i=1}^{n} c_i X_i\}_{n=1}^{\infty}$ is a cauchy sequence. By Proposition3.2.4, $f(X)$ exists. □

Proof of (ii). We set $f_n(X) := \Sigma_{i=1}^{n} c_i X^i$ for each $n \in \mathbb{N}$. By Proposition3.2.3, for any $K \in (0, R)$, $\{X \in M(n, \mathbb{C}) | ||X|| \leq K\}$ is compact. And,

$$\begin{aligned} & \sup_{||X|| \leq K} ||f_n(X) - f(X)|| && (3.2.1) \\ = & \sup_{||X|| \leq K} || \sum_{i=n+1}^{\infty} c_i X^i || \\ = & \sum_{i=n+1}^{\infty} |c_i| K^i \\ \to & \ 0 \ (n \to \infty) \end{aligned}$$

So $\{f_n\}_{n=1}^{\infty}$ uniformly converges to f on compact sets. By Weierstrass's theorem(See [19]), this implies that f is holomorphic. □

Proposition 3.2.8. *Let*

(S1) $X \in M(n, \mathbb{C})$

(S2) f, h are power series whose radius of convergence is not less than $R > 0$.

(S3) u is a power series whose radius of convergence is not less than $R' > 0$.

(A1) $||X|| < R$.

then the followings hold

(i) If $u = f + h$ and $R = R'$ then $u(X) = f(X) + h(X)$.

(ii) If $u = fh$ and $R = R'$ then $u(X) = f(X)h(X)$.

(iii) If $||f(X)|| < R'$ then $u \circ f(X) = u(f(X))$.

Proof. By Proposition3.2.5, clearly these Propositions hold in M.

By Proposition3.2.7, $u, f + h, fh, u \circ f, u(f(\cdot))$ are continuous on $M(n, \mathbb{C})$. So, by Proposition3.2.6, these Propositions hold at X. □

Proposition 3.2.9. *For any $X \in M(n, \mathbb{C})$*

$$det(exp(X)) = exp(tr(X)) \tag{3.2.2}$$

Proof. Because $det(exp(\cdot))$ and $exp(tr(\cdot))$ are continuous, by Proposition3.2.6, it is enough to show (3.2.2) for any $X \in M(n, \mathbb{C})$ such that X is diagonizable. Let us fix $X \in M(n, \mathbb{C})$ such that X is diagonizable. There is $P \in GL(n, \mathbb{C})$ such that $PXP^{-1} = \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n \end{pmatrix}$. And $exp(PXP^{-1}) = \begin{pmatrix} exp(\lambda_1) & 0 & ... & 0 \\ 0 & exp(\lambda_2) & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & exp(\lambda_n) \end{pmatrix}$ So

$$\begin{aligned} det(exp(X)) &= det(Pexp(X)P^{-1}) \\ &= det(exp(PXP^{-1})) \\ &= exp(\lambda_1)exp(\lambda_2)...exp(\lambda_n) \\ &= exp(\sum_{i=1}^{n} \lambda_i) \\ &= exp(tr(PXP^{-1})) \\ &= exp(tr(X)) && (3.2.3) \end{aligned}$$

□

Proposition 3.2.10 (Exponential and Logarithm of matrix)**.** *Let*

(S1) $log(X) := \Sigma_{i=1}^{\infty} \frac{(-1)^{i-1}(X-E_n)^i}{i!}$ *for* $X \in M(n,\mathbb{C})$ *such that* $||X|| < 1$.

then

(i) $exp(log(X)) = X$ *for any* $X \in M(n,\mathbb{C})$ *such that* $||X|| < 1$.

(ii) $log(exp(X)) = X$ *for any* $X \in M(n,\mathbb{C})$ *such that* $||X|| < 1$ *such that* $||X|| < log2$.

Proof. By (iii) of Proposition, (i) and (ii) hold. □

The following Proposition says exponential map is locally isomorphism.

Proposition 3.2.11.

(i) $exp(\cdot)$ *is* C^{∞} *isomorphism to some open set in some neighborhood of* O.

(ii) $log(E+\cdot)$ *is* C^{∞} *isomorphism to some open set in some neighborhood of* E.

Proof. See the corollary of inverse mapping theorem in Section2. □

Proposition 3.2.12 (Basic properties about Exponential of matrix)**.**

(i) $exp(X+Y) = exp(X)exp(Y)$ *for any* $X, Y \in M(n,\mathbb{C})$ *such that* $XY = YX$.

(ii) $exp(X)^m = exp(mX)$ *for any* $X \in M(n,\mathbb{C})$ *and* $m \in \mathbb{N}$.

(iii) $exp(tX) = \Sigma_{i=0}^{K} \frac{t^i X^i}{i!} + O(t^{K+1})$ $(t \to 0)$ *for any* $X \in M(n,\mathbb{C})$ *and* $K \in \mathbb{N}$.

(iv) $\frac{d}{dt} exp(tX) = exp(tX)X = Xexp(tX)$

proof of (i).

$$\begin{aligned} exp(X+Y) &= \Sigma_{j=0}^{\infty}\Sigma_{i=0}^{j} \frac{{}_jC_i X^i Y^{j-i}}{j!} \\ &= \Sigma_{j=0}^{\infty}\Sigma_{i=0}^{j} \frac{{}_jP_i}{i!}\frac{X^i Y^{j-i}}{j!} \\ &= \Sigma_{j=0}^{\infty}\Sigma_{i=0}^{j} \frac{j!}{(j-i)!i!}\frac{X^i Y^{j-i}}{j!} \end{aligned}$$

For any $M \in \mathbb{N}$

$$\begin{aligned} ||\Sigma_{i=0}^{M}\frac{X^i}{i!}\Sigma_{j=0}^{M}\frac{Y^j}{j!} &- \Sigma_{j=0}^{M}\Sigma_{i=0}^{j}\frac{j!}{(j-i)!i!}\frac{X^iY^{j-i}}{j!}|| \\ &= ||\Sigma_{0\le i\le M, 0\le j\le M, i+j>M}\frac{X^iY^j}{i!j!}|| \\ &\le \Sigma_{0\le i\le M, 0\le j\le M, i+j>M}\frac{||X||^i||Y||^j}{i!j!} \\ &= ||\Sigma_{i=0}^{M}\frac{||X||^i}{i!}\Sigma_{j=0}^{M}\frac{||Y||^j}{j!} \\ &\quad -\Sigma_{j=0}^{M}\Sigma_{i=0}^{j}\frac{j!}{(j-i)!i!}\frac{||X||^i||Y||^{j-i}}{j!}|| \end{aligned}$$

Because

$$\lim_{M\to\infty} ||\Sigma_{i=0}^{M}\frac{||X||^i}{i!}\Sigma_{j=0}^{M}\frac{||Y||^j}{j!} = exp(||X||)exp(||Y||)$$

and

$$\lim_{M\to\infty} \Sigma_{j=0}^{M}\Sigma_{i=0}^{j}\frac{j!}{(j-i)!i!}\frac{||X||^i||Y||^{j-i}}{j!}|| = exp(||X|| + ||Y||)$$

and $exp(||X||)exp(||Y||) = exp(||X|| + ||Y||)$, the following holds.

$$\lim_{M\to\infty} \Sigma_{i=0}^{M} \frac{||X||^i}{i!} \Sigma_{j=0}^{M} \frac{||Y||^j}{j!} - \Sigma_{j=0}^{M} \Sigma_{i=0}^{j} \frac{j!}{(j-i)!i!} \frac{||X||^i ||Y||^{j-i}}{j!} = 0$$

So

$$exp(X+Y) = \lim_{M\to\infty} \Sigma_{i=0}^{M} \frac{X^i}{i!} \Sigma_{j=0}^{M} \frac{Y^j}{j!} = exp(X)exp(Y)$$

□

proof of (ii). It is easy to show (ii) from (i) □

proof of (iii).

$$\begin{aligned} ||exp(tX) - \Sigma_{i=0}^{K} \frac{t^i X^i}{i!}|| &\leq ||\Sigma_{i=K+1}^{\infty} \frac{t^i X^i}{i!}|| \\ &= |t|^{K+1} ||\Sigma_{i=K+1}^{\infty} \frac{t^{i-K+1} X^i}{i!}|| \\ &\leq |t|^{K+1} ||X||^{K+1} \Sigma_{i=K+1}^{\infty} \frac{|t|^{i-K+1} ||X||^{i-K+1}}{i!} \\ &\leq |t|^{K+1} ||X||^{K+1} \Sigma_{i=K+1}^{\infty} \frac{|t|^{i-K+1} ||X||^{i-K+1}}{(i-K-1)!} \\ &= |t|^{K+1} ||X||^{K+1} exp(|t| ||X||) \end{aligned} \quad (3.2.4)$$

□

proof of (iv). By (i), for any $t_0 \in \mathbb{R}$

$$\begin{aligned} exp(tX) - exp(t_0 X) &= exp(t_0 X)(exp((t-t_0)X) - E) \\ &= (exp((t-t_0)X) - E)exp(t_0 X) \end{aligned}$$

By (iii),

$$exp((t-t_0)X) - E = X + o(t-t_0)$$

So (iv) holds. □

Proposition 3.2.13.

$$exp(tX)exp(tY) = exp(t(X+Y) + \frac{t^2[X,Y]}{2} + o(t^2))$$

Proof.

$$\begin{aligned} exp(tX)exp(tY) &= (E + tX + \frac{1}{2}t^2X^2 + O(t^3))(E + tY + \frac{1}{2}t^2Y^2 + O(t^3)) \\ &= E + t(X+Y) + \frac{1}{2}t^2(X^2 + Y^2 + 2XY) + o(t^3) \end{aligned}$$

So

$$\begin{aligned} log(exp(tX)exp(tY)) &= t(X+Y) + \frac{1}{2}t^2(X^2+Y^2+2XY) + O(t^3) \\ &\quad -\frac{1}{2}(t(X+Y) + \frac{1}{2}t^2(X^2+Y^2+2XY) + O(t^3))^2 \\ &\quad +O(t^3) \\ &= t(X+Y) + \frac{1}{2}t^2(X^2+Y^2+2XY) - \frac{1}{2}t^2(X+Y)^2 \\ &\quad +O(t^3) \\ &= t(X+Y) + \frac{1}{2}t^2(XY - YX) + O(t^3) \end{aligned}$$

By Proposition3.2.12,

$$exp(tX)exp(tY) = exp(t(X+Y) + \frac{1}{2}t^2(XY - YX) + O(t^3))$$

□

Proposition implies the following.

Proposition 3.2.14.

$$exp(tX)exp(tY)exp(-tX)exp(-tY) = exp(\frac{t^2[X,Y]}{2} + o(t^2))$$

3.3 Lie algebra

3.3.1 Definition of Lie algebra

Definition 3.3.1 (Lie algebra of Lie subgroup)**.** *Let G is a Lie subgroup of $GL(n,\mathbb{C})$. We set*

$$Lie(G) := \{X \in M(n,\mathbb{C}) | exp(tX) \in G \ (\forall t \in \mathbb{R})\}$$

We call $Lie(G)$ Lie algebra of G.

Definition 3.3.2 (Lie algebra of Lie group)**.** *Let G_1 is a Lie group and G_2 is a Lie subgroup of $GL(n,\mathbb{C})$ such that G_1 is locally isomorphic to G_2. We set $Lie(G_1) := Lie(G_2)$.*

By Proposition3.4.8, $Lie(G_1)$ is well-defined.

Definition 3.3.3 (General Lie algebra)**.** *Let*

(i) K be a field.

(ii) L be a vector space on K.

(iii) L has operation $[\cdot,\cdot]$ which satisfies the followings.

(a)Alternativity. $[X,X] = 0$ *for any* $X \in L$.

(b)Jacobi's Rule. $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ *for any* $X,\ Y,\ Z \in L$.

(c)Bilinearity. $[aX + bY, cZ + dW] = ac[X,Z] + ad[X,W] + bc[Y,Z] + bd[Y,W]$ *for any* $X,\ Y,\ Z,\ W \in L$ *and* $a,b,c,d \in K$.

then we call L a Lie algebra on K.

Clearly, the followings hold.

Proposition 3.3.4. *For any Lie albegra L,*

$$[X,Y] = -[Y,X] \ (\forall X,Y \in L)$$

Definition 3.3.5 (Lie subalgebra, ideal)**.** *Let L be a Lie algebra. We call $L' \subset L$ a Lie subalgebra of L if L' is a subvectorspace of L and $[L',L'] \subset L'$. And, if L' is a Lie subalgebra and $[L,L'] \subset L'$ then we call L' is an ideal of L. We call $\{0\}$ and L are trivial ideals.*

The following clearly holds.

Proposition 3.3.6. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are ideals of $\mathfrak{g}$. We denote the minimam ideal containing $\mathfrak{h}_1$ and $\mathfrak{h}_2$ by $\langle[\mathfrak{h}_1,\mathfrak{h}_2]\rangle$.*

Proposition 3.3.7. *Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{z} := \{X \in \mathfrak{g} | [X,Y] = 0 \ (\forall Y \in \mathfrak{g})\}$*

Definition 3.3.8 (Simple Lie algebra)**.** *Let $\mathfrak{g}$ be a Lie algebra. We call $\mathfrak{g}$ is a simple Lie algebra if $\mathfrak{g}$ has no non-trivial ideals and $\mathfrak{g}$ is not abelian.*

By Proposition3.3.13, the following clearly holds.

Proposition 3.3.9. *Let $\mathfrak{g}$ be a simple Lie algebra. Then $\langle[\mathfrak{g},\mathfrak{g}]\rangle = \mathfrak{g}$.*

Definition 3.3.10 (Direct sum of Lie algebras)**.** *Let L be a Lie algebra. And let $\mathfrak{g}_1,...,\mathfrak{g}_k$ be ideals of L and $L = \oplus_{i=1}^k \mathfrak{g}_i$. Then we say L is the direct sum of $\mathfrak{g}_1,...,\mathfrak{g}_k$.*

Definition 3.3.11 (Abelian Lie algebra)**.** *Let $\mathfrak{g}$ be a Lie algebra. We call $\mathfrak{g}$ is an abelian Lie algebra if $[\mathfrak{g},\mathfrak{g}] = 0$.*

Proposition 3.3.12. *Let $\mathfrak{z}$ is the center of a Lie algebra and fix any $X \in \mathfrak{z}$. Then $\langle X\rangle$ is an ideal of $\mathfrak{g}$ and irreducible.*

By Proposition3.3.13, the following clearly holds.

Proposition 3.3.13. *Let $\mathfrak{g}$ is a Lie algebra which is the direct sum of $\mathfrak{g}_1, ..., \mathfrak{g}_k$ which are ideals of $\mathfrak{g}$. Then if $i \neq j$ then*

$$[\mathfrak{g}_i, \mathfrak{g}_j] = \{0\}$$

Proposition 3.3.14. *Let $\mathfrak{g}$ is a Lie algebra which is the direct sum of $\mathfrak{g}_1, ..., \mathfrak{g}_k$ which are ideals of $\mathfrak{g}$. Let us fix any $i \in \{1, 2, ..., k\}$. For any $\mathfrak{h}$ which is an ideal of $\mathfrak{g}_i$, $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.*

Proof. Let us fix any $X \in \mathfrak{g}$ and $Y \in \mathfrak{g}_i$. There are $X_j \in \mathfrak{g}_j$ $(j = 1, 2, ., , , .k)$ such that $X = \sum_{j=1}^{k} X_j$. By Proposition3.3.13,

$$XY = X_i Y \in \mathfrak{g}_i$$

□

Definition 3.3.15 (Semisimple Lie algebra). *Let $\mathfrak{g}$ be a Lie algebra. We call $\mathfrak{g}$ is a semisimple Lie algebra if $\mathfrak{g}$ is a direct sum of finite simple Lie algebras.*

Definition 3.3.16 (Reductive Lie algebra). *Let $\mathfrak{g}$ be a Lie algebra. We call $\mathfrak{g}$ is a reductive Lie algebra if $\mathfrak{g}$ is a direct sum of finite simple Lie algebras and an abelian Lie algebras.*

Proposition 3.3.17 (quotient Lie algebra). *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ be an ideal of $\mathfrak{g}$. Let $\mathfrak{g}/\mathfrak{h}$ be the quotient vector space. We set for each $X, Y \in \mathfrak{g}$*

$$[X + \mathfrak{h}, Y + \mathfrak{h}] = [X, Y] + \mathfrak{h}$$

$[\cdot, \cdot]$ is the well-defined Lie bracket on $\mathfrak{g}/\mathfrak{h}$. So $\mathfrak{g}/\mathfrak{h}$ is a Lie algebra.

Proof. For any $X, Y \in \mathfrak{g}$ and $h_X, h_Y \in \mathfrak{h}$,

$$[X + h_X, Y + h_Y] = [X, Y] + [X, h_Y] - [Y + h_Y, h_X]$$

So $[X + h_X, Y + h_Y] \in [X, Y] + \mathfrak{h}$. This means that $[\cdot, \cdot]$ is the well-defined Lie bracket on $\mathfrak{g}/\mathfrak{h}$. □

Proposition 3.3.18 (Adjoint representation of a Lie algebra). *Let $\mathfrak{g}$ be a Lie algebra. We set for each $X \in \mathfrak{g}$*

$$ad(X)Y = [X, Y] \ (Y \in \mathfrak{g})$$

Then

$$ad(aX + bY) = a \cdot ad(X) + b \cdot ad(Y) \ (\forall a, \forall b \in \mathbb{R}, \forall X \in \mathfrak{g}, \forall Y \in \mathfrak{g}) \tag{3.3.1}$$

and

$$ad([X, Y]) = [ad(X), ad(Y)] \ (\forall X \in \mathfrak{g}, \forall Y \in \mathfrak{g}) \tag{3.3.2}$$

We call ad the adjoint representation of $\mathfrak{g}$.

Proof. By linearlity of Lie bracket, (3.3.1) holds. And for any $X, Y, Z \in \mathfrak{g}$

$$\begin{aligned} & [[X, Y], Z] \\ = \ & -[Z, [X, Y]] \\ = \ & [X, [Y, Z]] + [Y, [Z, X]] \\ = \ & [X, [Y, Z]] - [Y, [X, Z]] \\ = \ & (ad(X)ad(Y) - ad(Y)ad(X))Z \\ = \ & [ad(X), ad(Y)]Z \end{aligned}$$

So (3.3.2) holds. □

3.3.2 Basic properties of Lie algebra

Lemma 3.3.19. *Let*

(S1) $A : \mathbb{N} \ni n \mapsto A(n) \in M(n, \mathbb{C})$ *and* $B : \mathbb{N} \ni n \mapsto B(n) \in M(n, \mathbb{C})$.

(A1) $B(m) = O(\frac{1}{m^2})$

(A2) $S := sup_{m \in \mathbb{N}} ||A(m)||^m < \infty$

then

$$\{A(m)(E + B(m))\}^m = A(m)^m + O(\frac{1}{m})$$

Proof.

$$\begin{aligned}\{A(m)(E+B(m))\}^m &= A(m)(E+B(m))A(m)(E+B(m))...A(m)(E+B(m)) \\ &= A(m)^m + \sum_{k=1}^{m} C_k(m)\end{aligned}$$

Here, for each $k \in \{1,2,...,m\}$

$$C_k(m) := \sum_{i_1<i_2<...<i_k} A(m)^{i_1}B(m)A(m)^{i_2}B(m)...A(m)^{i_k}B(m)A(m)^{m-i_1-i_2...-i_k}$$

Then $||C_k(m)|| \leq {}_mC_k||A(m)||^m||B(t)||^k \leq \frac{S}{k!}m^kO(\frac{1}{m^2k}) = O(\frac{1}{m^k})$.

So $\sum_{k=1}^m ||C_k(m)|| = ||C_1(m)|| + \sum_{k=2}^m ||C_k(m)|| \leq O(\frac{1}{m}) + mO(\frac{1}{m^2}) = O(\frac{1}{m})$. □

Proposition 3.3.20. *Let G is a Lie sub group of $GL(n,\mathbb{C})$. Then $Lie(G)$ is a $\mathbb{R}$-vector space and for any $X, Y \in Lie(G)$ $[X,Y] \in Lie(G)$.*

Proof. There is W such that W is an open subset of $GL(n,\mathbb{C})$ and $1_G \in W$ and $W \cap G \subset V$.

By the definition of $Lie(G)$, For any $X \in Lie(G)$ and $a \in \mathbb{R}$, $aX \in Lie(G)$.

Let us fix any $X, Y \in Lie(G)$. By Proposition3.2,

$$exp(sX)exp(sY) = exp(s(X+Y)+O(s^2)) = exp(s(X+Y))(E+O(s^2)) \ (s \to 0)$$

So

$$\{exp(\frac{t}{m}(X+Y))(E+O(\frac{1}{m^2}))\}^m = exp(t(X+Y)) + O(\frac{1}{m})$$

This implies

$$exp(t(X+Y)) + O(\frac{1}{m}) = \{exp(\frac{t}{m}X)exp(\frac{t}{m}Y)\}^m \ (m \to \infty)$$

There is $\delta > 0$ such that $exp(s(X+Y) \in W$ $(\forall s \in (-\delta,\delta))$. Let us fix $s \in (-\delta,\delta)$. So for sufficient larget $m \in \mathbb{N}$ $exp(s(X+Y)) + O(\frac{1}{m}) \in W \cap G$. So $exp(s(X+Y)) + O(\frac{1}{m}) \in V$,

Because V is closed set, $exp(t(X+Y)) \in V$. Consequently $X+Y \in Lie(G)$.

Also, by similar argument to the above one,

$$exp(t[X,Y]) = \lim_{m\to\infty}\{exp(\frac{t}{m}X)exp(\frac{t}{m}Y)exp(\frac{-t}{m}X)exp(\frac{-t}{m}Y)\}^m$$

Consequently $[X,Y] \in Lie(G)$. □

From the proof of Proposition, the following holds.

Proposition 3.3.21. *Let G is a Lie subgroup of $GL(n,\mathbb{C})$ and V is a closed subset of $GL(n,\mathbb{C})$ and V is a neighborhood of 1_G. And we set*

$$\mathfrak{g}_V := \{X \in M(n,\mathbb{C}|exp(tX) \in V, \ |t| \ll 1\}$$

Then $\mathfrak{g}_V$ is a $\mathbb{R}$-vector space and for any $X, Y \in \mathfrak{g}_V$ $[X,Y] \in \mathfrak{g}_V$.

3.4 The structure of C^ω-class manifold of Lie group

3.4.1 Local coordinate system of Lie group

Lemma 3.4.1. *For $X_1, X_2, ..., X_m \in M(n,\mathbb{C})$,*

$$exp(X_1)exp(X_2)...exp(X_m) = E + X_1 + X_2 + ... + X_m + o(\sum_{i=1}^{m} ||X_i||)$$

Proof. For any i,

$$o(||X_i||) = o(\sum_{i=1}^{m} ||X_i||)$$

So, by the definition of exponential of matrix and Lemma3.4.1

$$\begin{aligned}
& exp(X_1)exp(X_2)...exp(X_m) \\
= & (E + X_1 + o(||X_1||))(E + X_2 + o(||X_2||))...(E + X_m + o(||X_m||)) \\
= & E + X_1 + X_2 + ... + X_m \\
& + \sum_{2\leq k\leq m, i_1<i_2<...<i_k} X_{i_1}X_{i_2}...X_{i_k} + o(\sum_{i=1}^{m} ||X_i||) \\
= & E + X_1 + X_2 + ... + X_m \\
& + \sum_{2\leq k\leq m, i_1<i_2<...<i_k} o(X_{i_1}) + o(\sum_{i=1}^{m} ||X_i||) \\
= & E + X_1 + X_2 + ... + X_m \\
& + \sum_{2\leq k\leq m, i_1<i_2<...<i_k} o(\sum_{i=1}^{m} ||X_i||) + o(\sum_{i=1}^{m} ||X_i||) \\
= & E + X_1 + X_2 + ... + X_m + o(\sum_{i=1}^{m} ||X_i||)
\end{aligned}$$

□

Lemma 3.4.2. *Let us fix any subvectorspace $V1$ and $V2$ of $\mathbb{C}^n$ such that $V_1 \oplus V_2 = \mathbb{C}^n$. Then V_1 and V_2 are closed subset.*

Proof. There is $P \in GL(n, \mathbb{C})$ such that $V_1 = P\{w \in \mathbb{C}^n | w_j = 0 \ (j = 1, 2, ..., dimV_1)\}P^{-1}$ and $V_2 = P\{w \in \mathbb{C}^n | w_j = 0 \ (j = dimV_1 + 1, ..., n)\}P^{-1}$ □

Lemma 3.4.3. *Let*

(S1) $G = GL(n, \mathbb{C})$.

(S2) $\mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_m$ *are vector subspaces of* $Lie(G)$ *such that*

$$Lie(G_2) = \oplus_{i=1}^{m} \mathfrak{g}_i$$

(S3) $\mathfrak{g}_i(\epsilon) := \{X \in Lie(G) | ||X|| < \epsilon\}$ $(i = 1, 2, ..., m, \epsilon > 0)$.

$$\begin{array}{rcl}
i : \oplus_{i=1}^{m} \mathfrak{g}_i(\epsilon) & \to & G \\
\Cup & & \Cup \\
(X_1, X_2, ..., X_m) & \mapsto & exp(X_1)exp(X_2)...exp(X_m)
\end{array}$$

then there is $\epsilon > 0$ such that $i(\oplus_{i=1}^{m} \mathfrak{g}_i(\epsilon))$ is an open set and $i| \oplus_{i=1}^{m} \mathfrak{g}_i(\epsilon)$ is C^ω-class isomorphism.

Proof. We set

$$\begin{array}{rcl}
j : G & \to & M(n, \mathbb{C}) \\
\Cup & & \Cup \\
y & \mapsto & log(y)
\end{array}$$

By Lemma3.4.1,

$$j \circ i(X_1, X_2, ..., X_m) = X_1 + X_2 + ... + X_m + o(||X_1|| + ||X_2|| + ... + ||X_m||)$$

So, the jacobian of $j \circ i$ at O is non-singular. By inverse function theorem(see Section2), the proposition holds. □

Lemma 3.4.4. *Let*

(S1) G_2 *is a Lie subgroup of* $GL(n, \mathbb{C})$.

Then for sufficient small $\epsilon > 0$,

$$G_2 \cap exp(B(O, \epsilon)) = exp(Lie(G_2) \cap B(O, \epsilon))$$

Proof of the right side $\subset$ the left side. It is trivial. □

Proof of the left side $\subset$ the right side. There is a vector subspace $\mathfrak{q}$ such that $M(n,\mathbb{C}) = Lie(G) \oplus \mathfrak{q}$. Proposition3.4.3, $i : Lie(G) \oplus \mathfrak{q} \ni (X,Y) \mapsto exp(X)exp(Y)$ is locally homeomorphism. Let us assume there is $\{\epsilon_k\}_{k=1}^{\infty} \subset (0,1)$ such that $\lim_{k\to\infty} \epsilon = 0$ and for each ϵ_k the left side $\subsetneq$ the right side. By Lemma3.4.3, there are $Z_k \in B(O,\epsilon_k)$ and $X_k \in Lie(G_2)$ and $Y_k \in \mathfrak{q}$ $(k = 1,2,...)$ such that for any k

$$exp(Z_k) = exp(X_k)exp(Y_k)$$

and

$$\lim_{k\to\infty} ||X_k|| = 0,\ \lim_{k\to\infty} ||Y_k|| = 0$$

and

$$||Y_k|| \neq 0$$

We can assume $||Y_k|| \leq 1$ for any k. Because $\overline{B(O,1)}$ is compact, there is a subsequence $\{Y_{\varphi(k)}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} \lceil \frac{1}{||Y_\varphi(k)||} \rceil Y_{\varphi(k)} = Y$. Clearly $||Y|| = 1$. By Proposition3.4.2, $Y \in \mathfrak{q}$. So $Y \notin Lie(G)$.

Because V is a neighborhood of 1_{G_2}, there is $\epsilon > 0$ such that $exp(B(O,\epsilon)) \cap G_2 \subset V$. Let us fix any $t \in (0,\epsilon)$.

$$exp(tY) = \lim_{k\to\infty} exp(t\lceil \frac{1}{||Y_\varphi(k)||} \rceil Y_{\varphi(k)})$$

Because $r_k := \lceil \frac{1}{||Y_\varphi(k)||} \rceil \to \infty$, $t = \lim_{k\to\infty} \frac{\lceil tr_k \rceil}{r_k}$. So

$$\begin{aligned} exp(tY) &= \lim_{k\to\infty} exp(\frac{\lceil tr_k \rceil}{r_k} r_k Y_{\varphi(k)}) \\ &= \lim_{k\to\infty} exp(Y_{\varphi(k)})^{\lceil tr_k \rceil} \end{aligned}$$

For any k,

$$exp(\lceil tr_k \rceil Y_{\varphi(k)}) = \{exp(-X_{\varphi(k)})exp(Z_{\varphi(k)})\}^{\lceil tr_k \rceil} \in G_2 \cap exp(B(O,\epsilon)) \subset V$$

Because V is closed set, $exp(tY) \in V$. So for any $t \in \mathbb{R}$

$$exp(tY) = exp(\frac{t}{\lceil \frac{t}{\delta} \rceil + 1} Y)^{\lceil \frac{t}{\delta} \rceil + 1} \in G_2$$

So $Y \in Lie(G_2)$. This is contradiction. □

Proposition 3.4.5. *Let G be a topological group and G_0 be a connected component of G which contains 1_G. Then G_0 is closed normal subgroup of G.*

Proof. Because $\bar{G_0}$ is connected, $\bar{G_0} = G_0$. So G_0 is closed. Because $x \mapsto x^{-1}$ is isomorphism, G_0^{-1} is connected and $1_G \in G_0^{-1}$. So $G_0^{-1} \subset G_0$. Because $x \mapsto gx$ is isomorphism, for any $g \in G_0$, gG_0 is connected and contains 1_G. So for any $g \in G_0$, $gG_0 \subset G_0$. This implies that G_0 is subgroup of G. And for any $g \in G_0$, gG_0g^{-1} is connected and contains 1_G.. So for any $g \in G_0$, $gG_0g^{-1} \subset G_0$. This implies that G_0 is a normal subgroup of G. □

Proposition 3.4.6. *Let*

- *(S1) G_1 is a connected Lie group which is locally isomorphic to a Lie subgroup of $GL(n,\mathbb{C})$ G_2.*
- *(S2) G_0 is a connected component of G_1 which contains 1_{G_1}.*
- *(A1) N is a connected open neighborhood of 1_{G_1}.*
- *(S3) $N_m := \{n_1n_2...n_m | n_i \in N,\ i = 1,2,...,m\}$ for each $m \in \mathbb{N}$.*

then

- *(i) G_0 is closed and open subset of G_1.*
- *(ii) $G_0 = \cup_{i=1}^{\infty} N_i$.*
- *(iii) Any connected component of G_1 is closed and open subset of G_1.*

(iv) G_1 *satisfies the second axiom of countability. Specially,* G_1 *is paracompact.*

(v) G_1 *is separable.*

(vi) G_1 *is* σ*-compact.*

(vii) G_0 *is path connected.*

Proof of (i) and (ii). By Lemma3.4.4, we can assume $N = \eta(exp(Lie(G_2) \cap B(O, \epsilon)))$ for some $\epsilon > 0$ and $N = N^{-1}$. We set $H := \cup_{i=1}^{\infty} N_i$. By continuity of multiple operation in G_1, for each $i \in \mathbb{N}$, N_i is connected. Because $1_{G_1} \in N_i$ for any $i \in \mathbb{N}$, H is connected. So,

$$H \subset G_0$$

Because N_m is an open subset for each $m \in \mathbb{N}$, H is an open subset. Let us fix any $g \in H^c$. If we assume $gN \cap H \neq \phi$, then there is $m \in \mathbb{N}$ and there are $n_0 \in N$ and $n_1, n_2, ..., n_m \in N$ such that $gn_0 = n_1 n_2 .. n_m$. So $g \in N_m N^{-1} = N_m N = N_{m+1}$. This implise $g \in H$. This is a contradiction. So $gN \cap H = \phi$. This means H is a closed subset of G_1. Because $G_0 \subset H \cup H^c$ and H is open and H^c is open and G_0 is connected and $G_0 \cap H \neq \phi$, $G_0 \cap H^c = \phi$. This means

$$G_0 \subset H$$

So $G_0 = H$. □

Proof of (iii). Let us fix and set any connected component of G_1 C. And let us fix $g_0 \in C$. Clearly $C = g_0 G_0$. Because L_{g_0} is isomorphism, C is open and closed. □

Proof of (iv)(v). In the proof of (ii), we set $N' := \eta(exp(Lie(G_2) \cap \overline{B(O, \epsilon)}))$. By (ii), $G_0 = \cup_{n=1}^{\infty} N'_n$. Because N'_n is compact for any $n \in \mathbb{N}$, clearly, G_0 satisfies the second axiom of countability. Because $\overline{B(O, \epsilon)})$ is separable, N' is separable. Because N'_n is separable for any $n \in \mathbb{N}$, clearly, G_0 is separable. And, by (S1) and (iii), G_1 satisfies the second axiom of countability and G_1 is separable. □

Proof of (vi). Let $\{X_i\}_{i=1}^{\infty}$ is a sequence of all connected components of G. Let fix $\{x_i\}_{i=1}^{\infty}$ such that $x_i \in X_i$ $(\forall i)$. In (A1), we can assume that N is relative compact. Then $G = \cup_{m=1}^{\infty} \cup_{k=1}^{m} x_k \bar{N}_m$ and $\cup_{k=1}^{m} x_k \bar{N}_m$ is compact $(\forall m \in \mathbb{N})$. So, G is σ-compact. □

Proof of (vii). (vii) is from (i). □

From the proof of Lemma3.4.4, by Proposition3.1.5, the following holds.

Lemma 3.4.7. *Let*

(S1) G_2 *is a Lie subgroup of* $GL(n, \mathbb{C})$.

(A1) W *is a neighborhood of* 1_{G_2} *in* G_2.

(S2) $\mathfrak{g}_W := \{X \in M(n, \mathbb{C}| exp(tX) \in W \ |t| \ll 1\}$.

Then for sufficient small $\epsilon > 0$,

$$W \cap exp(B(O, \epsilon)) = exp(\mathfrak{g}_W \cap B(O, \epsilon))$$

Proposition 3.4.8. *Let* G *is a Lie subgroup of* $GL(n, \mathbb{C})$ *and* W *is a neighborhood of* 1_G. *Then*

$$Lie(G) = \{X \in M(n, \mathbb{C}) | exp(tX) \in W \ (0 \leq t << 1)\}$$

Proof. By Proposition3.1.5, there is V such that V is a closed subset of $GL(n, \mathbb{C})$ and V is a neighborhood of 1_G and $V \subset W$. Clearly $\mathfrak{g}_V \subset \mathfrak{g}_W$ and $\mathfrak{g}_V \subset Lie(G)$. We assume that there is $X \in Lie(G) \setminus \mathfrak{g}_V$. By Proposition3.3.21, $\langle X \rangle \cap \mathfrak{g}_V = \{0\}$. By Lemma3.4.3, there is $\delta > 0$ such that

$$(-\delta, \delta) \times (B(O, \delta) \cap \mathfrak{g}_V) \ni (t, Y) \to exp(tX) exp(Y) \in GL(n, \mathbb{C})$$

is injective. By Lemma3.4.7, $\{exp(tX) exp(\mathfrak{g}_V \cap B(O, \delta))\}_{t \in (-\delta, \delta)}$ is a family of neighborhood of some point of G. Because $\{exp(tX) exp(\mathfrak{g}_V \cap B(O, \delta))\}_{t \in (-\delta, \delta)}$ are disjoint, G does not satisfy the second axiom. This contradicts with Proposition3.4.6. □

By Lemma3.4.7 and Proposition3.4.8, the following holds.

Lemma 3.4.9. *Let*

(S1) G_2 *is a Lie subgroup of* $GL(n, \mathbb{C})$.

(A1) W *is a neighborhood of* 1_{G_2} *in* G_2.

(S2) $\mathfrak{g}_W := \{X \in M(n, \mathbb{C}|exp(tX) \in W \ |t| \ll 1\}$.

Then for sufficient small $\epsilon > 0$,

$$W \cap exp(B(O, \epsilon)) = exp(Lie(G_2) \cap B(O, \epsilon)) \tag{3.4.1}$$

Theorem 3.4.10 (von Neumann-Cartan's theorem I)**.** *Let*

(S1) G_1 *is a Lie group which is isomorphic to a Lie subgroup* G_2 *of* $GL(n, \mathbb{C})$.

(S2) $\mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_m$ *are vector subspaces of* $Lie(G_2)$ *such that*

$$Lie(G_2) = \oplus_{i=1}^m \mathfrak{g}_i \tag{3.4.2}$$

(S3) $\mathfrak{g}_i(\epsilon) := \{X \in Lie(G_2)|||X|| < \epsilon\}$ $(i = 1, 2, ..., m, \epsilon > 0)$.

(S4) *For any* $x \in G_2$

$$\begin{array}{rcl} i_x : \oplus_{i=1}^m \mathfrak{g}_i(\epsilon) & \to & G_2 \\ \Cup & & \Cup \\ (X_1, X_2, ..., X_m) & \mapsto & xexp(X_1)exp(X_2)...exp(X_m) \end{array} \tag{3.4.3}$$

(S5) $\psi := i_e$

(S6) $\phi := exp(\cdot)$

then

(i) G_1 *is a* C^ω*-manifold and* $\{\eta_z \circ \phi\}_{z \in G_1}$ *is a local coordinate system.*

(ii) $\{\eta_z \circ \psi\}_{z \in G_1}$ *is a local corrdinate system which is equivalent to* $\{\eta_z \circ \phi\}_{z \in G_1}$.

(iii) *There are open neighborhood of* 1_{G_1} U *and open neighborhood of* 1_{G_2} V *and* $\tau : U \to V$ *is a* C^ω*-class homeomorphism.*

STEP1. Showing i_x *is locally injective.* We set

$$\begin{array}{rcl} j_x : G_2 & \to & M(n, \mathbb{C}) \\ \Cup & & \Cup \\ y & \mapsto & log(x^{-1}y) \end{array} \tag{3.4.4}$$

By Lemma3.4.1,

$$j_x \circ i_x(X_1, X_2, ..., X_m) = X_1 + X_2 + ... + X_m + o(||X_1|| + ||X_2|| + ... + ||X_m||) \tag{3.4.5}$$

So, the jacobian of $j_x \circ i_x$ at O is non-singular. By inverse function theorem(see [?]), i_x is locally injective. □

STEP2. Constructing local corrdinates system of G_2. By Lemma3.4.9, there is $\epsilon > 0$ such that

$$V_\epsilon := exp(Lie(G) \cap B(O, \epsilon)) = V \cap exp(B(O, \epsilon)) \tag{3.4.6}$$

Clearly V_ϵ is an open neighborhood of 1_{G_2}. By (3.4.6), for any $X_0 \in Lie(G) \cap B(O, \epsilon)$ and $\delta > 0$ such that $B(X_0, \delta) \subset B(O, \epsilon)$,

$$exp(Lie(G) \cap B(X_0, \delta)) = V \cap exp(B(X_0, \delta)) \tag{3.4.7}$$

Because the topology of V is equal to the relative topology respect to $GL(n, \mathbb{C})$, $i_e : Lie(G) \cap B(O, \epsilon) \to G_2 \cap exp(B(O, \epsilon))$ is an continous and open map. By STEP1, i_e is a homeomorphism.

And, for any $x \in G_2$, $i_x : Lie(G_2) \cap B(O, \epsilon) \to xV_\epsilon$ is homeomorphism. □

STEP3. Constructing local corrdinates system of G_1. There is $\delta > 0$ such that

$$V_\delta V_\delta^{-1} V_\delta \subset V_\epsilon \tag{3.4.8}$$

$U_\delta := \eta(V_\delta)$. For any $x' \in G_1$, $\phi'_x : Lie(G_2) \cap B(O, \delta) \ni X \mapsto x'\eta(exp(X)) \in x'U_\delta$. Clearly ϕ'_x is homeomorphism. By Proposition, U_ϵ and V_ϵ satisfy the conditions in Definition3.1.1. □

STEP4. Showing (i). Let us assume $zU_\delta \cap wU_\delta \neq \phi$ and let us fix any $X \in \phi_z^{-1}(zU_\delta \cap wU_\delta)$ and let us set $Y := \phi_w^{-1}(\phi_z(X))$. Then

$$Y = log(\tau(w^{-1}z\eta(exp(X))) \tag{3.4.9}$$

There are $u_x, u_y \in U_\delta$ and $v_x, v_y \in V_\delta$ such that

$$zu_x = wu_y$$

and

$$\eta(v_x) = u_x,\ \eta(v_y) = u_y$$

By (3.4.8),

$$v_x^{-1} \in V_\epsilon \tag{3.4.10}$$

So

$$\eta(v_x^{-1}) = \eta(v_x)^{-1}$$

This implies

$$u_y u_x^{-1} = \eta(v_y)\eta(v_x^{-1})$$

By (3.4.10),

$$\eta(v_y)\eta(v_x^{-1}) = \eta(v_y x_x^{-1})$$

So

$$Y = log(\tau(\eta(v_y x_x^{-1})\eta(exp(X)))) \tag{3.4.11}$$

Because $v_y x_x^{-1} exp(X) \in V_\epsilon$,

$$\eta(v_y x_x^{-1})\eta(exp(X)) = \eta(v_y x_x^{-1} exp(X))$$

So

$$Y = log(v_y x_x^{-1} exp(X)) \tag{3.4.12}$$

Consequently, $\phi_w^{-1} \circ \phi_z$ is C^ω-class. □

STEP5. Showing (iii). It is possible to show (iii) by from STEP1. to STEP4. □

STEP6. Showing $\psi^{-1} \circ \phi$ is locally C^ω-homeomorphism. It is possible to show STEP6 by STEP1. □

STEP7. Showing (ii). If $zU_\delta \cap wU_\delta \neq \phi$,

$$\phi^{-1} \circ \tau_w \circ \eta_z \circ \phi = \phi^{-1} \circ \psi \circ \psi^{-1} \circ \tau_w \circ \eta_z \circ \psi \circ \psi^{-1} \circ \phi$$

and

$$\psi^{-1} \circ \tau_w \circ \eta_z \circ \phi = \psi^{-1} \circ \tau_w \circ \eta_z \circ \psi \circ \psi^{-1} \circ \phi$$

So by STEP6, (iii) holds. □

Proposition 3.4.11. *Let G be a Lie group. Then there is an open neighborhood U such that U has no subgroups without $\{e\}$.*

Case when $Lie(G) = \{0\}$. By von-Neumann Cartan theorem, $\{e\}$ is an open neighborhood. □

Case when $Lie(G) \neq \{0\}$. There is $\epsilon > 0$ such that $Exp : Lie(G) \cap B(O, 2\epsilon) \ni X \mapsto Exp(X) \in Exp(Lie(G) \cap B(O, 2\epsilon))$ is a diffeomorphism and $Exp(Lie(G) \cap B(O, 2\epsilon))$ is an open subset of G. We set $U := Exp(Lie(G) \cap B(O, \epsilon))$. Let us any $Exp(X) \in U$ such that $X \in Lie(G) \cap B(O, \epsilon) \setminus \{0\}$. We set $g := Exp(\lfloor \frac{||X||}{\epsilon} \rfloor X)$. Then $\epsilon \leq \lfloor \frac{||X||}{\epsilon} \rfloor ||X|| < 2\epsilon$. So, $g \notin U$. This implies that U has no subgroups without $\{e\}$. □

3.4.2 Analycity of Lie group

Definition 3.4.12 (One-parameter group). *We call $g \in C(\mathbb{R}, G)$ a one-parameter group of G if $g(s+t) = g(s)g(t)$ (for any $s, t \in \mathbb{R}$).*

Proposition 3.4.13. *Let G_1 be a Lie group which is isomorphic to a Lie subgroup G_2 of $GL(n, \mathbb{C})$. Let us assume τ is a local isomorphism from G_1 to G_2. And let $g \in C(\mathbb{R}, G)$ be a one-parameter group of G. Then there is $\epsilon > 0$ and such that there is the unique $X \in Lie(G_2)$ such that*

$$\tau(g(s)) = exp(sX) \ \forall s \in (-\epsilon, \epsilon) \tag{3.4.13}$$

Existence. Let us fix $\tau : U \to V$ is a local isomorphism and $\epsilon > 0$ and $i : Lie(G_2) \cap B(O, 2\epsilon) \to G_2 \cap exp(B(O, 2\epsilon))$ be a homeomorphism and $\delta > 0$ such that $g((-2\delta, 2\delta)) \subset U$. There is the one-parameter subgroup h such that $h|(-2\delta, 2\delta) = \tau \circ g|(-2\delta, 2\delta)$.

If $h \equiv 1_{G_2}$, then O satisfies (3.4.13). Else if $h \equiv 1_{G_2}$, there is $t_0 \in (0, \delta)$ and $X_1 \in Lie(G_2) \cap B(O, \epsilon)$ such that $1_{G_2} \neq h(t_0) = exp(X_1)$. We set $X_0 := \dfrac{X_1}{t_0}$.

There is $Y_1 \in Lie(G_2) \cap B(O, \epsilon)$ such that

$$h(\frac{t_0}{2}) = exp(Y_1)$$

Then $exp(X_1) = h(t_0) = exp(2Y_1)$. Because $2Y_1 \in Lie(G_2) \cap B(O, 2\epsilon)$, $X_1 = 2Y_1$. So,

$$h(\frac{t_0}{2}) = exp(\frac{1}{2}X_1)$$

And there is $Y_1 \in Lie(G_2) \cap B(O, \epsilon)$ such that

$$h(\frac{t_0}{4}) = exp(Y_2)$$

Then $exp(Y_1) = h(\dfrac{t_0}{2}) = exp(2Y_2)$. Because $2Y_2 \in Lie(G_2) \cap B(O, 2\epsilon)$, $Y_1 = 2Y_2$. So,

$$h(\frac{t_0}{4}) = exp(\frac{1}{2}Y_1) = exp(\frac{1}{4}X_1)$$

So, by mathematical induction,

$$h(\frac{t_0}{2^m}) = exp(\frac{1}{2^m}X_1) \ (\forall m \in \mathbb{N})$$

By calculating powers of both sides,

$$h(t_0\frac{k}{2^m}) = exp(t_0\frac{k}{2^m}X_0) \ (\forall k, m \in \mathbb{N})$$

Because $\{t_0\dfrac{k}{2^m} | k, m \in \mathbb{N} \text{ such that } \dfrac{k}{2^m} \leq 1\}$ is dense in $[0, \delta]$,

$$h(t) = exp(tX_0) \ (\forall t \in (-\delta, \delta))$$

□

Uniqueness. Let us fix any $X, Y \in Lie(G_2)$ such that $exp(tX) = exp(tY)$ ($\forall t \in \mathbb{R}$). If there is $a \in \mathbb{R}$ such that $X = aY$, $exp(t(a-1)Y) = E$ ($\forall t \in \mathbb{R}$). By (i) of Theorem3.4.10, $a = 1$ or $Y = 0$.

If there is X and Y are linear independent, there are $Z_1, Z_2, ..., Z_r$ such that $Z_1, Z_2, ..., Z_r, X, -Y$ are the basis of $Lie(G_2)$. $exp(tX) = exp(tY)$ implies $exp(tX)exp(t(-Y)) = e$. This contradicts with (ii) of Theorem3.4.10. □

Theorem 3.4.14. *Let*

(S1) $G_{1,1}$ be a Lie group which is isomorphic to a Lie subgroup $G_{1,2}$ of $GL(n, \mathbb{C})$.

(S2) $G_{2,1}$ be a Lie group which is isomorphic to a Lie subgroup $G_{2,2}$ of $GL(n, \mathbb{C})$.

(A1) $\Phi \in C(G_{1,1}, G_{2,1})$ is a homomorphism.

then

(i) There is a homomorphisim of Lie algebras $\iota : Lie(G_{1,1}) \to Lie(G_{2,1})$ such that

$$\Phi(\eta_1(exp(tX)) = \eta_2(exp(t\iota(X))) \ (|t| \ll 1) \tag{3.4.14}$$

(ii) Φ *is* C^ω*-class.*

(iii) If Φ *is a local isomorphism, then* ι *is an isomorphism.*

STEP1. constructing ι*.* For each $X \in Lie(G_{1,1})$, by Proposition3.4.13, there is only one Y such that

$$\Phi(\eta_1(exp(tX))) = \eta_2(exp(tY)) \ (any\ t\ such\ that\ |t| \ll 1)$$

We set $\iota(X) := Y$. □

STEP2. Showing ι *is a linear.* For any $X \in Lie(G_{1,1})$ and $a \in \mathbb{R}$, clearly $\iota(aX) := a\iota(X)$.

For any $X,\ Y \in Lie(G_{1,1})$ and $t \in \mathbb{R}$ such that $|t| \ll 1$,

$$\begin{aligned}
& \Phi(\eta_1(exp(t(X+Y)))) \\
= \ & \Phi(\eta_1(\lim_{m\to\infty}(exp(\frac{t}{m}X)exp(\frac{t}{m}Y))^m)) \\
= \ & \Phi(\lim_{m\to\infty}\eta_1((exp(\frac{t}{m}X)exp(\frac{t}{m}Y))^m)) \\
= \ & \lim_{m\to\infty}\Phi(\eta_1((exp(\frac{t}{m}X)exp(\frac{t}{m}Y))^m)) \\
= \ & \lim_{m\to\infty}\Phi(\eta_1((exp(\frac{t}{m}X)exp(\frac{t}{m}Y)))^m) \\
= \ & \lim_{m\to\infty}\Phi(\eta_1((exp(\frac{t}{m}X)exp(\frac{t}{m}Y))))^m \\
= \ & \lim_{m\to\infty}\{\Phi(\eta_1(exp(\frac{t}{m}X)))\Phi(\eta_1(exp(\frac{t}{m}Y)))\}^m \\
= \ & \lim_{m\to\infty}\{\eta_2(exp(\frac{t}{m}\iota(X)))\eta_2(exp(\frac{t}{m}\iota(Y)))\}^m \\
= \ & \lim_{m\to\infty}\{\eta_2(exp(\frac{t}{m}\iota(X))exp(\frac{t}{m}\iota(Y)))\}^m \\
= \ & \lim_{m\to\infty}\eta_2(\{exp(\frac{t}{m}\iota(X))exp(\frac{t}{m}\iota(Y))\}^m) \\
= \ & \eta_2(\lim_{m\to\infty}\{exp(\frac{t}{m}\iota(X))exp(\frac{t}{m}\iota(Y))\}^m) \\
= \ & \eta_2(t(\iota(X)+\iota(Y)))
\end{aligned}$$

So

$$\iota(X+Y) = \iota(X) + \iota(Y)$$

□

STEP2. Showing (ii). Let ψ_i is the local corrdinate of $G_{i,2}$ in von Neumann-Cartan's theorem($i = 1, 2$). By (i), for any $x \in G_{1,1}$ and $X \in Lie(G_{1,1})$ such that $||X|| \ll 1$

$$\Phi(\eta_{x,1} \circ \psi_1^{-1}(X)) = \Phi(x)\eta_2(\psi_2^{-1}(\iota(X))$$

This implies

$$\psi_2(\tau_{\Phi(x),2}(\Phi(\eta_{x,1} \circ \psi_1^{-1}(X)))) = \iota(X)$$

Because ι is a linear mapping, Φ is C^ω. □

STEP3. Showing $\iota([X,Y]) = [\iota(X), \iota(Y)]$. By Proposition3.2.14, for any $X,\ Y \in Lie(G_{1,1})$ and $t \in \mathbb{R}$ such that $|t| \ll 1$,

$$\begin{aligned}
& \Phi(\eta_1(exp(t([X,Y])))\\
= \ & \Phi(\eta_1(\lim_{m\to\infty}(exp(\frac{\sqrt{t}}{m}X)exp(\frac{\sqrt{t}}{m}Y)exp(\frac{-\sqrt{t}}{m}X)exp(\frac{-\sqrt{t}}{m}Y))^m))\\
= \ & \Phi(\lim_{m\to\infty}\eta_1((exp(\frac{\sqrt{t}}{m}X)exp(\frac{\sqrt{t}}{m}Y)exp(\frac{-\sqrt{t}}{m}X)exp(\frac{-\sqrt{t}}{m}Y))^m))\\
= \ & \lim_{m\to\infty}\Phi(\eta_1((exp(\frac{\sqrt{t}}{m}X)exp(\frac{\sqrt{t}}{m}Y)exp(\frac{-\sqrt{t}}{m}X)exp(\frac{-\sqrt{t}}{m}Y))^m))\\
= \ & \lim_{m\to\infty}\Phi(\eta_1((exp(\frac{\sqrt{t}}{m}X)exp(\frac{\sqrt{t}}{m}Y)exp(\frac{-\sqrt{t}}{m}X)exp(\frac{-\sqrt{t}}{m}Y)))^m)\\
= \ & \lim_{m\to\infty}\Phi(\eta_1((exp(\frac{\sqrt{t}}{m}X)exp(\frac{\sqrt{t}}{m}Y)exp(\frac{-\sqrt{t}}{m}X)exp(\frac{-\sqrt{t}}{m}Y))^m))\\
= \ & \lim_{m\to\infty}\Phi(\eta_1((exp(\frac{\sqrt{t}}{m}X)exp(\frac{\sqrt{t}}{m}Y)exp(\frac{-\sqrt{t}}{m}X)exp(\frac{-\sqrt{t}}{m}Y))))^m\\
= \ & \lim_{m\to\infty}\{\Phi(\eta_1(exp(\frac{\sqrt{t}}{m}X)))\Phi(\eta_1(exp(\frac{\sqrt{t}}{m}Y)))\Phi(\eta_1(exp(-\frac{\sqrt{t}}{m}X)))\Phi(\eta_1(exp(-\frac{\sqrt{t}}{m}Y)))\}^m\\
= \ & \lim_{m\to\infty}\{\eta_2(exp(\frac{\sqrt{t}}{m}\iota(X)))\eta_2(exp(\frac{\sqrt{t}}{m}\iota(Y)))\eta_2(exp(-\frac{\sqrt{t}}{m}\iota(X)))\eta_2(exp(-\frac{\sqrt{t}}{m}\iota(Y)))\}^m\\
= \ & \lim_{m\to\infty}\{\eta_2(exp(\frac{\sqrt{t}}{m}\iota(X))exp(\frac{\sqrt{t}}{m}\iota(Y))exp(-\frac{\sqrt{t}}{m}\iota(X))exp(-\frac{\sqrt{t}}{m}\iota(Y)))\}^m\\
= \ & \lim_{m\to\infty}\eta_2(\{exp(\frac{\sqrt{t}}{m}\iota(X))exp(\frac{\sqrt{t}}{\iota}(Y))exp(-\frac{\sqrt{t}}{m}\iota(X))exp(-\frac{\sqrt{t}}{\iota}(Y))\}^m)\\
= \ & \eta_2(\lim_{m\to\infty}\{exp(\frac{\sqrt{t}}{m}\iota(X))exp(\frac{\sqrt{t}}{\iota}(Y))exp(-\frac{\sqrt{t}}{m}\iota(X))exp(-\frac{\sqrt{t}}{\iota}(Y))\}^m)\\
= \ & \eta_2(t[\iota(X), \iota(Y)])
\end{aligned}$$

□

Proposition 3.4.15. *Let*

- *(S1)* $G_{1,1}$ *be a Lie group which is isomorphic to a Lie subgroup* $G_{1,2}$ *of* $GL(n,\mathbb{C})$.
- *(S2)* $G_{2,1}$ *be a Lie group which is isomorphic to a Lie subgroup* $G_{2,2}$ *of* $GL(n,\mathbb{C})$.
- *(S3)* $G_{3,1}$ *be a Lie group which is isomorphic to a Lie subgroup* $G_{3,2}$ *of* $GL(n,\mathbb{C})$.
- *(A1)* $f : G_{1,1} \to G_{2,1}$ *is a homomorphism of Lie groups.*
- *(A2)* $g : G_{2,1} \to G_{3,1}$ *is a homomorphism of Lie groups.*
- *(S4) By Proposition3.4.14, homomorphisms of Lie algebras derived from* $f \circ g, f, g$, *respectively. We define* $\Phi(f \circ g), \Phi(f), \Phi(g)$ *are homomorphisms of Lie algebras derived from* $f \circ g, f, g$, *respectively.*

then

$$\Phi(f \circ g) = \Phi(g) \circ \Phi(f) \tag{3.4.15}$$

Proof. Let us fix any $X \in Lie(G_{1,1})$. Because for $t \in \mathbb{R}$ such that $|t| \ll 1$

$$\begin{aligned}
& \eta_3(exp(t\Phi(g \circ f)X)\\
= \ & g \circ f(\eta_1(exp(tX)))\\
= \ & g(\eta_2(expt\Phi(f)X))\\
= \ & \eta_3(exp(t\Phi(g)\Phi(f)X))
\end{aligned}$$

$\Phi(f \circ g) = \Phi(g) \circ \Phi(f)$. □

By Theorem3.4.14, any inner automorphism of G_1 is C^ω-class. By von-Neumann Cartan's theorem, This implies the following two Proposition.

Proposition 3.4.16. *Let*

- *(S1)* G_1 *is a Lie group which is isomorphic to a Lie subgroup* G_2 *of* $GL(n,\mathbb{C})$.

(S2) For sufficient small open neighborhood of 1_{G_2} V and $z \in G_1$, we set $\mu_z : V \ni g \mapsto gz \in G_1$.

then

(i) $\{\mu_z \circ \phi\}_{z\in G_1}$ is a local corrdinate system of G_1 which is equivalent to $\{\eta_z \circ \phi\}_{z\in G_1}$.

(ii) $\{\mu_z \circ \psi\}_{z\in G_1}$ is a local corrdinate system of G_1 which is equivalent to $\{\eta_z \circ \psi\}_{z\in G_1}$.

Proposition 3.4.17. *Let*

(S1) G_1 is a Lie group which is isomorphic to a Lie subgroup G_2 of $GL(n, \mathbb{C})$.

then for any $g \in G_1$,

(i) $l_g : G_1 \ni x \mapsto gx \in G_1$ is C^ω-class homeomorphism.

(ii) $r_g : G_1 \ni x \mapsto xg \in G_1$ is C^ω-class homeomorphism.

These Propositions imply the following theorem.

Theorem 3.4.18 (von Neumann-Cartan's theorem II)**.** *Let*

(S1) G_1 is a Lie group which is isomorphic to a Lie subgroup G_2 of $GL(n, \mathbb{C})$.

(S2) $\mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_m$ are vector subspaces of $Lie(G_2)$ such that

$$Lie(G_2) = \oplus_{i=1}^m \mathfrak{g}_i$$

(S3) $\mathfrak{g}_i(\epsilon) := \{X \in Lie(G_2) | ||X|| < \epsilon\}$ $(i = 1, 2, ..., m, \epsilon > 0)$.

(S4) For any $x \in G_2$

$$\begin{array}{rcl} i_x : \oplus_{i=1}^m \mathfrak{g}_i(\epsilon) & \to & G_2 \\ \Psi & & \Psi \\ (X_1, X_2, ..., X_m) & \mapsto & x exp(X_1) exp(X_2) ... exp(X_m) \end{array}$$

then $G_1 \times G_1 \ni (x, y) \mapsto xy^{-1} \in G_1$ is C^ω-class.

Proposition 3.4.19 (Exponential mapping of Lie algebra)**.** *Let*

(S1) G_1 is a Lie group which is isomorphic to a Lie subgroup G_2 of $GL(n, \mathbb{C})$.

(S2) $\epsilon > 0$ and $exp(Lie(G_1) \cap B(O, \epsilon))$.

(S3) For each $X \in Lie(G_1)$, set $Exp(X) := \eta(exp(\frac{X}{m}))^m$ for $m \in \mathbb{N}$ such that $\frac{X}{m} \in B(O, \epsilon)$.

then the followings hold.

(i) Exp is well-defined and continuous.

Proof of (i). Let us fix any $m, m' \in \mathbb{N}$ such that $\frac{X}{m} \in B(O, \epsilon)$ and $\frac{X}{m'} \in B(O, \epsilon)$. Then $\frac{iX}{mm'} \in B(O, \epsilon)$ $i = 0, 1, ..., max(m, m')$. By the Definition of locally isomorphism(Definition3.1.1),

$$\eta(exp(\frac{t}{m}X))^m = \eta(exp(\frac{t}{mm'}X))^{mm'} = \eta(exp(\frac{t}{m'}X))^{m'}$$

So Exp is well-defined. Because η and exp are continuous and G_1 is topological group, Exp is continuous. □

3.5 Correspondence between Lie groups and Lie algebras

3.5.1 Tangent space of Lie Groups

Proposition 3.5.1. *Let*

(S1) G_1 is a Lie group which is isomorphic to a Lie subgroup G_2 of $GL(n, \mathbb{C})$.

(S2) For each $X \in Lie(G_1)$,

$$\iota(X)(f) := \frac{d}{dt}|_{t=0} f(\eta(exp(tX))) \ (f \in C^\infty(1_{G_1}))$$

then $\iota(Lie(G_1)) \subset T_{1_{G_1}}(G_1)$ *and* $\iota : Lie(G_1) \to T_{1_{G_1}}(G_1)$ *is a isomorphism of vector spaces.*

STEP0:Proof of $\iota(Lie(G_1)) \subset T_{1_{G_1}}(G_1)$. By Leibniz product rule in calculas, $\iota(Lie(G_1)) \subset T_{1_{G_1}}(G_1)$. □

STEP1:Proof of linearity of ι. Let us fix any $X \in Lie(G_1)$ and $a \in \mathbb{R}$. For the formula of the compositition of $f(\eta(exp(\cdot X)))$ and $a\cdot$, $\iota(aX) = a\iota(X)$

And let us fix any $Y \in Lie(G_1)$. By Lemma3.4.1,

$$f(\eta(\psi(t(X+Y)))) = f(\eta(\varphi(\varphi^{-1}\psi(t(X+Y))))) = f(\eta(\varphi((tX,tY)+o(t))))$$

By the chain rule, $\iota(X+Y)(f) = \frac{d}{dt}_{|t=0} f(\eta(\varphi(tX,tY)))$. By applying the chain rulte to the composition of $(u,w) \mapsto f(\eta(\varphi(uX,wY)))$ and $t \mapsto (tX,tY)$,

Because $f(\eta(exp(t(X+Y)))) = f(\eta(exp(tX)exp(tY)+o(t)))$,

$$\frac{d}{dt}_{|t=0} f(\eta(\varphi(tX,tY))) = \iota(X)(f) + \iota(X)(f)$$

□

STEP2:Proof of that ι *is injective.* Let us fin any $X \in Lie(G_1)$ such that $X \neq O$. By linearity of ι, it is enought to show $\iota(X) \neq 0$. There is $X_2, X_3, ..., X_r \in Lie(G_1)$ such that $X, X_2, X_3, ..., X_r$ is a basis of $Lie(G_1)$. Here, $r := Lie(G_1)$. Let us set $f_X(\eta(\psi(t_1,t_2,...,t_r))) := t_1$ for $|t_1| \ll 1, ..., |t_r| \ll 1$. Clearly $f_X \in C^\infty(1_{G_1})$ and $\iota(X)(f_X) = 1$. So $\iota(X) \neq 0$. □

STEP3:Proof of that ι *is surjective.* By Proposition3.4.10, $\dim T_{1_{G_1}} = Lie(G_1)$. By this and STEP1 and STEP2, ι is surjective. □

3.5.2 Homomorphism of Lie groups

Theorem 3.5.2. *Let*

- *(S1)* $G_{1,1}$ *be a Lie group which is isomorphic to a Lie subgroup* $G_{1,2}$ *of* $GL(n,\mathbb{C})$.
- *(S2)* $G_{2,1}$ *be a Lie group which is isomorphic to a Lie subgroup* $G_{2,2}$ *of* $GL(n,\mathbb{C})$.
- *(A1)* $\Phi \in C(G_{1,1}, G_{2,1})$ *is a homomorphism.*

then

- *(i)* $d\Phi_e(i_1(X)) = i_2(\iota(X))$ $(\forall X \in Lie(G_{1,1})$. *Here,* $i_i : Lie(G_{i,1}) \to T_e(G_{i,1})$ $(i=1,2)$ *are isomorphisms of two vector spaces.*
- *(ii)* $\Phi(Exp(X)) = Exp(i_2^{-1}(d\Phi_e(i_1(X))))$ $(\forall X \in Lie(G_{1,1}))$

STEP1. Showing (i). Let us fix any $X \in Lie(G_{1,1})$ and $f \in C^\infty(1_{G_{2,1}})$. Then

$$f(\Phi(\eta_1(exp(tX)))) = f(\eta_2(expt\iota(X)))\ (\forall t : |t| \ll 1)$$

Differentiating both sides by t and setting $t = 0$,

$$d\Phi_e(\iota_1(X))(f) = i_2(\iota(X))(f)$$

□

STEP2. Showing (ii). Let us fix any $X \in Lie(G_{1,1})$. For sufficient large $m \in \mathbb{N}$,

$$\begin{aligned}
\Phi(Exp(X)) &= \Phi(Exp(\frac{1}{m}X))^m \\
&= \Phi(\eta_1(exp(\frac{1}{m}X)))^m \\
&= \eta_2(exp(\iota(\frac{1}{m}X)))^m \\
&= \eta_2(exp(i_2^{-1}(i_2(\iota(\frac{1}{m}X)))))^m \\
&= \eta_2(exp(d\Phi_e(i_1((\frac{1}{m}X)))))^m \\
&= Exp(d\Phi_e(i_1((\frac{1}{m}X))))^m \\
&= Exp(d\Phi_e(i_1((m\frac{1}{m}X)))) \\
&= Exp(d\Phi_e(i_1((X))))
\end{aligned}$$

□

3.5.3 Invariant vector fields of Lie Groups

It is easy to show the following proposition.

Proposition 3.5.3 (Regular representation on $C^\infty(G)$)**.** *Let G_1 be a Lie group which is locally isomorphic to a linear Lie subgroup G_2. For $g \in G_1$ and $f \in C^\infty(G_1)$, we set*

$$\pi_L(g)f(x) := f(g^{-1}x),\ \pi_R(g)f(x) := f(xg),\ (x \in G_1) \tag{3.5.1}$$

Then π_L and π_R are representation of G_1. We call π_L the left regular representation of G_1 and π_R the right regular representation of G_1

Proof. By

$$\begin{aligned} & \pi_L(g_1)\pi_L(g_2)f(x) \\ = & \pi_L(g_2)f(g_1^{-1}x) \\ = & f(g_2^{-1}g_1^{-1}x) \\ = & f((g_1g_2)^{-1}x) \\ = & \pi_L(g_1g_2)f(x) \end{aligned}$$

and

$$\begin{aligned} & \pi_R(g_1)\pi_R(g_2)f(x) \\ = & \pi_R(g_2)f(xg_1) \\ = & f(xg_1g_2) \\ = & \pi_R(g_1g_2)f(x) \end{aligned}$$

π_L and π_R are representation of G_1. □

Definition 3.5.4 ($\mathscr{D}(M)$)**.** *Let M be a C^∞-class manifold. Denote the set of all C^∞-class vector fields by $\mathfrak{X}$. Denote the algebra on $\mathbb{R}$ generated by $C^\infty(M,\mathbb{R})$ and $\mathfrak{X}(M)$ with the operation of $End_{\mathbb{C}}(C^\infty(M))$ by $\mathscr{D}(M)$.*

Definition 3.5.5 (Invariant vector field on a Lie group)**.** *Let G_1 be a Lie group which is locally isomorphic to a Lie subgroup G_2. We call $P \in \mathscr{D}(G_1)$ an left invariant differential operation if $\pi_L(g)P = P\pi_L(g)$ for any $g \in G_1$. We call $P \in \mathscr{D}(G_1)$ an right invariant differential operation if $\pi_R(g)P = P\pi_R(g)$ for any $g \in G_1$. If $P \in \mathfrak{X}(G_1)$ then we call P a left invariant vector field on G_1 by $\mathfrak{X}_L(G_1)$. If $P \in \mathfrak{X}(G_1)$ then we call P a right invariant vector field on G_1. We denote the set of all left invariant differential fields on G_1 by by $\mathfrak{X}_L(G_1)$. We denote the set of all right invariant differential fields on G_1 by by $\mathfrak{X}_R(G_1)$.*

The following clearly holds.

Proposition 3.5.6. *Let G_1 be a Lie group which is locally isomorphic to a Lie subgroup G_2. Then $\mathfrak{X}_L(G_1)$ and $\mathfrak{X}_R(G_1)$ are algebras on $\mathbb{R}$.*

Proposition 3.5.7. *Let*

(S1) G_1 is a Lie group which is isomorphic to a Lie subgroup G_2 of $GL(n,\mathbb{C})$.

(S2) For each $X \in Lie(G_1)$,

$$\iota_L(X)(f)(x) := \frac{d}{dt}|_{t=0} f(x\eta(exp(tX)))\ (f \in C^\infty(1_{G_1},\ x \in G_1)) \tag{3.5.2}$$

and

$$\iota_R(X)(f)(x) := \frac{d}{dt}|_{t=0} f(\eta(exp(-tX))x)\ (f \in C^\infty(1_{G_1},\ x \in G_1)) \tag{3.5.3}$$

then the followings hold.

(i) ι_L is an isomorphism of Lie algebras between $Lie(G_1)$ and $\mathfrak{X}_L(G_1)$. In particular, for anly $X, Y \in Lie(G_1)$

$$[\iota_L(X), \iota_L(Y)] = \iota_L([X,Y]) \tag{3.5.4}$$

(ii) ι_R is an isomorphism of Lie algebras between $Lie(G_1)$ and $\mathfrak{X}_R(G_1)$.

STEP1. $\iota_L(Lie(G_1)) \subset \mathfrak{X}_L(G_1)$. By analiticity of multiple operation of G_1 and the product rule in calculas, $\iota_L(Lie(G_1)) \subset \mathfrak{X}_L(G_1)$. For any $g \in G_1$ and $f \in C^\infty(G_1)$ and $x \in G_1$,

$$\begin{aligned} & \pi_L(g)\iota_L(X)(f)(x) \\ = & \iota_L(X)(f)(g^{-1}x) \\ = & \frac{d}{dt}f((g^{-1}x)\eta(exp(tX)))|_{t=0} \\ = & \frac{d}{dt}f(g^{-1}(x\eta(exp(tX))))|_{t=0} \\ = & \frac{d}{dt}\pi_L(g)f(x\eta(exp(tX)))|_{t=0} \\ = & \iota_L(X)\pi_L(g)f(x) \end{aligned} \tag{3.5.5}$$

So $\iota_L(X)$ is left invariant. □

STEP2. $\iota_R(Lie(G_1)) \subset \mathfrak{X}_R(G_1)$. It is easy to show this by the similar method to STEP1. □

STEP3. ι_L *and* ι_R *are* $\mathbb{R}$*-linear and injective.* It is easy to show this by the similar method to Proposition3.5.1. □

STEP4. ι_L *and* ι_R *are surjective.* Let us fix any $F \in \mathfrak{X}_L(G_1)$. By Proposition3.5.1, there is $X \in Lie(G_1)$ such that

$$F(f)(e) = \iota(X)(f)\ (\forall f \in C^\infty(G_1),\ \forall x \in G_1) \tag{3.5.6}$$

Because F is a left invariant vector field, for any $x \in G_1$,

$$\begin{aligned} & F(f)(x) = \\ = & \pi_L(x^{-1})(F(f))(e) \\ = & F(\pi_L(x^{-1})(f))(e) \\ = & \frac{d}{dt}\pi_L(x^{-1})(f)(\eta(exp(tX)))|_{t=0} \\ = & \frac{d}{dt}f(x\eta(exp(tX)))|_{t=0} \\ = & \iota_L(X)(f)(x) \end{aligned} \tag{3.5.7}$$

□

STEP5. Calculas of $\iota([X, Y])$. Let us fix any $f \in C^\infty(1_{G_1})$.

By Proposition3.2.14,

$$\begin{aligned} & \iota([X, Y])(f) \\ = & \frac{d}{dt}f(\eta(exp(t[X, Y])))|_{t=0} \\ = & \frac{d}{dt}f(\eta(exp(\sqrt{t}X)exp(\sqrt{t}Y)exp(-\sqrt{t}X)exp(-\sqrt{t}Y)))|_{t=0} \end{aligned} \tag{3.5.8}$$

□

STEP6. Taylor expansion of $f(\eta(exp(t_1X_1)exp(t_2X_2)exp(t_3X_3)exp(t_4X_4)))$. By the definition of ι_L, for any $i_4 \in \mathbb{Z} \cap [0, \infty)$,

$$\begin{aligned} & \iota_L(X_4)^{i_4}(f)(exp(t_1X_1)exp(t_2X_2)exp(t_3X_3)) \\ = & (\frac{\partial}{\partial t_4})^{i_4}f(\eta(exp(t_1X_1)exp(t_2X_2)exp(t_3X_3)exp(t_4X_4)|_{t_4=0} \end{aligned} \tag{3.5.9}$$

By repeating the above discussion in the same manner below, for any $i_1, i_2, i_3, i_4 \in \mathbb{Z} \cap [0, \infty)$,

$$\begin{aligned} & \iota_L(X_1)^{i_1}\iota_L(X_2)^{i_2}\iota_L(X_3)^{i_3}\iota_L(X_4)^{i_4}(f)(e) \\ = & (\frac{\partial}{\partial t_1})^{i_1}...(\frac{\partial}{\partial t_4})^{i_4}f(\eta(\Pi_{k=1}^4 exp(t_kX_k))|_{\mathbf{t=0}} \end{aligned} \tag{3.5.10}$$

So,

$$
\begin{aligned}
& f(exp(t_1X_1)exp(t_2X_2)exp(t_3X_3)exp(t_4X_4)) \\
= \ & f(e) \\
+ \ & \sum_{k=1}^{4} \iota_L(X_k)(f) \\
+ \ & \sum_{t_1+...+t_4=2} \frac{1}{i_1!}\frac{1}{i_2!}\frac{1}{i_3!}\frac{1}{i_4!}\iota_L(X_1)^{i_1}...\iota_L(X_4)^{i_4}f(e)t^{i_1}...t^{i_4} \\
+ \ & o(|\boldsymbol{t}|^2)
\end{aligned}
\tag{3.5.11}
$$

□

STEP7. Showing $\iota_L([X,Y]) = [\iota_L(X), \iota_L(Y)]$. In we set $t_1 = t_2 = -t_3 = -t_4 = t$ and $X_1 = -X_3 = X$ and $X_2 = -X_4 = Y$ in (3.5.11),

$$
\begin{aligned}
& f(exp(\sqrt{t}X)exp(\sqrt{t}Y)exp(-\sqrt{t}X)exp(-\sqrt{t}Y)) \\
= \ & f(e) \\
+ \ & [\iota(X), \iota(X)](f)t \\
+ \ & o(|\boldsymbol{t}|)
\end{aligned}
\tag{3.5.12}
$$

By (3.5.8),

$$\iota([X,Y])(f) = [\iota(X), \iota(X)](f) \tag{3.5.13}$$

□

STEP4 in the proof of Proposition3.5.7 implies the following Proposition.

Proposition 3.5.8. *Let G_1 be a Lie group which is locally isomorphic to a Lie subgroup G_2. Let us fix any $F_1, F_2 \in \mathfrak{X}_L(G_1)$ such that $F_1(f)(e) = F_2(f)(e)$ $(\forall f \in C^\infty(e))$. Then $F_1 = F_2$.*

3.5.4 Taylor expansion of C^ω-class function

STEP6 in the proof of Proposition3.5.7 implies the following Proposition.

Proposition 3.5.9. *Let*

(S1) G_1 be a Lie group which is locally isomorphic to a Lie subgroup G_2.

(S2) f be a C^∞-class function at a neighborhood of 1_{G_1}.

(S3) $X_1, ..., X_m \in Lie(G_1)$.

(S4) $g(\boldsymbol{t}) := f(\sum_{i=1}^m t_iX_i)$.

Then

$$(\frac{\partial}{\partial t_1})^{i_1}...(\frac{\partial}{\partial t_m})^{i_m}g(0) = \iota_L(X_1)^{i_1}...\iota_L(X_m)^{i_m}f \tag{3.5.14}$$

Theorem 3.5.10. *Let*

(S1) $G_{1,1}$ is a Lie group which is isomorphic to a Lie subgroup $G_{1,2}$ of $GL(n,\mathbb{C})$.

(S2) $G_{2,1}$ is a Lie group which is isomorphic to a Lie subgroup $G_{2,2}$ of $GL(n,\mathbb{C})$.

then the followings are equivalent.

(i) $Lie(G_{1,1})$ and $Lie(G_{2,1})$ are isomorphic.

(ii) $G_{1,1}$ and $G_{2,1}$ are locally isomorphic.

Proof of (ii) $\implies$ (i). If (ii), by the same argument of the proof of Proposition3.4.14 and Lemma3.4.7 and von Neumann-Cartan's theorem, (ii) $\implies$ (i). □

Proof of (i) $\implies$ (ii). Let $\Phi : Lie(G_{1,1} \to Lie(G_{2,1}$ be an isomorphism. Let $X_{1,1}, ..., X_{1,m}$ be a basis of $Lie(G_{1,1}$. And let us set $X_{2,i} := \Phi(X_{1,i})$ $(i = 1, 2, ..., m)$. We set $e_j : (-\epsilon, \epsilon)^m \ni (t_1, ..., t_m) \to \Pi_{i=1}^m exp(t_i X_{j,i})$ $(j = 1, 2)$. There is $\epsilon > 0$ such that $e_j((-\epsilon, \epsilon)^m)$ is an open subset of G_j and $e_j((-\epsilon, \epsilon)^m) \subset V_j$ and e_j is homeomorphism$(j = 1, 2)$.

We set $\Psi : \eta_1(e_1((-\epsilon, \epsilon)^m)) \to \eta_2(e_2((-\epsilon, \epsilon)^m))$ by $\Psi(e_1(\boldsymbol{t})) := e_2(\boldsymbol{t})$. There is $\delta > 0$ such that $e_j((-\delta, \delta)^m)e_j((-\delta, \delta)^m) \subset e_j((-\epsilon, \epsilon)^m)$ $(j = 1, 2)$. We set $\phi_{j,i} : (-\delta, \delta)^{2m} \to (-\epsilon, \epsilon)$ by

$$e_j(\boldsymbol{x})e_j(\boldsymbol{y}) = e_j(\phi_{j,1}(\boldsymbol{x}, \boldsymbol{y}), ..., \phi_{j,m}(\boldsymbol{x}, \boldsymbol{y})) \tag{3.5.15}$$

$(j = 1, 2)$. We set $\psi_{j,i}(e_j(\boldsymbol{x})e_j(\boldsymbol{y})) := \phi_{j,i}(\boldsymbol{x}, \boldsymbol{y})$. By von Neumann-Cartan's theorem, $\phi_{\{j,i\}}$ are relal analytic functions. So, for each j, i there are $C_{j,i,I,J}$ $I, J \in \mathbb{Z}^m$

$$\phi_{j,i}(\boldsymbol{x}, \boldsymbol{y}) = \sum C_{j,i,I,J} t^{(I,J)} \tag{3.5.16}$$

We will show $\phi_{1,i} = \phi_{2,i}$ $(i = 1, 2, ..., m)$. By Proposition3.5.9,

$$C_{j,i,I,J} = \iota_L(X_1)^{i_1} ... \iota_L(X_m)^{i_m} \iota_L(X_1)^{j_1} ... \iota_L(X_m)^{j_m} \psi_{j,i}(0) \tag{3.5.17}$$

Let us fix $k, l \in \{1, 2, ..., m\}$. Because Φ is an isomorphism, there is $c_{k,l,1}, ..., c_{k,l,m} \in \mathbb{R}$ such that

$$[X_{j,k}, X_{j,l}] = \sum_{i=1}^{m} c_{k,l,i} X_{j,i} \tag{3.5.18}$$

So, by (3.5.4),

$$\iota_L(X_{j,k})\iota_L(X_{j,l}) = \iota_L(X_{j,l})\iota_L(X_{j,k}) + \sum_{i=1}^{m} c_{k,l,i}\iota_L(X_{j,i}) \tag{3.5.19}$$

By repeating apply of this equation to $\iota_L(X_1)^{i_1} ... \iota_L(X_m)^{j_m}$, $C_{1,i,I,J} = C_{2,i,I,J}$. So $\phi_{1,i} = \phi_{2,i}$ $(i = 1, 2, ..., m)$.

We set $W_j := \eta_j(e_j((-\delta, \delta)^2 m))$ $j = 1, 2$. Because $\phi_{1,i} = \phi_{2,i}$ $(i = 1, 2, ..., m)$, for each $x, y \in W_1$

$$xy \in W_1 \iff \Psi(x)\Psi(y) \in W_2 \tag{3.5.20}$$

and if $xy \in W_1$

$$\Psi(xy) = \Psi(x)\Psi(y) \tag{3.5.21}$$

Consequently, $G_{1,1}$ and $G_{2,1}$ are locally isomorphic. □

3.5.5 Differential representation

Clearly the following holds.

Proposition 3.5.11 (Definition of differential representation of a continuous representation of Lie group)**.** *Let*

- *(S1) G_1 is a Lie group which is locally isomorphic to a Lie subgroup of $GL(n, \mathbb{C})$. G_2 has at most countable connected components.*
- *(S2) (π, V) is a finite dimensional continuous representation of G_1.*
- *(S3) $P := \{v_1, v_2, ..., v_r\}$ is a basis of V.*
- *(S4) For each $f \in End_{\mathbb{C}}(V)$, denote the representation matrix with respect to P by $\Phi(f)$.*
- *(S5) By $\Phi|GL(V) : GL(V) \to GL(n, \mathbb{C})$, introduces a topology of $GL(V)$.*

Then

- *(i) $\Phi|GL(V) : GL(V) \to GL(n, \mathbb{C})$ is an isomorphism of topological groups. So, $GL(V)$ is a Lie group.*
- *(ii) $\pi : G_1 \to GL(V)$ is an homomorphism of Lie groups.*
- *(iii) $Lie(GL(V)) = M(n\mathbb{C})$. By Proposition3.4.14, π introduces the homomorphism from $Lie(G_1)$ to $M(n\mathbb{C})$. we denote this homomorphism by $d\pi_e$. We call $d\pi_e$ the differential representation of π.*
- *(iv) $d\pi$ is continuous.*
- *(v)*

$$d\pi(X) = \frac{d}{dt}|_{t=0}|\pi(Exp(tX)) \; (\forall X \in Lie(G_1))$$

Proof of (iv). Because $d\pi$ is a linear mapping from $Lie(G_1)$ to $M(n\mathbb{C})$, $d\pi$ is continuous. □

Proof of (v). Let us fix any $X \in Lie(G_1)$. From the definition of $d\pi$,

$$\pi(Exp(tX)) = Exp(td\pi(X)) \ (\forall t \in \mathbb{R})$$

Then (v) holds. □

The following clearly holds.

Proposition 3.5.12. *The followings are settings and assumptions.*

- *(S1) G is a connected Lie group.*
- *(S2) (π, V) is a finite dimensional continuous representation of G.*
- *(S3) W is a $\mathbb{C}$-linear subspace of V.*

Then W is G-invariant if and only if W is $d\pi(Lie(G))$- invariant.

Proposition 3.5.13 (Adjoint representation of a Lie group)**.** *Let*

- *(S1) G_1 is a Lie group which is locally isomorphic to a Lie subgroup of $GL(n, \mathbb{C})$. G_2 has at most countable connected components.*
- *(S2) For each $g \in G_1$, we define $\sigma(g) \in Auto(G)$ by $\sigma(g)(x) := gxg^{-1}$ $(x \in G_1)$.*

Then

- *(i) For any $g \in G_1$, $\sigma(g)$ is an automorhism of a Lie group. By Proposition3.4.14, we denote the endmorphism of $Lie(G_1)$ by $Ad(g)$.*
- *(ii) $Ad(G_1) \subset GL(Lie(G_1))$*
- *(iii) $(Ad, GL(Lie(G_1)))$ is a continuous representation of G_1 on $\mathbb{R}$.*

Proof of (i). Because $\sigma(g^{-1}) = \sigma(g)^{-1}$ and analyticity of the group operation on G_1, (i) holds. □

Proof of (ii). Because $\sigma(1_{G_1}) = id_{G_1}$, $Ad(1_{G_1}) = id_{Lie(G_1)}$. Let us fix any $g, h \in G_1$. Because $\sigma(gh) = \sigma(g)\sigma(h)$, $Ad(gh)$ is the homomorphism of a Lie algebra $Lie(G_1)$ derived from $\sigma(g)\sigma(h)$. By Proposition3.4.15, $Ad(gh) = Ad(g)Ad(h)$. So, $Ad(G_1) \subset GL(Lie(G_1))$. □

Proof of (iii). Let us fix $v := (v_1, v_2, ..., v_r)$ which is a basis of $Lie(G_1)$. We denote the representation matrix of $Ad(g)$ respect to v by $R(g)$. Let us fix $\epsilon > 0$ such that $exp(B(O, \epsilon) \cap Lie(G_1)) \subset V$. Let us fix $\delta > 0$ such that $\{vY | Y \in B(0, 2\delta) \cap \mathbb{C}^r\} \subset B(O, \epsilon) \cap Lie(G_1)$. For any $Y \in B(0, 1) \cap \mathbb{C}^r$, $exp(\delta Ad(g)vY) = \tau(g\eta(exp(\delta Y))g^{-1})$. So,

$$vR(g)Y = \frac{1}{\delta} log(\tau(g\eta(exp(\delta Y))g^{-1})) \tag{3.5.22}$$

By setting $Y = e_1$, ..., $Y = e_r$, $vR(\cdot)$ is continuous. Because v is $N \times r$-matrix and $rank(v) = r$, $R(\cdot)$ is continuous. So, $(Ad, Lie(G_1))$ is a continuous representation of G_1. □

Proposition 3.5.14. *Here are the settings and assumptions.*

- *(S1) G_1 is a Lie group which is locally isomorphic to a Lie subgroup of $GL(n, \mathbb{C})$.*

Then

- *(i) $dAd = ad$.*
- *(ii) $Ad(Exp(X)) = Exp(ad(X))$ $(\forall X \in Lie(G_1)$.*

Proof of (i). Let us assume $i : Lie(G_1) \to T_e(G_1)$ be an isomorphism of vector spaces in Proposition3.5.1. Let us fix any $X, Y \in Lie(G_1)$ and $s, t \in \mathbb{R}$ such that $|s| \ll 1, |t| \ll 1$ and $f \in C^\infty(e)$. Then

$$f(Exp(sAd(Exp(tX))Y)) = f(Exp(tX)Exp(sY)Exp(-tX))$$

And, by Proposition3.5.2,

$$Ad(Exp(tX)) = exp(tdAd(X))$$

Because

$$\begin{aligned} & \frac{d}{dt}|_{t=0}\frac{d}{ds}|_{s=0}f(Exp(tX)Exp(sY)Exp(-tX)) \\ = & \frac{d}{ds}|_{s=0}\frac{d}{dt}|_{t=0}f(\eta(exp(sY)+st[X,Y]+O(t^2))) \\ = & \frac{d}{ds}|_{s=0}i(s[X,Y])(f) \\ = & i([X,Y])(f) = i(ad(X)Y)(f) \end{aligned}$$

and

$$\begin{aligned} & \frac{d}{dt}|_{t=0}\frac{d}{ds}|_{s=0}f(Exp(sAd(Exp(tx))Y)) \\ = & \frac{d}{dt}|_{t=0}i(Ad(Exp(tX))(Y))(f) \\ = & \frac{d}{dt}|_{t=0}i(exp(tdAd(X)(Y))(f) \\ = & \frac{d}{dt}|_{t=0}i(E+tdAd(X)(Y)+O(t^2))(f) \\ = & \frac{d}{dt}|_{t=0}i(E)(f)+ti(dAd(X)(Y))(f)+O(t^2) \\ = & i(dAd(X)(Y))(f) \end{aligned}$$

$i(dAd(X)(Y))(f) = i(ad(X)Y)(f)$. So, $dAd = ad$. □

Proof of (ii). By (3.5.5) and (i),

$$Ad(Exp(X)) = Exp(dAd(X)) = Exp(ad(X))$$

□

Proposition 3.5.15. *Here are the settings and assumptions.*

(S1) G is a linear Lie group of $GL(n,\mathbb{C})$.

Then for any $g \in G$

(i) The representation matrix of $Ag(g)$ is $g \otimes (g^T)^{-1}$ with basis $\{E_{i,j}\}_{i,j}$.

(ii) $det(Ag(g)) = 1$.

Proof of (i). We set $h := g^{-1}$. Let us fix any i_0, j_0 and i, j. Then

$$(gE_{i_0,j_0}g^{-1})_{i,j} = (gE_{i_0,j_0}g^{-1})_{i,j} = \sum_l (gE_{i_0,j_0})_{i,l}h_{l,j} = \sum_{k,l} g_{i,k}(E_{i_0,j_0})_{k,l}h_{l,j} = g_{i,i_0}h_{j_0,j} = g_{i,i_0}h^T_{j,j_0}$$

So, the representation matrix of $Ag(g)$ is $g \otimes (g^T)^{-1}$. □

Proof of (ii). By Proposition2.1.11 and (i), (ii) holds. □

3.5.6 Baker-Campbell-Hausdorff formula

Proposition 3.5.16. *Here are the settings and assumptions.*

(S1) $S, T \in M(n,\mathbb{C})$.

Then

$$\frac{d}{ds}|_{s=0}exp(-S)exp(S+sT) = \frac{E-exp(-ad(S))}{ad(S)}T = \sum_{p=0}(-1)^p\frac{ad(S)^p}{(p+1)!}T$$

STEP1. Simplifieing S. Clearly

$$\frac{d}{ds}|_{s=0}exp(-S)exp(S+sT)$$

and

$$\sum_{p=0}(-1)^p\frac{ad(S)^p}{(p+1)!}T$$

are continuous respects to S. For any $P \in GL(n, \mathbb{C})$

$$\begin{aligned} & P\frac{d}{ds}|_{s=0} exp(-S)exp(S+sT)P^{-1} \\ = & \frac{d}{ds}|_{s=0} exp(-PSP^{-1})exp(PSP^{-1}+sPTP^{-1}) \end{aligned}$$

and

$$\begin{aligned} & P\sum_{p=0}(-1)^p \frac{ad(S)^p}{(p+1)!}TP^{-1} \\ = & \sum_{p=0}(-1)^p \frac{ad(PSP^{-1})^p}{(p+1)!}PTP^{-1} \end{aligned}$$

So, we can assume S is a diagonal matrix. □

STEP2. Linearity respects to T. By Wierstrass's theorem,

$$\begin{aligned} & exp(-S)exp(S+sT) \\ = & exp(-S) \lim_{m\to\infty} \frac{d}{ds}|_{s=0} \sum_{i=0}^{m} \frac{(S+sT)^i}{i!} \end{aligned}$$

We set

$$L_m(T) := exp(-S)\frac{d}{ds}|_{s=0} \sum_{i=0}^{m} \frac{(S+sT)^i}{i!}$$

Because

$$\begin{aligned} & \frac{d}{ds}|_{s=0}(S+sT)^i \\ = & \frac{d}{ds}|_{s=0} \sum_{j=0} sS^jTS^{i-j-1} + o(s) \\ = & \sum_{j=0} S^jTS^{i-j-1} \end{aligned}$$

$L_m(\cdot)$ is linear for any $m \in \mathbb{N}$. Because $L_m(\cdot)$ normed converges to

$$\frac{d}{ds}|_{s=0} exp(-S)exp(S+s\cdot)$$

$\frac{d}{ds}|_{s=0} exp(-S)exp(S+s\cdot)$ is linear. □

STEP3. Simplifying T. By STEP2, we can assume $T = E_{i,j}$. □

STEP4. Showing this equation. If $[S,T]=0$, the both side equals to T. So, we can assume $[S,T] \neq 0$. We set $\lambda_1, ..., \lambda_n$ by

$$S = \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n \end{pmatrix}$$

We set $\lambda = \lambda_i - \lambda_j$. Then

$$ST = \lambda T$$

Because $[S,T] \neq 0$, $\lambda_i \neq \lambda_j$ and $i \neq j$. Because $\lambda_j T$ and T are commutative, by replacing S by $S - \lambda_j T$, we can assume $\lambda_j = 0$. Then

$$TS = T^2 = O$$

So

$$ad(S)T = \lambda T$$

$$
\begin{aligned}
& \frac{d}{ds}|_{s=0} exp(-S)exp(S+sT) \\
= & \frac{d}{ds}|_{s=0} exp(-S)\{\sum_{i=1}^{m} s\frac{S^{i-1}}{i!}T + o(1)\} \\
= & exp(-S)\sum_{i=1}^{m}\frac{S^{i-1}}{i!}T \\
= & exp(-S)\sum_{i=1}^{m}\frac{\lambda^{i-1}}{i!}T \\
= & exp(-\lambda)\sum_{i=1}^{m}\frac{\lambda^{i-1}}{i!}T \\
= & exp(-\lambda)\frac{exp\lambda - 1}{\lambda}T \\
= & \frac{1-exp(-\lambda)}{\lambda}T \\
= & \sum_{i=1}^{m}(-1)^{i+1}\frac{\lambda^{i-1}}{i!}T \\
= & \sum_{i=1}^{m}(-1)^{i+1}\frac{ad(S)^{i-1}}{i!}T
\end{aligned}
$$

Consequently, this Proposition holds. □

Proposition 3.5.17. *Let*

(S1) $S, T \in M(n, \mathbb{C})$.

Then

(i) If $|t| < \dfrac{log2}{||X|| + ||Y||}$ *then* $Z(t) := log(exp(tX)exp(tY))$ *converges.*

(ii) We set $\{Z_m\}_{m=1}^{\infty}$ *by* $Z(t) = \sum_{m=1}^{\infty} Z_m t^m$ *then*

$$Z_1 = X + Y$$

and for any $m \in \mathbb{N} \cap [2, \infty)$

$$Z_m = \sum_{\epsilon \in \{0,1\}^{m-2}} C_\epsilon ad(W_{\epsilon_1})...ad(W_{\epsilon_{m-2}})ad(X)Y \tag{3.5.23}$$

Here $W_0 := X$ *and* $W_1 := Y$ *and* $C_\epsilon \in \mathbb{Q}$ *and* C_ϵ *does not* X, Y.

(iii) If $||X|| + ||Y|| < log2$ *then* $Z := \sum_{m=1}^{\infty} Z_m$ *exists and* $exp(X)exp(Y) = expZ$.

Proof of (i). If $|t| < \dfrac{log2}{||X|| + ||Y||}$ then

$$
\begin{aligned}
& ||exp(tX)exp(tY) - E|| \\
\leq & \lim_{m\to\infty} ||\sum_{i=0}^{m}\frac{1}{i!}t^i X^i|\sum_{i=0}^{m}\frac{1}{i!}t^i Y^i - E|| \\
\leq & \lim_{m\to\infty} |\sum_{i=0}^{m}\frac{1}{i!}|t|^i||X||^i\sum_{i=0}^{m}\frac{1}{i!}|t|^i||Y||^i - 1| \\
\leq & |exp|t|||X|||exp|t|||Y|| - 1| \\
\leq & |exp|t|(||X|| + ||Y||) - 1| \\
< & 1
\end{aligned}
$$

So, if $|t| < \dfrac{log2}{||X|| + ||Y||}$ then $log(exp(tX)exp(tY))$ converges. □

Proof of (ii). By Proposition3.5.14,

$$\begin{aligned} & \frac{d}{dt}exp(Z(t)) \\ = & \frac{d}{dt}exp(tX)exp(tY) \\ = & exp(tX)Xexp(tY) + exp(tX)exp(tY)Y \\ = & exp(tX)exp(tY)exp(-tY)Xexp(tY) + exp(tX)exp(tY)Y \\ = & exp(Z(t))(exp(-tY)Xexp(tY) + Y) \\ = & exp(Z(t))(exp(-tad(Y))X + Y) \end{aligned}$$

So

$$exp(-Z(t))\frac{d}{dt}exp(Z(t)) = exp(-tad(Y))X + Y$$

Because

$$\begin{aligned} & exp(-Z(t))\frac{d}{dt}exp(Z(t)) \\ = & exp(-Z(t))\frac{d}{ds}|_{s=0}exp(Z(t+s)) \\ = & exp(-Z(t))\frac{d}{ds}|_{s=0}exp(Z(t) + sZ'(t) + o(s)) \\ = & exp(-Z(t))\frac{d}{ds}|_{s=0}exp(Z(t) + sZ'(t)) + o(s) \\ = & exp(-Z(t))\frac{d}{ds}|_{s=0}exp(Z(t) + sZ'(t)) \end{aligned}$$

by Proposition3.5.16,

$$\sum_{p=0}(-1)^p\frac{ad(Z(t))^p}{(p+1)!}Z'(t) = exp(-tad(Y))X + Y$$

So,

$$Z'(t) = \sum_{p=1}(-1)^{p+1}\frac{ad(Z(t))^p}{(p+1)!}Z'(t) + exp(-tad(Y))X + Y$$

Because

$$\sum_{p=1}(-1)^{p+1}\frac{ad(Z(t))^p}{(p+1)!}Z'(t)$$

has no constant,

$$Z_1 = X + Y$$

We assume $Z_1, ..., Z_m$ satisfies the condition (3.5.23). Because

$$Z(t) = Z_1t + Z_2t^2 + ... + Z_mt^m + ...$$

and

$$Z'(t) = t + 2Z_2t + ... + mZ_mt^{m-1} + (m+1)Z_{m+1}t^m...$$

$$(m+1)Z_{m+1} = \sum_{k=1}^{m}\sum_{i_1+...+i_k+(l-1)=m-1} lZ_{i_1}...Z_{i_k}Z_l + \frac{(-1)^m}{m!}ad(Y)^mX$$

Because of (3.5.6) and the assumption of this mathematical induction,

$$\begin{aligned} & (m+1)Z_{m+1} \\ = & \sum_{\epsilon\in\{0,1\}^{m-1}} D_{1,\epsilon}ad(W_{\epsilon_1})...ad(W_{\epsilon_{m-1}})ad(X)X \\ & + \sum_{\epsilon\in\{0,1\}^{m-1}} D_{2,\epsilon}ad(W_{\epsilon_1})...ad(W_{\epsilon_{m-1}})ad(X)Y \\ & + \sum_{\epsilon\in\{0,1\}^{m-1}} D_{3,\epsilon}ad(W_{\epsilon_1})...ad(W_{\epsilon_{m-1}})ad(Y)X \\ & + \sum_{\epsilon\in\{0,1\}^{m-1}} D_{4,\epsilon}ad(W_{\epsilon_1})...ad(W_{\epsilon_{m-1}})ad(Y)Y \end{aligned}$$

Because $ad(X)X = 0$ and $ad(Y)Y = 0$ and $ad(Y)X = -ad(X)Y$,

$$\begin{aligned}&(m+1)Z_{m+1}\\ =&\sum_{\epsilon\in\{0,1\}^{m-1}}(D_{2,\epsilon}-D_{3,\epsilon})ad(W_{\epsilon_1})...ad(W_{\epsilon_{m-1}})ad(X)Y\end{aligned}$$

So Z_{m+1} satisfies the condition (3.5.23). □

3.5.7 Analytic subgroup

Theorem 3.5.18 (Analytic subgroup)**.** *Let*

(S1) G_1 is a Lie group which is locally isomorphic to a linear Lie subgroup G_2 of $GL(n,\mathbb{C})$.

(S2) $\mathfrak{h}$ be a Lie subalgebra of $Lig(G_1)$.

Then there is H such that H is a subgroup of G_1 and H is a Lie group and $Lie(H) = \mathfrak{h}$. We say H is a analytic subgroup of G whose Lie algebra is $\mathfrak{h}$.

STEP1. Construction of H. There are $X_1,...,X_k,...,X_m,...,X_N \in M(n,\mathbb{C})$ such that $N = n^2$ and $X_1,...,X_N$ is a basis of $M(n,\mathbb{C})$ $X_1,...,X_k,...,X_m$ is a basis of $Lie(G_1)$ and $X_1,...,X_k$ is a basis of $\mathfrak{h}$. By von Neumann-Cartan's theorem, there is $\epsilon > 0$ such that

$$e : (-\epsilon,\epsilon)^m \ni t \mapsto Exp(\sum_{i=1}^{m} t_iX_i) \in G_1$$

is a C^ω-class homeomorphism to an open subset of U and

$$E : (-\epsilon,\epsilon)^N \ni t \mapsto Exp(\sum_{i=1}^{N} t_iX_i) \in GL(n\mathbb{C})$$

is a C^ω-class homeomorphism to an open subset of $GL(n\mathbb{C})$. We set

$$H := \{Exp(X_1)...Exp(X_l)|X_1,...,X_l \in \mathfrak{h},\ l \in \mathbb{N}\}$$

Clearly H is subgroup of G_1. □

STEP2. Constructing the topology of H. We set the topology of H whose fundamental neighborhood system of H is $\{hExp(B_k(O,s\epsilon))|0 \le s < 1,\ h \in H\}$. We will show $\{hExp(B_k(O,s\epsilon))|0 \le s < 1,\ h \in H\}$ satisfies the aixoms of a fundamental neighborhood system.

Let us fix any $exp(\sum_{i=1}^k t_iX_i)$ such that $t \in (-s\epsilon,s\epsilon)^k$. We will show there is $\delta > 0$ such that

$$exp(\sum_{i=1}^{k} t_iX_i)exp(\sum_{i=1}^{k}(-\delta,\delta)X_i) \subset exp(\sum_{i=1}^{k}(-s\epsilon,s\epsilon)X_i) \tag{3.5.24}$$

There is $\epsilon_1 > 0$ such that $t + (-\epsilon_1,\epsilon_1)^k \subset (-s\epsilon,s\epsilon)^k$. There is $\delta \in (0,\epsilon)$ such that

$$exp(\sum_{i=1}^{k} t_iX_i)exp(\sum_{i=1}^{k}(-\delta,\delta)X_i) \subset exp(\sum_{i=1}^{k} t_iX_i + \sum_{i=1}^{N}(-\epsilon_1,\epsilon_1)X_i)$$

By the continuity of exp and log, we can assume

$$log(exp(\sum_{i=1}^{k} t_iX_i))exp(\sum_{i=1}^{k}(-\delta,\delta)X_i)) \subset \sum_{i=1}^{N}(-\epsilon,\epsilon)X_i$$

By Baker-Campbell-Hausdorff formula,

$$log(exp(\sum_{i=1}^{k} t_iX_i)exp(\sum_{i=1}^{k}(-\delta,\delta)X_i)) \subset \sum_{i=1}^{N}(-\epsilon,\epsilon)X_i \cap \mathfrak{h}$$

Because $exp|(\sum_{i=1}^{N}(-\epsilon,\epsilon)X_i)$ is injective,

$$\begin{aligned} & exp(\sum_{i=1}^{k} t_iX_i))exp(\sum_{i=1}^{k}(-\delta,\delta)X_i) \\ \subset\ & exp(\sum_{i=1}^{k}(-\epsilon,\epsilon)X_i \cap \sum_{i=1}^{k} t_iX_i + \sum_{i=1}^{N}(-\epsilon_1,\epsilon_1)X_i) \\ =\ & exp(\sum_{i=1}^{k} t_iX_i + \sum_{i=1}^{k}(-\epsilon_1,\epsilon_1)X_i) \\ \subset\ & exp(\sum_{i=1}^{k}(-s\epsilon,s\epsilon)X_i) \end{aligned}$$

Let us fix any $h_1, h_2 \in H$ such that

$$h_1Exp(B_k(O,s_1\epsilon)) \cap h_2Exp(B_k(O,s_2\epsilon)) \neq \phi$$

Then there is $u_1 \in Exp(B_k(O,s_1\epsilon))$ and $u_2 \in Exp(B_k(O,s_2\epsilon))$ such that $h_1u_1 = h_2u_2$. By (3.5.24), there is $\delta > 0$ such that $u_1Exp(B_k(O,\delta)) \subset Exp(B_k(O,s_1\epsilon))$ and $u_2Exp(B_k(O,\delta)) \subset Exp(B_k(O,s_2\epsilon))$.

$$\begin{aligned} & h_1Exp(B_k(O,s_1\epsilon)) \supset h_1u_1Exp(B_k(O,\delta)) \\ =\ & h_1u_2Exp(B_k(O,\delta)) \subset h_2Exp(B_k(O,s_2\epsilon)) \end{aligned}$$

Consequently, $\{hExp(B_k(O,s\epsilon))|0 \leq s < 1,\ h \in H\}$ satisfies the aixoms of a fundamental neighborhood system.

□

STEP3. Showing properties of H. Clearly $Exp : \mathfrak{h} \to H$ is continuous. Because $B_k(O,\epsilon)$ is connected and Exp is continuous, $Exp(B_k(O,\epsilon))$ is a connected. So H is connected. And clearly H is Housdorff space. □

STEP4. Showing H is a topological group. It is enough to show continuity of the multiple operation and the inverse operation of H. Let us fix any $g_1, g_2 \in H$ and $s \in [0,1)$. We set $g := g_1^{-1}g_2$. It is enouth to show for sufficient small $s_1, s_2 \in [0,1)$ $\{g_1Exp((B_k(O,s_1\epsilon))\}^{-1}g_2Exp((B_k(O,s_2\epsilon))$ is contained $gExp((B_k(O,s\epsilon))$. For sufficient small $X, Y \in \mathfrak{h}$,

$$\begin{aligned} & \{g_1Exp(X)\}^{-1}g_2Exp(Y) \\ =\ & Exp(-X)gExp(Y) \\ =\ & gg^{-1}Exp(-X)gExp(Y) \\ =\ & gExp(-Ad(g^{-1})X)Exp(Y) \end{aligned}$$

By the defitnition of H, there are $Z_1, ..., Z_k \in \mathfrak{h}$ such that

$$g^{-1} = exp(Z_1)...exp(Z_k)$$

So, by Proposition3.5.14,

$$\begin{aligned} & Ad(g^{-1})X \\ =\ & Ad(exp(Z_1))...Ad(exp(Z_k))X \\ =\ & exp(ad(Z_1))...exp(ad(Z_k))X \end{aligned}$$

By Proposition3.4.2, $\mathfrak{h}$ is a closed subset of $M(n,\mathbb{C})$. So, $Ad(g^{-1})X \in \mathfrak{h}$. By Baker-Campbell-Hausdorff's formula, for sufficient small $X, Y \in \mathfrak{h}$,

$$Exp(-Ad(g^{-1})X)Exp(Y) \in Exp((B_k(O,s\epsilon))$$

So, the multiple operation and the inverse operation of H are continuous.

□

STEP5. Showing H is a Lie group. We can assume $\tau(e((-\epsilon,\epsilon)^m)) \subset V$. By Baker-Campbell-Hausdorff's formula, there is $\epsilon_1 > 0$ such that

$$\tau(e([-\epsilon_1,\epsilon_1]^k \times \{0\}^{m-k}))\tau(e([-\epsilon_1,\epsilon_1]^k \times \{0\}^{m-k})) \subset \tau(e((-\epsilon,\epsilon)^k \times \{0\}^{m-k}))$$

We set $V_H := \tau(e([-\epsilon_1, \epsilon_1]^k \times \{0\}^{m-k}))$. Clearly V_H is a neighborhood of the unit element in H and $V_H \subset V$. Because $\tau(e([-\epsilon_1, \epsilon_1]^k \times \{0\}^{m-k}))$ is compact subset of $GL(n, \mathbb{C})$, V_H is closed subset of $GL(n, \mathbb{C})$. We will show the topology of V_H is equal to the relative topology of $GL(n, \mathbb{C})$. It is enough to show for any $t \in [-\epsilon_1, \epsilon_1]^k$ such that for any $\alpha < \epsilon$

$$V_H \cap exp(\sum_{i=1}^k t_i X_i) exp(\sum_{i=1}^k (-\alpha, \alpha) X_i) = V_H \cap exp(\sum_{i=1}^k t_i X_i) exp(\sum_{i=1}^N (-\alpha, \alpha) X_i)$$

Let us fix any $t \in [-\epsilon_1, \epsilon_1]^k$ and $\alpha < \epsilon$ and

$$exp(\sum_{i=1}^k t_i X_i) u \in exp(\sum_{i=1}^k t_i X_i) exp(\sum_{i=1}^N (-\alpha, \alpha) X_i) \cap V_H$$

Because $exp(\sum_{i=1}^k -t_i X_i) exp(\sum_{i=1}^k [-\epsilon_1, \epsilon_1] X_i) \subset exp(\sum_{i=1}^k (-\epsilon, \epsilon) X_i)$ and exp is injective in $\sum_{i=1}^N (-\epsilon, \epsilon) X_i$,

$$u \in exp(\sum_{i=1}^k (-\epsilon, \epsilon) X_i)$$

So,

$$exp(\sum_{i=1}^k t_i X_i) u \in exp(\sum_{i=1}^k t_i X_i) exp(\sum_{i=1}^k (-\alpha, \alpha) X_i)$$

Consequently, H is a Lie group. Clearly $Lie(H) = \mathfrak{h}$. □

Proposition 3.5.19. *Let G be a Lie group and H is a closed subgroup of G. Then H is a Lie group.*

STEP1. Showing that H has at most countable connected components. For any $h \in H$, the connected component of H which contains h(called H_h) is contained some connected component of G.So, H has at most countable connected components. □

STEP2. Showing that H is a Lie group. We set

$$\mathfrak{h} := \{X \in M(n, \mathbb{C}) | Exp(tX) \in U \cap H \ (|t| \ll 1)\}$$

Because $U \cap H$ is closed, by the argument which is similar to the proof of Proposition3.3.2, $\mathfrak{h}$ is a Lie algebra. And clearly $\mathfrak{h}$ is a Lie subalgebra of $Lie(G)$. Let us take $X_1, ..., X_k, ..., X_m, ..., X_N$ which is a basis of $M(n, \mathbb{C})$ such that $X_1, ..., X_k$ is a basis of $\mathfrak{h}$ and $X_1, ..., X_m$ is a basis of $Lie(G)$. Because $U \cap H$ is closed and H satisfies the second countable axiom, by the argument which is similar to the proof of Lemma3.4.9 and Baker-Campbell-Hausdorff formula,

$$Exp(\mathfrak{h} \cap \sum_{i=1}^k (-\epsilon, \epsilon) X_i) = Exp(\sum_{i=1}^m (-\epsilon, \epsilon) X_i) \cap H = Exp(\sum_{i=1}^N (-\epsilon, \epsilon) X_i) \cap H$$

We set

$$V_H := Exp(\mathfrak{h} \cap \sum_{i=1}^k [-\frac{1}{2}\epsilon, \frac{1}{2}\epsilon] X_i)$$

So, by the argument which is similar to the proof of Theorem3.5.18, V_H is closed neighborhood of e and the relative topology of V_H to G is equal to the relative topology of V_H to $GL(n, \mathbb{C})$. So, by Proposition3.4.8, H is a Lie group and $\mathfrak{h} = Lie(H)$. □

3.6 Invariant measure

3.6.1 Existence of Invariant measure

Definition 3.6.1 (Baire measure)**.** *Let X be a locally compact Housdorff space. We say μ is a Baire measure on X if*

$$C_c(X) \subset L^1(X, \mu)$$

Definition 3.6.2 (Invariant measure)**.** *Let G be a locally compact topological group. We say μ is a left invariant measure on G if for any $f \in C_c(G)$ and any $g_0 \in G$*

$$\int_G f(g_0 g) d\mu(g) = \int_G f(g) d\mu(g)$$

We say μ is a right invariant measure on G or a right Haar measure on G if for any $f \in C_c(G)$ and any $g_0 \in G$

$$\int_G f(g g_0) d\mu(g) = \int_G f(g) d\mu(g)$$

We say G is unimodular if there is a left and right Haar measure on G. We call a left and right Haar measure on G a Haar measure on G.

We say μ is a right invariant measure on G

Notation 3.6.3. *Let G be a Lie group and $g_0 \in G$. For each $g \in G$ and $x \in G$, $L_{g_0}(x) := g_0 x$.*

Definition 3.6.4 (Left invariant form)**.** *Let*

- *(S1) G is a Lie group and $m := Lie(G)$.*
- *(S2) ω is a m-form on G.*

We say ω is left invariant if for any $g \in G$ $dL_g \omega = \omega$. Here, for eahc $v_1, ..., v_m \in T_x(G)$,

$$(dL_g \omega)_x(v_1, ..., v_m) := \omega_{gx}(dL_g v_1, ..., dL_g v_m)$$

Lemma 3.6.5. *Let G be a Lie group and $m := Lie(G)$. And let us ω_e a antisymmetric m-th tensor at 1_G and $\omega \neq 0$. For each $x \in G$ and $v_1, ...v_m \in T_x(G)$,*

$$\omega_x(v_1, ..., v_m) := \omega_e(dL_x^{-1} v_1, ..., dL_x^{-1} v_m)$$

Then ω is a C^ω-class left invariant form.

Proof. Let us fix any $g, x \in G$ and $v_1, ..., v_m \in T_x(G)$.

$$\begin{aligned}
& (L_g \omega)_x(v_1, ..., v_m) \\
= \ & \omega_{gx}(dL_g v_1, ..., dL_g v_m) \\
= \ & \omega_e(dL_{gx}^{-1} dL_g v_1, ..., dL_{gx}^{-1} dL_g v_m) \\
= \ & \omega_e(dL_x^{-1} dL_g^{-1} dL_g v_1, ..., dL_x^{-1} dL_g^{-1} dL_g v_m) \\
= \ & \omega_e(dL_x^{-1} v_1, ..., dL_x^{-1} v_m) = \omega_x(v_1, ..., v_m)
\end{aligned}$$

□

Lemma 3.6.6. *Let*

- *(S1) G be a Lie group.*
- *(S2) ω be a C^ω-class left invariant form.*
- *(S3) $g \in G$.*
- *(S4) (U_α, ψ_α) and (U_β, ψ_β) are local coordinates on G and $gU_\beta \cap U_\alpha \neq \phi$.*
- *(S5) For any $x \in U_\alpha$ and $y \in U_\beta$*

$$\omega_x = \Phi_\alpha(x) d\phi_{\alpha,1} \wedge ... \wedge d\phi_{\alpha,m}, \ \omega_y = \Phi_\beta(y) d\phi_{\beta,1} \wedge ... \wedge d\phi_{\beta,m}$$

Then, for any $x \in U_\beta \cap L_g^{-1} U_\alpha$,

$$\Phi_\beta(x) = det(J(\psi_\alpha \circ L_g \circ \phi_\beta)(\psi_\beta(x)))\Phi_\alpha(gx)$$

Proof. Let us fix any $x \in U_\beta \cap L_g^{-1} U_\alpha$. Then

$$\omega_x = \Phi_\beta(x)(d\phi_{\beta,1} \wedge ... \wedge d\phi_{\beta,m})_x$$

and

$$\omega_{gx} = \Phi_\alpha(gx)(d\phi_{\alpha,1} \wedge ... \wedge d\phi_{\alpha,m})_{gx}$$

So,

$$\omega_x((\frac{\partial}{\partial\psi_{\beta,1}})_x, ..., (\frac{\partial}{\partial\psi_{\beta,m}})_x) = \omega_{gx}(dL_g((\frac{\partial}{\partial\psi_{\beta,1}})_x), ..., dL_g((\frac{\partial}{\partial\psi_{\beta,m}})_x))$$

and

$$\omega_{gx}(dL_g((\frac{\partial}{\partial\psi_{\beta,1}})_x), ..., dL_g((\frac{\partial}{\partial\psi_{\beta,m}})_x)) = detJ(\psi_\alpha \circ L_g \circ \phi_\beta)(\psi_\beta(x))$$

These implies that

$$\Phi_\beta(x) = \Phi_\alpha(gx)detJ(\psi_\alpha \circ L_g \circ \phi_\beta)(\psi_\beta(x))$$

□

By following the argument of the proof of Lemma3.6.6 in reverse, we can show the following proposition.

Lemma 3.6.7. *Here are settings and assumptions.*

(S1) G *is a Lie group.*

(S2) $\{U_\alpha, \psi_\alpha\}_{\alpha\in\Lambda}$ *is a system of local corrdinates of* G*.*

(S3) $\{\Phi_\alpha\}_{\alpha\in\Lambda}$ *is a family such that* $\Phi_\alpha \in C^\infty(U_\alpha, \mathbb{R})$ $(\forall\alpha \in \Lambda)$*.*

(A1) Then, for any $g \in G$ *and* $x \in U_\beta \cap L_g^{-1}U_\alpha$*,*

$$\Phi_\beta(x) = det(J(\psi_\alpha \circ L_g \circ \phi_\beta)(\psi_\beta(x)))\Phi_\alpha(gx)$$

(S4) We set

$$\omega_x = \Phi_\alpha(x)d\phi_{\alpha,1} \wedge ... \wedge d\phi_{\alpha,m} \ (x \in U_\alpha, \alpha \in \Lambda)$$

Then ω *is well-defined and* C^ω *left-invariant form.*

Proposition 3.6.8. *Here are settings and assumptions.*

(S1) G *is a Lie group.*

(S2) ω *is a* C^∞ *class form on* G *such that* $\omega_g \neq 0$ $(\forall g \in G)$

(S3) μ *is the measure on* G *induced by* ω*.*

(A1) μ *is left invariant.*

Then ω *is a left invariant form.*

Proof. By Lemma3.6.6, There is a $\{U_\alpha, \psi_\alpha\}_{\alpha\in\Lambda}$ is a system of local corrdinates of G preserving the orientation of G and $\Phi_\alpha > 0$ on U_α $(\forall\alpha \in \Lambda)$ and $det(\phi_\alpha^{-1} \circ L_g \circ \psi_\beta) > 0$. Let us fix any $g \in G$ and $U_\beta \cap g^{-1}U_\alpha \neq \phi$. Let us fix any $f \in C_c(gU_\beta \cap U_\alpha)$. Because μ is left invariant,

$$\int_{U_\beta\cap g^{-1}U_\alpha} f(gx)d\mu(x) = \int_G f(gx)d\mu(x) = \int_G f(x)d\mu(x) = \int_{gU_\beta\cap U_\alpha} f(x)d\mu(x) = \int_{\psi_\alpha^{-1}(gU_\beta\cap U_\alpha)} f(\psi_\alpha(x))\Phi_\alpha(\psi_\alpha(x))dx$$

By change-of-variables formula for integral

$$\int_{U_\beta\cap g^{-1}U_\alpha} f(gx)d\mu(x) = \int_{\psi_\beta^{-1}(U_\beta\cap g^{-1}U_\alpha)} f(g\psi_\beta(y))\Phi_\beta(\psi_\beta(y))dy$$
$$= \int_{\psi_\alpha^{-1}(gU_\beta\cap U_\alpha)} f(\psi_\alpha(x))\Phi_\beta(g^{-1}\psi_\alpha(x))|det(\phi_\beta \circ L_g \circ \psi_\alpha)|^{-1}dx$$

So, for any $g \in G$ and $x \in U_\beta \cap L_g^{-1}U_\alpha$,

$$\Phi_\beta(x) = |det(J(\psi_\alpha \circ L_g \circ \phi_\beta)(\psi_\beta(x)))|\Phi_\alpha(gx)$$

Because $det(J(\psi_\alpha \circ L_g \circ \phi_\beta)(\psi_\beta(x))) > 0$,

$$\Phi_\beta(x) = det(J(\psi_\alpha \circ L_g \circ \phi_\beta)(\psi_\beta(x)))\Phi_\alpha(gx)$$

So, ω is left invariant form. □

Lemma3.6.6 implies the following.

Lemma 3.6.9. *Let G be a Lie group in which there is a left invariant form ω. Then G is orientable and ω is C^ω-class .*

Proof. By replacing two variables if necessary, there is a local coordinate system $\{U_\alpha, \psi_\alpha\}_{\alpha\in\Lambda}$ such that $\Phi_\alpha > 0$ $(\forall\alpha \in \Lambda)$. By Lemma3.6.6, $\{U_\alpha, \psi_\alpha\}_{\alpha\in\Lambda}$ preserves the orientation of G. □

Lemma 3.6.10. *Let*

- *(S1) M is a paracompact C^∞-class manifold.*
- *(S2) $H : M \to M$ is a C^∞-class homeomorphism.*
- *(S3) $\{U_\alpha\}_{\alpha\in\Lambda}$ is a open covering of M.*
- *(S4) f is a C^∞-class function on M.*
- *(A1) supp(f) is compact and there is $\alpha \in \Lambda$ such that supp$(f) \subset U_\alpha$.*

Then there are $\{U_{\beta_i}\}_{i=1}^N$ and $\{f_i\}_{i=1}^N \subset C^\infty(M)$ such that $\{H(U_{\beta_i})\}_{i=1}^N$ is a covering of supp(f) and

$$f = \sum_{i=1}^N f_i$$

and

$$supp(f_i) \subset U_\alpha,\ supp(f_i \circ H) \subset U_{\beta_i}\ (i = 1, 2, ..., N)$$

Proof. Because *supp*(f) is compact, there are $\{U_{\beta_i}\}_{i=1}^N$ such that $\{H(U_{\beta_i})\}_{i=1}^N$ is a covering of *supp*(f). Because *supp*(f) is paracompact and $\{H(U_{\beta_i})\}_{i=1}^N$ is a open covering of *supp*(f), there is $\{h_i\}_{i=1}^N \subset C^\infty(M)$ such that $\{h_i\}_{i=1}^N$ is a partition of unity which is subordinate to $\{H(U_{\beta_i})\}_{i=1}^N$. We set $f_i := h_i$ $(i = 1, 2, ..., N)$. Clearly $\{f_i\}_{i=1}^N$ satisfies the conditions in this Proposition. □

By Riesz-Markov-Kakutani representation theorem[15], any left invariant measure induces a measure.

Theorem 3.6.11. *Let*

- *(S1) G be a Lie group.*

Then

- *(i) There is C^∞-class left invariant form ω on G.*
- *(ii) G is orientable by ω.*
- *(iii) The measure induced from ω is left invariant. Specially, G has a left invariant measure.*

Proof. (i) is from Lemma3.6.5. (ii) is from Lemma3.6.9. We will show (iii). We set $m := Lie(G)$. Let us fix $f \in C_c^\infty(G)$ and $g_0 \in G$. For $x \in G$,

$$(L_{g_0} f)(x) := f(g_0x)$$

By (ii) and the second contable axiom, there is $\{U_i, \psi_i, V_i, \Phi_i, \rho_i\}_{i=1}^\infty$ such that $\{U_i, \psi_i\}_{i=1}^\infty$ is a local coordinate system of G and $\{U_i, \psi_i\}_{i=1}^\infty$ is local finite and for each i $V_i \in \mathcal{O}(\mathbb{R}^m)$

$$\psi_i : U_i \to V_i$$

is an homeomorphism and $\{U_i, \psi_i\}_{i=1}^\infty$ preserves a orientation of G and for each i and $x \in U_i$

$$\omega_x = \Phi_i(x)(d\psi_{i,1} \wedge ... \wedge d\psi_{i,m})_x$$

and $\Phi_i > 0$ and $\{\rho_i\}_{i=1}^\infty$ is a partition of unity subordinating $\{U_i\}_{i=1}^\infty$. We set for each i, $f_i := f\rho_i$. By Lebesgue's convergence theorem,

$$\int_G f\omega = \sum_{i=1}^\infty \int_G f_i\omega,\ \int_G L_{g_0} f\omega = \sum_{i=1}^\infty \int_G L_{g_0} f_i\omega$$

So, it is enough to show for each i

$$\int_G f_i\omega = \int_G L_{g_0} f_i\omega$$

By Lemma 3.6.10, we can assume that for each i, there is j such that *supp*$(L_{g_0} f_i) \subset U_j$. Because *supp*$(f_i)$ is compact, there is an open set U_i' such that

$$supp(f_i) \subset U_i' \subset U_i$$

and

$$supp(L_{g_0} f_i) = L_{g_0}^{-1} supp(f_i) \subset L_{g_0}^{-1} U_i' \subset U_j$$

We set $\phi_i := \psi_i^{-1}$ and $V_i := \psi_i(U_i)$ and $\phi_j := \psi_j^{-1}$ and $V_j := \psi_j(U_j)$. By change-of-variables formula for integral and Lemma3.6.6,

$$\begin{aligned}
& \int_G L_{g_0} f_i \omega = \int_{\psi_j(L_{g_0}^{-1} U_i')} f_i(g_0 \phi_j(x)) \Phi_j(x) dx \\
= & \int_{\psi_j(L_{g_0}^{-1} U_i')} f_i(\phi_i(\psi_i(g_0 \phi_j(x)))) \Phi_j(x) dx \\
= & \int_{\psi_j(L_{g_0}^{-1} U_i')} f_i(\phi_i(\psi_i \circ L_{g_0} \circ \phi_j(x))) \\
& \times det(J(\psi_i \circ L_{g_0} \circ \phi_j))(\psi_j \circ L_{g_0}^{-1} \phi_i \circ \psi_i \circ L_{g_0} \circ \phi_j(x))))^{-1} \\
& \times \Phi_j(\psi_j \circ L_{g_0}^{-1} \phi_i \circ \psi_i \circ L_{g_0} \circ \phi_j(x)))) \\
= & \int_{V_i'} f_i(\phi_i(y)) det(J(\psi_i \circ L_{g_0} \circ \phi_j))(\psi_j \circ L_{g_0}^{-1} \circ \phi_i(y))^{-1} \\
& \times \Phi_j(\psi_j \circ L_{g_0}^{-1} \phi_i(y)) dy \\
= & \int_{V_i'} f_i(\phi_i(y)) \Phi_i(y) dy \\
= & \int_G f_i \omega
\end{aligned}$$

□

3.6.2 Haar measure

Theorem 3.6.12. *Let*

- *(S1) G be a Lie group with $m := dimLie(G)$.*
- *(S2) ω^L is a left invariant m-form and ω^R is a right m-form on G.*
- *(A1) $\omega_e^L = \omega_e^R$.*
- *(S3) dg_L is the left invariant measure induced from ω^L. dg_R is the right invariant measure induced from ω^R.*

Then

- *(i) $\omega^R = det(Ad(\cdot))\omega^L$.*
- *(ii) $dg_R = |det(Ad(\cdot))| dg_L$. We set $\Delta_L(\cdot) := |det(Ad(\cdot))|$ and $\Delta_R(\cdot) := |det(Ad(\cdot))|^{-1}$.*

Proof. It is enough to show (i). Let us fix any $g \in G$. and $v \in T_g(G)$ and $u := dL_g^{-1} v$. Then

$$\begin{aligned}
& \omega_g^R(v) = \omega_g^R(dL_g u) = \omega_e(dR_g dL_g u) = \omega_e(\iota(Ad(g)\iota^{-1}(u))) = det(Ad(g))\omega_e(u) \\
= & det(Ad(g))\omega_e(dL_g^{-1} v) = det(Ad(g))\omega^L(v)
\end{aligned}$$

This implies (i). □

Proposition 3.6.13. *Any compact Lie group is unimodular.*

Proof. Let us fix any G be a compact Lie group. Clearly, $|det(Ad(G))|$ is compact subgroup of $\mathbb{R}_{>0}^{\times}$. So, $|det(Ad(G))| = \{1\}$. □

3.6.3 Integral on all inverse elements

Proposition 3.6.14. *Let*

- *(S1) G is a Lie group.*
- *(S2) $I : G \ni g \mapsto g^{-1} \in G$.*
- *(S3) $f \in C_c(G)$.*

(S4) ω be a left invariant and right invariant form on G.

then

$$\int_G f(g^{-1})\omega = \int_G f(g)\omega$$

STEP1. Construction of a left invariant form. We set $m := dim(Lie(G))$. Let us fix $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$ a system of local coordinates which preserves the orientation of G. Let us fix $\{a_\alpha\}_{\alpha\in\Lambda}$ such that for any $\alpha \in \Lambda$ $a_\alpha \in C^\infty(U_\alpha)$ and

$$\omega|U_\alpha = a_\alpha d\psi_\alpha^1 \wedge ... \wedge d\psi_\alpha^m$$

Then $\{(I(U_\alpha), \psi_\alpha \circ I^{-1})\}_{\alpha\in\Lambda}$ a system of local coordinates of G. For any $\alpha, \beta \in \Lambda$ such that $(I(U_\alpha) \cap (I(U_\beta) \neq \phi$,

$$\psi_\alpha \circ I^{-1} \circ (\psi_\beta \circ I^{-1})^{-1} = \psi_\alpha \circ \psi_\beta^{-1}$$

So, $\{(I(U_\alpha), \psi_\alpha \circ I^{-1})\}_{\alpha\in\Lambda}$ preserves the orientation of G.

We set ω' by

$$\omega'_g(u_1, u_2, ..., u_m) := \omega_{I^{-1}(g)}((dI)^{-1}_{I^{-1}(g)}u_1, ..., (dI)^{-1}_{I^{-1}(g)}u_m)$$

We will show ω' is left invarinat. Because ω is right invariant,

$$\begin{aligned}
&\omega'_{(L_x)(g)}((dL_x)_g v_1, ...(dL_x)_g v_m) = \omega'_{xg}((dL_x)_g v_1, ...(dL_x)_g v_m) = \omega_{I(xg)}((dI)^{-1}_{I(xy)}(dL_x)_g v_1, ..., (dI)^{-1}_{I(xy)}(dL_x)_g v_m) \\
=\ & \omega_{I(xg)}((dI)^{-1}_{I(xy)}(dL_I(x))^{-1}_g v_1, ..., (dI)^{-1}_{I(xy)}(dL_I(x))^{-1}_g v_m) = \omega_{I(xg)}(d(L_{I(x)} \circ I))^{-1}_{I(xy)}v_1, ..., d(L_{I(x)} \circ I))^{-1}_{I(xy)}v_m) \\
=\ & \omega_{I(xg)}(d(I \circ R_x)^{-1}_{I(xy)}v_1, ..., d(I \circ R_x)^{-1}_{I(xy)}v_m) = \omega_{R_{I(x)(I(g))}}(d(I \circ R_x)^{-1}_{I(xy)}v_1, ..., d(I \circ R_x)^{-1}_{I(xy)}v_m) \\
=\ & \omega_{R_{I(x)(I(g))}}((dR_x)^{-1}_{R_{I(x)(I(g))}}(dI)^{-1}_{I(g)}v_1, ..., (dR_x)^{-1}_{R_{I(x)(I(g))}}(dI)^{-1}_{I(g)}v_m) \\
=\ & \omega_{R_{I(x)(I(g))}}((dR_{I(x)})_{I(g)}(dI)^{-1}_{I(g)}v_1, ..., (dR_{I(x)})^{-1}_{I(g)}(dI)^{-1}_{I(g)}v_m) \\
=\ & \omega_{I(g)}(dI)^{-1}_{I(g)}v_1, ..., (dI)^{-1}_{I(g)}v_m) = \omega_{I^{-1}(g)}(dI)^{-1}_{I^{-1}(g)}v_1, ..., (dI)^{-1}_{I^{-1}(g)}v_m) = \omega'_g(v_1, ..., v_m)
\end{aligned}$$

So, ω' is left invariant. So, there is $C \in \mathbb{R}$ such that $\omega' = C\omega$. □

STEP2. Display of X using local coordinates.

$$\begin{aligned}
&\omega'_g(u_1, u_2, ..., u_m) = \omega_{I^{-1}(g)}((dI)^{-1}_{I^{-1}(g)}u_1, ..., (dI)^{-1}_{I^{-1}(g)}u_m) = \omega_e(d(L_{I^{-1}(g)})^{-1}_e(dI)^{-1}_{I^{-1}(g)}u_1, ..., d(L_{I^{-1}(g)})^{-1}_e(dI)^{-1}_{I^{-1}(g)}u_m) \\
=\ & \omega_e(d(I \circ L_{I^{-1}(g)})^{-1}_e u_1, ..., d(I \circ L_{I^{-1}(g)})^{-1}_e u_m) = \omega_e(d(L_g)^{-1}_e u_1, ..., d(L_g)^{-1}_e u_m)
\end{aligned}$$

For any $u_1, ..., u_m \in T_g(G)$,

$$\begin{aligned}
&\omega'_g(u_1, u_2, ..., u_m) = \omega_{I^{-1}(g)}((dI)^{-1}_{I^{-1}(g)}u_1, ..., (dI)^{-1}_{I^{-1}(g)}u_m) = \omega_{I^{-1}(g)}((dI)^{-1}_{I^{-1}(g)}u_1, ..., (dI)^{-1}_{I^{-1}(g)}u_m) \\
=\ & a_\alpha(I^{-1}(g))d\psi_\alpha^1 \wedge ... \wedge d\psi_\alpha^m((dI)^{-1}_{I^{-1}(g)}u_1, ..., (dI)^{-1}_{I^{-1}(g)}u_m) \\
=\ & a_\alpha(I^{-1}(g))d\psi_\alpha^1 \circ (dI)^{-1}_{I^{-1}(g)} \wedge ... \wedge d\psi_\alpha^1 \circ (dI)^{-1}_{I^{-1}(g)}(v_1, ..., v_m) \\
=\ & a_\alpha(I^{-1}(g))d(\psi_\alpha \circ I^{-1})^1_{I^{-1}(g)} \wedge ... \wedge d(\psi_\alpha \circ I^{-1})^m_{I^{-1}(g)}(v_1, ..., v_m)
\end{aligned}$$

this proposition holds. So,

$$\int_G f(g^{-1})\omega = \int_G f(g)\omega'$$

By setting $f = 1$, $\omega' = \omega$. So,

$$\int_G f(g^{-1})\omega = \int_G f(g)\omega$$

□

By the proof of Proposition3.6.14, the following holds.

Proposition 3.6.15. *Let*

(S1) G is a Lie group.

(S2) $I : G \ni g \mapsto g^{-1} \in G$.

(S3) $f \in C_c(G)$.

(S4) ω be a left invariant on G.

then

$$\int_G f(g^{-1})\omega = \int_G f(g)\Delta_R(g)\omega$$

3.6.4 $L^p(G)$

Proposition 3.6.16. *Let G be a Lie group. Then $L^p(G)$ is separable for any $p \in \mathbb{N} \cap [1, \infty)$.*

Proof. By Proposition3.4.6 there is $\{U_i\}_{i=1}^{\infty}$ which is a local finite open covering of G_1 and $\{\varphi_i\}_{i=1}^{\infty}$ is a partition of unity with respect to $\{U_i\}_{i=1}^{\infty}$ and for any i U_i is C^∞-class homeomorphic to $(0,1)^m$. For each i, $L^2(U_i)$ is separable. So, there is $\{f_{i,k}\}_{i,k} \subset C^\infty(G)$ such that $supp(f_{i,k}) \subset U_i$ $(\forall i, \forall k)$ and $\{f_{i,k}|U_i\}_k$ is dense in $L^p(U_i)$ $(\forall i)$. We set $A := \{\sum_{i=1}^N f_{i,k_i} | k_i \in \mathbb{N}\ (i = 1, 2, ..., N),\ N \in \mathbb{N}\}$. Clearly A is separable.

Let us fix any $f \in L^p(G)$. Let us fix any $\epsilon > 0$. Because $\lim_{N \to \infty} f * \chi_{\cup_{i=1}^N U_i} = f$ and $f \in L^p(G)$, by Lebesgue's convergence theorem, there is $N \in \mathbb{N}$ such that

$$||f - f * \chi_{\cup_{i=1}^N U_i}|| < \frac{\epsilon}{2}$$

We set $f_1 := f * \chi_{U_1}$ and $f_i := f * \chi_{U_i \setminus \cup_{k=i-1}^N U_k}$ $(i = 1, 2, ..., N)$. Then $f * \chi_{\cup_{i=1}^N U_i} = \sum_{i=1}^N f_i$. There are $f_{i,k_1}, ..., f_{i,k_N}$ such that $||f_i - f_{i,k_i}|| < \dfrac{\epsilon}{2N}$ $(i = 1, 2, ..., N)$. Clearly

$$||f * \chi_{\cup_{i=1}^N U_i} - \sum_{i=1}^N f_{i,k_i}|| < \frac{\epsilon}{2}$$

So, $||f - \sum_{i=1}^N f_{i,k_i}|| < \epsilon$. Consequently, $L^p(G_1)$ is separable. □

By the proof of Proposition3.6.16, the following holds.

Proposition 3.6.17. *Let G be a Lie group. Then there is at most countable subset of $C_c(G)$ which is dense in $L^p(G)$.*

3.6.5 Convolution

Definition 3.6.18 (Convolution of function and linear functional)**.** *Let*

(S1) G be a Lie group.

(S2) $f \in C_c(G)$.

(S3) T is a $\mathbb{C}$-linear functional on $C_c(G)$.

Then

$$T * f(x) := T(\tau_x(f))\ (x \in G)$$

Here,

$$\tau_x(f)(y) = f(xy^{-1})\ (x, y \in G)$$

Notation 3.6.19 (Dirac delta function δ_x)**.** *Let G be a topological group and $x \in G$. We set δ_x by*

$$\delta_x(f) := f(x)\ (f \in C(G))$$

Definition 3.6.20 (Convolution of functions)**.** *Let G be a Lie group. Let us fix dg_r which is a right invariant measure on G. Let us fix $f, g \in C(G)$ and assume $supp(f)$ or $supp(g)$ is compact. We set*

$$f * g(x) := \int_G f(xy^{-1}) g(y) dg_r(y)\ (x \in G)$$

Proposition 3.6.21. *We succeed notations in Definition3.6.20. Then*

(i) $f * g \in C(G)$

(ii) If $f_1, f_2 \in C_c(G)$ then $f_1 * f_2 \in C_c(G)$ and $supp(f_1 * f_2) \subset supp(f_1) supp(f_2)$

(iii) If $f_3, f_3 \in C_c(G)$ then $(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3)$.

Proof of (i). Firstly let us assume $g \in C_c(G)$. Let us fix any $x \in G$ and $\epsilon > 0$.

$$f * g(x) = \int_G f(xy^{-1}) g(y) dg_r(y) = \int_{supp(g)} f(xy^{-1}) g(y) dg_r(y)$$

We set $K := dg_r(supp(g))$. Becase $f, g \in C(G)$, for each $y \in supp(g)$, there is $U_{x,y}$ and V_y such that $U_{x,y}$ is an open neighborhood of x and V_y is an open neighborhood of y and

$$|f(zw^{-1})g(w) - f(xw^{-1})g(w)| < \frac{\epsilon}{K+1} \ (\forall z \in U_{x,y}, \forall w \in V_y)$$

Because $supp(g)$ is compact, there are $V_{y_1}, ..., V_{y_n}$ such that $supp(g) \subset \cup_{i=1}^n V_{y_i}$. We set $U_x := \cap_{i=1}^n U_{x,y_i}$. Then clearly

$$|f(zw^{-1})g(w) - f(xw^{-1})g(w)| < \frac{\epsilon}{K+1} \ (\forall z \in U_x, \forall w \in V_y)$$

So,

$$|f * g(z) - f * g(x)| < \epsilon \ (\forall z \in U_x)$$

This means $f * g$ is continuous.

Firstly let us assume $f \in C_c(G)$. Let us fix any $x \in G$.

$$f * g(x) = \int_G f(xy^{-1})g(y)dg_r(y) = \int_G f((yx^{-1})^{-1})g(yx^{-1}x)dg_r(y) = \int_G f(y^{-1})g(yx)dg_r(y)$$
$$= \int_{supp(f)^{-1}} f(y^{-1})g(yx)dg_r(y)$$

So, we can prove continuity of $f * g$ by the argument which is similar to the proof in case $g \in C_c(G)$. □

Proof of (iii). Let us fix any $x \in G$.

$$(f_1 * f_2) * f_3(x) = \int_G f_1 * f_2(xy^{-1})f_3(y)dg_r(y) = \int_G \int_G f_1(xy^{-1}z^{-1})f_2(z)dg_r(z)f_3(y)dg_r(y)$$
$$= \int_G \int_G f_1(x(zy)^{-1})f_2(zyy^{-1})dg_r(z)f_3(y)dg_r(y) = \int_G \int_G f_1(xz^{-1})f_2(zy^{-1})dg_r(z)f_3(y)dg_r(y)$$

by Fubini Theorem

$$= \int_G f_1(xz^{-1}) \int_G f_2(zy^{-1})f_3(y)dg_r(y)dg_r(z) = \int_G f_1(xz^{-1})f_2 * f_3(z)dg_r(z) = f_1 * (f_2 * f_3)(x)$$

□

3.7 Basic Notions of Lie algebras

Definition 3.7.1 (Automorphism Group $Aut(\mathfrak{g})$)**.** *Let*

(S1) $\mathfrak{g}$ *is a Lie algebra.*

We set

$$\mathfrak{g} := \{a \in GL(\mathfrak{g}) | [aX, aY] = a[X, Y] \ (\forall X, Y \in \mathfrak{g})\}$$

and call it the automorphism group of $\mathfrak{g}$.

The following is clear.

Proposition 3.7.2 (Inner Automorphism Group $Int(\mathfrak{g})$)**.** *Let*

(S1) $\mathfrak{g}$ *is a Lie algebra.*

(S2) $ad(\mathfrak{g}) := \{ad(X) | X \in \mathfrak{g}\}$.

Then $ad(\mathfrak{g})$ *is a Lie algebra. Let* $Int(\mathfrak{g})$ *denote the analytical subgroup of* $ad(\mathfrak{g})$ *in* $GL(\mathfrak{g})$*. We call it the inner automorphism group of* $\mathfrak{g}$.

Definition 3.7.3 (Killing form)**.** *Let* $\mathfrak{g}$ *be a Lie algebra. We set*

$$\langle X, Y \rangle := Trace(ad(X)ad(Y))$$

3.8 Connected component of Lie group

Proposition 3.8.1. *Let*

(S1) G_1 *is a Lie group which is locally isomorphic to a linear Lie subgroup of* $GL(n, \mathbb{C})$ *and* G_1 *be connected.*

(A1) There is open neighborhood of 1_{G_1} U *such that for any* $x, y \in U$ $xy = yx$.

Then G_1 *is commutative.*

Proof. By Proposition3.4.6, we can assume that for any $g \in G_1$ there are $g_1, ..., g_M \in U$ such that $g = g_1 \cdot g_2 ... g_M$. Let us fix any $g = g_1 \cdot g_2 ... g_M$ and $h = h_1 \cdot h_2 ... h_N$ such that $g_1, ..., g_M, h_1, ..., h_N \in U$.

$$\begin{aligned} gh &= g_1 \cdot g_2 ... g_M \cdot h_1 \cdot h_2 ... h_N \\ &= h_1 \cdot h_2 ... h_N \cdot g_1 \cdot g_2 ... g_M \\ &= hg \end{aligned} \tag{3.8.1}$$

□

Proposition 3.8.2. *Let*

(S1) G_1 *be a Lie group which is locally isomorphic to a linear Lie subgroup of* $GL(n, \mathbb{C})$.

(S2) $G_{1,0}$ *be the connected component of* G_1.

Then $G_{1,0}$ *is path-connected.*

Proof. For sufficient small $\epsilon > 0$, $N(\epsilon) := Exp(B(O, \epsilon))$ is path-connected. Clearly, finite multiple of $N(\epsilon)$ is path-connected. So, by Proposition3.4.6, $G_{1,0}$ is path-connected. □

3.9 Reductive Lie group

Definition 3.9.1 (Reductive Lie group)**.** *Let* $G \subset GL(n, \mathbb{C})$ *be a linear Lie group. We say* G *is a reductive Lie group if for any* $g \in G$ $\bar{g}^T \in G$. *Let* G *be a Lie group. We say* G *is reductive if* G *is locally isomorphic to a reductive linear Lie group and* G *has finite connected components.*

The following definition is from [6].

Definition 3.9.2 (Harish Chandra Class)**.** *The followings are settings.*

(S1) G *be a Lie group with the reductive Lie algebra* $\mathfrak{g}$.

(S2) $\mathfrak{g}_1 := \langle [\mathfrak{g}, \mathfrak{g}] \rangle$.

We say G *is said to belong to Harish-Chandra's class if it satisfies the following conditions.*

(i) G *has finitely many connected components.*

(ii) $Ad(g) \in Int(\mathfrak{g})$ *for any* $g \in G$.

(iii) The analytic subgroup G_1 *with* $\mathfrak{g}_1$ *has finite center.*

The followings clearly hold.

Proposition 3.9.3. *Let* $G \subset GL(n, \mathbb{C})$ *be a linear Lie group and* G *be reductive. Then*

(i)

$$G = \{\bar{g}^T | g \in G\}$$

(ii)

$$Lie(G) = \{\bar{X}^T | X \in Lie(G)\}$$

Proof of (i). For any $g \in G$, $g = \overline{\bar{g}^T}^T$. So the above equation holds. □

Proof of (ii). For any $X \in Lie(G)$, $exp(t\bar{X}^T) = \overline{exp(tX)}^T$. So $Lie(G) = \{\bar{X}^T | X \in Lie(G)\}$. □

Proposition 3.9.4. *Let $\mathfrak{g}$ be a Lie algebra. We set*

$$(X,Y) := ReTr(X^T\bar{Y}) \ (X,Y \in \mathfrak{g})$$

then

(i) $(\cdot,\cdot)$ is an inner product on $\mathfrak{g}$.

(ii) $(ad(X)Y,Z) = (Y, ad(\bar{X}^T)Z)$ for any $X,Y,Z \in \mathfrak{g}$.

Proof of (i). For any $X,Y \in \mathfrak{g}$,

$$\begin{aligned}(Y,X) &= ReTr(Y^T\bar{X}) = ReTr(\bar{X}^TY) \\ &= ReTr(X^T\bar{Y}) = (X,Y) = \overline{(X,Y)}\end{aligned}$$

Also,

$$(X,X) = \sum_{i,j} |x_{i,j}|^2$$

So, (i) holds. □

Proof of (ii). Because $Tr(X^TY^T\bar{Z}) = Tr(\bar{Z}X^TY^T)$,

$$\begin{aligned}&(ad(X)Y,Z) = ReTr((XY-YX)^T\bar{Z}) \\ =\ & ReTr((Y^TX^T - X^TY^T)\bar{Z}) = ReTr(Y^TX^T\bar{Z} - Y^T\bar{Z}X^T) \\ =\ & ReTr(Y^T\overline{ad(\bar{X}^T)Z}) = (Y, ad(\bar{X}^T)Z) \qquad (3.9.1)\end{aligned}$$

So, (ii) holds. □

Lemma 3.9.5. *Let $\mathfrak{g}$ be a Lie algebra and $\bar{\mathfrak{g}}^T = \mathfrak{g}$. For any $\mathfrak{h}$ which is an ideal of $\mathfrak{g}$, $\mathfrak{h}^\perp$ is also ideal. Here, we assume the inner product of $\mathfrak{g}$ is $(\cdot,\cdot)$.*

Proof. Let us fix any $X \in \mathfrak{g}$, $Y \in \mathfrak{h}^\perp$, $Z \in \mathfrak{h}$. By the assumption, $ad(\bar{X}^T)Z \in \mathfrak{h}$. By Proposition3.9.4,

$$(ad(X)Y,Z) = (Y, ad(\bar{X}^T)Z) = 0 \qquad (3.9.2)$$

So $(ad(X)Y \in \mathfrak{h}^\perp$. □

Proposition 3.9.6. *Let G_1 is a reductive Lie group such that G_1 is locally isomorphic to G_2 which is linear Lie group of $GL(n,\mathbb{C})$. Then $Lie(G_1)$ is a reductive Lie algebra. And we denote the center of $Lie(G_1)$ by $\mathfrak{z}$ and denote $\langle[Lie(G_1), Lie(G_1)]\rangle$ by $\mathfrak{g}_1$. Then*

$$Lie(G_1) = \mathfrak{z} \oplus \mathfrak{g}_1 \qquad (3.9.3)$$

and $\mathfrak{g}_1$ is a semisimple Lie algebra or $\{0\}$.

Proof. We set $\mathfrak{g} := Lie(G_1)$. If $Lie(G_1)$ has no trivial ideal, then $Lie(G_1)$ is reductive. Otherwise, $Lie(G_1)$ has a trivial ideal $\mathfrak{h}$. By Proposition3.9.5, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$. We set $\mathfrak{h}_1 := \mathfrak{h}$ and $\mathfrak{h}_2 := \mathfrak{h}^\perp$. If $\mathfrak{h}_1$ has a subset which is a not trivial ideal of $\mathfrak{h}_1$, by Proposition3.9.5, the subset is a not trivial ideal of $\mathfrak{g}$. By repeating the above argument, there are $\mathfrak{g}_1, ..., \mathfrak{g}_r, \mathfrak{g}_{r+1}, .., .\mathfrak{g}_m$ such that $\mathfrak{g}_1, ..., \mathfrak{g}_r, \mathfrak{g}_{r+1}, .., .\mathfrak{g}_m$ are ideals of $\mathfrak{g}$ and $\mathfrak{g}_1, ..., \mathfrak{g}_r$ are one-dimensional abelian Lie algebras and $\mathfrak{g}_{r+1}, ..., \mathfrak{g}_m$ are simple Lie algebras. So $\mathfrak{g}$ is reductive. Clearly $\mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_r$ is the center of $\mathfrak{g}$. Clearly $\langle[\mathfrak{g},\mathfrak{g}]\rangle \subset \langle[\mathfrak{g}_{r+1},\mathfrak{g}_{r+1}]\rangle \oplus ... \oplus \langle[\mathfrak{g}_m,\mathfrak{g}_m]\rangle$. So $\langle[\mathfrak{g},\mathfrak{g}]\rangle \subset \mathfrak{g}_{r+1} \oplus ... \oplus \mathfrak{g}_m$. Because for each $j \in \{r+1, ..., m\}$ $\mathfrak{g}_j$ is simple Lie algebra, $\langle[\mathfrak{g}_j,\mathfrak{g}_j]\rangle = \mathfrak{g}_j$. So $\mathfrak{g}_{r+1} \oplus ... \oplus \mathfrak{g}_m \subset \langle[\mathfrak{g},\mathfrak{g}]\rangle$. □

Proposition 3.9.7. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_m = \mathfrak{h}_1 \oplus ... \oplus \mathfrak{h}_n$ and $\mathfrak{g}_i$ and $\mathfrak{h}_j$ are ideal of $\mathfrak{g}$ and simple Lie algebras. Then $m = n$ and there is $\sigma : \{1,2,...,m\} \to \{1,2,...,m\}$ such that σ is bijective and $\mathfrak{g}_{\sigma(i)} = \mathfrak{h}_i$ $(\forall i \in \{1,2,...,m\})$.*

Proof. For each i, $\mathfrak{g}_1 \supset \langle[\mathfrak{g}_1,\mathfrak{g}_1]\rangle = \langle[\mathfrak{g}_1,\mathfrak{h}_1]\rangle \oplus ... \oplus \langle[\mathfrak{g}_1,\mathfrak{h}_n]\rangle$. Because $\langle[\mathfrak{g}_1,\mathfrak{g}_1]\rangle$ is not zero, there is $\sigma(1)$ such that $\langle[\mathfrak{g}_1,\mathfrak{h}_{\sigma(1)}]\rangle$ is not zero. Because $\langle[\mathfrak{g}_1,\mathfrak{h}_{\sigma(1)}]\rangle \subset \mathfrak{h}_1$ and $\mathfrak{h}_{\sigma(1)}$ is simple and $\mathfrak{g}_1$ is simple, $\mathfrak{g}_1 = \langle[\mathfrak{g}_1,\mathfrak{h}_{\sigma(1)}]\rangle = \mathfrak{h}_{\sigma(1)}$. By repeating the above argument, □

3.10 Discrete subgroup and Abelian Lie group

Definition 3.10.1 (Discrete subgroup)**.** *Let G is a topological group. We call $H \subset G$ a discrete subgroup of G if H is a subgroup of G and the relative of H to G is equal to the discrete topology.*

Proposition 3.10.2. *Let*

(S1) G_2 is a Lie group which is locally isomorphic to a linear Lie subgroup of $GL(n, \mathbb{C})$.

(S2) H is a subgroup of G_1.

then the followings equivalent.

(i) H is a discrete subgroup of G_1.

(ii) There is an open neighborhood of 1_{G_1} U such that $U \cap H = \{1_{G_1}\}$.

(iii) H is a closed subgroup of G_1 and H is a Lie group which is locally isomorphic to $\{1_{G_2}\}$. And $Lie(H) = \{0\}$.

Proof of that (i) $\implies$ (ii): Because $\{1_{G_1}\}$ is an open set of relative topology, there is an oen set U such that $\{1_{G_1}\} = U \cap H$. □

Proof of that (ii) $\implies$ that H is closed set: There is U_1 such that U_1 is open neighborhood of 1_{G_1} and $U_1^{-1}U_1 \subset U$. There is U_2 such that U_2 is open neighborhood of 1_{G_1} and $U_2^{-1} \subset U_1$ and $U_2 \subset U_1$. Let us assume there is $g \in \bar{H} \setminus H$. There is $u \in U_2$ and $h \in H$ such that $gu = h$. So $g \in hU_1$. Because G_1 is a Housdorff space, there is U_3 such that U_3 is an open neighborhood of 1_{G_1} and $U_3 \subset U_2$ and $h^{-1}g \notin U_3^{-1}$. So $h \notin gU_3$. Because $g \in \bar{H}$, there is $h_2 \neq h$ such that $h_2 \in gU_3$. So there is $u_3 \in U_3$ such that $h_2 = gu_3$. So $h_2u_3^{-1} = hu^{-1}$. Because $h^{-1}h_2 \in U_2^{-1}U_3 \subset U$. So $h^{-1}h_2 \in U \cap H = \{1_{G_1}\}$. This implies $h = h_2$. This is contradiction. □

Proof of that H is a Lie group: Because of (ii), H is locally isomorphic to $\{1_{G_2}\}$. Because $\{1_{G_2}\}$ is a linear Lie group of $GL(n, \mathbb{C})$, H is a Lie group. □

Proof of that (ii) $\implies$ that $Lie(H) = \{0\}$: By von-Neumann-Cartan's theorem, exp is locally injective. So $Lie(H) = \{0\}$. □

Proof of that (iii) $\implies$ (ii): By von Neumann-Cartan's theorem, there is $\epsilon > 0$ such that

$$exp(B(O, \epsilon)) \cap \tau(H \cap U) = exp(Lie(H) \cap B(O, \epsilon)) = \{1_{G_2}\} \tag{3.10.1}$$

So

$$\begin{aligned} & \eta(exp(B(O, \epsilon) \cap V) \cap H \\ = \; & \eta(exp(B(O, \epsilon)) \cap \tau(H \cap U)) \\ = \; & exp(Lie(H) \cap B(O, \epsilon)) = \{1_{G_1}\} \end{aligned} \tag{3.10.2}$$

This means (ii). □

Proof of that (ii) $\implies$ (i): For any $h \in H$, $\{h\} = hU \cap H$. This means (i). □

Proposition 3.10.3. *Let us fix any H which is a discrete subgroup of $\mathbb{R}^n$. Then there are linearly independent subset $X_1, ..., X_r \subset \mathbb{R}^n$ such that $H = \sum_{i=1}^{r} \mathbb{Z}X_i$. $r = 0$ means $H = \{0\}$.*

Proof of that $n = 1$. We can assume $H \neq \{0\}$. There is $Y \in H \setminus \{0\}$. We set $t_0 := inf\{t > 0 | tY \in H\}$. We assume $t_0 = 0$. There is $\{t_i\} \subset (0, \infty)$ such that $\lim_{i \to \infty} t_i = 0$ and $t_iY \in H$ $(\forall i)$. Let us fix any $t > 0$. $tY = \lim_{i \to \infty} \lceil \frac{t}{t_i} \rceil t_iY$. Because H is closed, $tY \in H$. This implies $\mathbb{R}Y \subset H$ and $Y \neq \mathbf{0}$. This contradicts with H is a discrete subgroup.

So $t_0 > 0$. We set $X_1 := t_0Y$. We assume there is $X \in H \setminus \mathbb{Z}X_1$. There is $t \in H \setminus \mathbb{Z}$ such that $X = tX_1$. $(t - \lceil t \rceil)t_0Y = (t - \lceil t \rceil)X_1 \in H$. This contradicts with the definition of t_0. □

Proof of that $n > 1$. We assume the Proposition is true if $n < N$ and $N \leq 1$. Let us take $X_1 \in H$ as in the $N = 1$ case. $(0, 1)X_1 \cap H = \phi$.

There is $X_2, ..., X_N \in \mathbb{R}^N$ such that $X_1, X_2, ..., X_N$ is a basis of $\mathbb{R}^N$. We set $H' := \{\boldsymbol{t'} \in \mathbb{R}^{N-1} | \exists s \in \mathbb{R} \text{ such that } sX_1 + \sum_{i=2}^{N} tX_i \in H\}$. Clearly H' is a subgroup of $\mathbb{R}^{N-1}$.

We assume H' is a not discrete subgroup of $\mathbb{R}^{N-1}$. By the same argument as above, there is a sequence $\{\boldsymbol{t_i}'\}_{i=1}^{\infty} \subset H'$ such that $\lim_{i \to \infty} \boldsymbol{t_i}' = 0$. Because $X_1 \in H$, there is a sequence $\{s_i\}_{i=1}^{\infty} \subset [-\frac{1}{2}, \frac{1}{2}]$ such that $s_iX_1 + \sum_{i=2}^{N} t_iX_i \in H$ $(\forall i)$. We can assume there is $s_0 \in [-\frac{1}{2}, \frac{1}{2}]$ such that $\lim_{i\to\infty} s_i = s_0$. Because H is closed, $s_0X_1 \in H$. By the definition of X_1, $s_0 = 0$. Because $s_iX_1 + \sum_{j=2}^N t_{i,j}X_j \in H \setminus \{0\}$ $(\forall i)$ and $\lim_{i\to\infty} s_iX_1 + \sum_{j=2}^N t_{i,j}X_j = 0$. This means H is a not discrete subgroup. This is contradiction. So H' is a discrete subgroup.

By the assumption of the mathematical induction, there is $Z_1, ..., Z_r \in \mathbb{R}^{N-1}$ such that $Z_1, ..., Z_r$ are linear independent and $H' = \sum_{i=1}^r \mathbb{Z}Z_i$. There are $s_1, ..., s_r \in \mathbb{R}$ such that $X'_{i+1} := s_iX_1 + \sum_{j=1}^r Z_{i,j}X_j \in H$ $(\forall i)$. Because

$$(X_1, X'_2, ..., X'_{r+1}) = (X_1, ..., X_N)\begin{pmatrix} 1 & s_1 & ... & s_r \\ 0 & z_{1,1} & ... & z_{r,1} \\ ... & ... & ... & ... \\ 0 & z_{1,N-1} & ... & z_{r,N-1} \end{pmatrix} \quad (3.10.3)$$

and the rank of $\begin{pmatrix} 1 & s_1 & ... & s_r \\ 0 & z_{1,1} & ... & z_{r,1} \\ ... & ... & ... & ... \\ 0 & z_{1,N-1} & ... & z_{r,N-1} \end{pmatrix}$ is $(r+1)$, $X_1, X'_2, ..., X'_{r+1}$ are linear independent.

Let us fix any $X \in H$. Because $X_1, X_2, ..., X_N$ is a basis of $\mathbb{R}^N$, there are s and $t_2, ..., t_N$ such that $X = sX_1 + t_2X_2 + ... + t_NX_N$. Because $(t_2, ..., t_N) \in H'$, there are $m_2, ..., m_N \in \mathbb{Z}$ such that $(t_2, ..., t_N)^T = m_2Z_2 + ... + m_NZ_N$.

Because $X - \sum_{i=1}^r X'_i \in \mathbb{R}X_1 \cap H = \mathbb{Z}X_1$, $X \in \mathbb{Z}X_1 + \sum_{i=1}^r \mathbb{Z}X'_i$. Consequently, $H = \mathbb{Z}X_1 + \sum_{i=1}^r \mathbb{Z}X'_i$. □

Proposition 3.10.4. *Let*

(S1) G_1 is a Le group which is locally isomorphic to a Lie subgroup of $GL(n, \mathbb{C})$.

(A1) G_1 is connected.

Then the followings are equivalent.

(i) G_1 is abelian.

(ii) $Lie(G_1)$ is abelian.

STEP1. Showing (i) $\Longrightarrow$ (ii). Let us fix any $X, Y \in Lie(G_1)$. Because

$$\begin{aligned} & exp(t(X+Y) + t^2[X,Y] + O(t^3)) \\ = \ & exp(tX)exp(tY) \\ = \ & exp(t(X+Y) + t^2[Y,X] + O(t^3)) \end{aligned} \quad (3.10.4)$$

, $[X, Y] = [Y, X]$. So $Lie(G_1)$ is abelian. □

STEP2. Showing (ii) $\Longrightarrow$ (i). There is $\epsilon > 0$ such that $exp(B(O,\epsilon))exp(B(O,\epsilon)) \subset V$. Let us fix any $g, h \in \eta(exp(B(O,\epsilon)))$. There is $X, Y \in B(O,\epsilon)$ such that $g = \eta(exp(X))$, $h = \eta(exp(Y)0$. Because X and Y are commutative,

$$\begin{aligned} gh &= \eta(exp(X))\eta(exp(Y)) \\ &= \eta(exp(X)exp(Y)) \\ &= \eta(exp(X+Y)) = \eta(exp(Y+X)) \\ &= \eta(exp(Y)exp(X)) = \eta(exp(Y))\eta(exp(X)) = hg \end{aligned} \quad (3.10.5)$$

By Proposition3.8.1, G_1 is abelian. □

Proposition 3.10.5. *Let*

(S1) G_1 is a Lie group.

(A1) G_1 is abelian.

(A2) G_1 is connected.

(S2) $N := dimLie(G_1)$.

Then there is $r \in \{1, 2, ..., n\}$ such that $\mathbb{T}^r \times \mathbb{R}^{N-r}$ is C^ω-class isomorhic as Lie group to G.

STEP1. Showing that $Exp : Lie(G_1) \to G_1$ is continuous and surjective. There is $\epsilon > 0$ such that for any $g \in G$ there are $exp(X_1), ..., exp(X_M) \in V_\epsilon := exp(B(O, \epsilon))$ which satisfies $g = exp(X_1)...exp(X_M)$. Because $Lie(G_1)$ and G_1 are commutative, $Exp : Lie(G) \to G_1$ is homomorphism of topological group.

Because Exp is a locally isomorphism from $Lie(G_1) \cap B(O, \epsilon) \to \eta(exp(B(O, \epsilon))) \cap V^\circ$, by Proposition3.4.6, Exp is surjective. □

STEP2. Showing that $Exp^{-1}(\{1_G\})$ is a discrete subgroup of $\mathbb{R}^N$. By von-Neumann-Cartan's theorem, there is $\epsilon > 0$ such that $exp^{-1}(\{1_G\}) \cap B(O, \epsilon) = O$. So $exp^{-1}(\{1_G\})$ is a discrete subgroup of $\mathbb{R}^N$. □

STEP3. exp is an open map. Because G is abelian, for any $X \in Lie(G)$ $exp(B(X, \epsilon)) = exp(X)exp(B(O, \epsilon))$. Because $exp(B(O, \epsilon))$ is open, exp is an open map. □

STEP4. Construction of a isomorphism of Lie groups. By Proposition3.10.3, there are $X_1, ..., X_N \in Lie(G)$ and r such that $X_1, ..., X_N$ is a basis of $Lie(G)$ and

$$exp^{-1}(\{1_G\}) = \sum_{i=1}^{r} \mathbb{Z}X_i \tag{3.10.6}$$

We set $i : \mathbb{T}^r \times \mathbb{R}^{N-r} \to G$ by

$$i(exp(i2\pi\theta_1), ..., exp(i2\pi\theta_r), \boldsymbol{t}) := exp(\sum_{i=1}^{r} \theta_i X_i + \sum_{i=r+1}^{N} t_i X_i) \tag{3.10.7}$$

By STEP3, i is an open map. So i is homeomorphism and isomoriphism of topological groups. By Proposition3.4.14, i is a C^ω-class isomorphism of Lie groups. □

3.11 Nilpotent Lie group

Definition 3.11.1 (Nilpotent Lie algebra, Lie group)**.** *Let G be a Lie group and $\mathfrak{g} := Lie(G)$. We set*

$$\mathfrak{g}_0 := \mathfrak{g},\ \mathfrak{g}_i := [\mathfrak{g}_{i-1}, \mathfrak{g}]\ (i = 1, 2, ...) \tag{3.11.1}$$

We call $\mathfrak{g}$ is a Nilpotent Lie algebra if there is $n \in \mathbb{N}$ such that $\mathfrak{g}_n = \{0\}$. We call G is a Nilpotent Lie group if G is $Lie(G)$ is a Nilpotent Lie algebra.

Proposition 3.11.2. *Let G be a Lie subgroup of $GL(n\mathbb{C})$ and G be a connected Nilpotent Lie group. Then $Exp : Lie(G) \to G$ is surjective.*

Proof. Let us fix any $g \in G$. By Proposition3.4.6, there are $X_1, ..., X_m \in Lie(G)$ such that $g = exp(X_1)exp(X_2)...exp(X_m)$.

Let us fix any $X, Y \in Lie(G)$. By Baker-Campbell-Hausdorff formula, there is a polynomial $Z(t)$ sucht that for $|t| \ll 1$

$$exp(tX)exp(tY) = exp(Z(t)) \tag{3.11.2}$$

Because $exp(\cdot X)exp(\cdot Y)$ is holomorphic, the power series of $exp(\cdot X)exp(\cdot Y)$ is equal to the power series of $exp(Z(t))$. The convergence radius of the power series of $exp(Z(t))$ is ∞. By identity theorem of holomorphic function(see [19]),

$$exp(X)exp(Y) = exp(Z(1))$$

So exp is surjective. □

3.12 Solvable Lie group

Definition 3.12.1 (Solvable Lie algebra, Lie group)**.** *Let G be a Lie group and $\mathfrak{g} := Lie(G)$. We set*

$$\mathfrak{g}_0 := \mathfrak{g},\ \mathfrak{g}_i := [\mathfrak{g}_{i-1}, \mathfrak{g}_{i-1}]\ (i = 1, 2, ...) \tag{3.12.1}$$

We call $\mathfrak{g}$ is a Solvable Lie algebra if there is $n \in \mathbb{N}$ such that $\mathfrak{g}_n = \{0\}$. We call G is a Solvable Lie group if G is $Lie(G)$ is a Solvable Lie algebra.

Clearly the following holds.

Proposition 3.12.2. *Any nilpotent Lie algebra is solvable.*

3.13 Complex Lie group and Holomorphic representation

From the definition and property of $\mathbb{C}$, the following holds.

Proposition 3.13.1 (Complexification)**.** *Here are settings and assumptions.*

(S1) $\mathfrak{g} \subset M(n, \mathbb{C})$ *is a Lie algebra.*

Then

$$\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} := \{X + iY | X, Y \in \mathfrak{g}\}$$

is a $\mathbb{C}$ *vector space with respect to*

$$(a + ib)(X + iY) := (aX - bY)$$

We call $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ *the complexification of* $\mathfrak{g}$.

From the definition and property of $\mathbb{C}$ and the definition of complexification, the following holds.

Proposition 3.13.2. *Here are settings and assumptions.*

(S1) $\mathfrak{g} \subset M(n, \mathbb{C})$ *is a Lie algebra.*

(S2) $f : \mathfrak{g} \to \mathfrak{g}$ *is a* $\mathbb{R}$ *linear map.*

If we define $F : \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ *by*

$$\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} := \{X + iY | X, Y \in \mathfrak{g}\}$$

then F *is a* $\mathbb{C}$ *linear map.*

Clearly the following holds by Proposition7.1.1.

Proposition 3.13.3. *Here are settings and assumptions.*

(S1) $\mathfrak{g} \subset M(n, \mathbb{C})$ *is a Lie algebra.*

(S2) $f : \mathfrak{g} \to \mathfrak{g}$ *is a* $\mathbb{R}$ *linear map.*

(A1) There is a basis of $\mathfrak{g}$ *which is* $\{X_i\}_{i=1}^n \cup \{iX_i\}_{i=1}^n$ *for some* $\{X_i\}_{i=1}^n \subset \mathfrak{g}$.

(A2) $\{X_i\}_{i=1}^n$ *is a basis of the complexification of* $\mathfrak{g}$.

(S3) F *is the complexification of* f.

(A3) All eigenvalues of F *are distinct.*

Then

$$det(f) = |det(F)|^2$$

Definition 3.13.4 (Complexification and Real Form of Lie Algebra)**.** *The followings are settings and assumptions.*

(S1) $\mathfrak{g}$ *is a complex Lie algebra.*

(S2) $\mathfrak{h}$ *is a real Lie algebra that is real subalgebra of* $\mathfrak{g}$.

(A1) $\mathfrak{g} = \mathfrak{h} + i\mathfrak{h}$ *and* $\mathfrak{h} \cap i\mathfrak{h} = \{0\}$.

Then we say $\mathfrak{h}$ *is a real form of* $\mathfrak{g}$ *and* $\mathfrak{g}$ *is a complexification of* $\mathfrak{h}$.

Definition 3.13.5 (Complex Lie group)**.** *We call* G *is a complex Lie group if* G *is a Lie group and* G *is a complex manifold and multiple operation and inverse operation of* G *are holomorphic.*

Definition 3.13.6 (Complex Structure)**.** *The followings are settings and assumptions.*

(S1) $\mathfrak{g}$ *is a real Lie algebra.*

We say $J \in Eng_{\mathbb{R}}(\mathfrak{g})$ *is a complex structure of* $\mathfrak{g}$ *if*

$$J^2 = -id, \ [X, JY] = J[X, Y] \ (\forall X, Y)$$

The followings are clearly hold.

Proposition 3.13.7. *The followings hold.*

(i) For any real Lie algebra $\mathfrak{g} \subset M(n, \mathbb{C})$, $\mathfrak{g}$ has a complex structure below.

$$J : \mathfrak{g} \ni X \mapsto iX \in \mathfrak{g}$$

(ii) If a real Lie algebra$\mathfrak{g}$ has a complex structure J, then $\mathfrak{g}_{\mathbb{C}} := \{X + iJY | X, Y \in \mathfrak{g}\}$ is a complex Lie algebra.

Definition 3.13.8 (Real Form of a Complex Lie algebra)**.** *The followings are settings and assumptions.*

(S1) $\mathfrak{g}$ is a real Lie algebra which has a complex structre J.

(S2) $\mathfrak{g}_{\mathbb{C}} := \{X + JY | X, Y \in \mathfrak{g}\}$.

(A1) There is a sub Lie algebra of $\mathfrak{g}$ $\mathfrak{g}_0$ such that

$$\mathfrak{g}_0 + J\mathfrak{g}_0 = \mathfrak{g}, \mathfrak{g}_0 \cap J\mathfrak{g}_0 = \phi$$

Then we call $\mathfrak{g}_0$ a real form of $\mathfrak{g}$.

The following is clear.

Proposition 3.13.9. *The followings are settings and assumptions.*

(S1) $\mathfrak{g}$ is a complex Lie algebra that has a complex structre J.

(S2) $\mathfrak{g}_{\mathbb{C}} := \{X + JY | X, Y \in \mathfrak{g}\}$ has a real form $\mathfrak{g}_0$.

(S3) $\mathfrak{h}$ is a complex Lie algebra.

(S4) Φ is a homomorphism from $\mathfrak{g}_0$ to $\mathfrak{h}$ as real Lie algebras.

Then Φ can be uniquely extended on $\mathfrak{g}$.

Proposition 3.13.10. *The followings are settings and assumptions.*

(S1) G is a connected Lie group.

(A1) $\mathfrak{g} := Lie(G)$ has a complex structre J.

(A2) $\mathfrak{g}_{\mathbb{C}} := \{X + JY | X, Y \in \mathfrak{g}\}$ has a real form $\mathfrak{g}_0$.

Then there is $\epsilon > 0$, G is a complex Lie group, which has a local coordinate system

$$\phi_g : B(0, \epsilon) \ni (x_1 + y_1 i, ..., x_n + y_n i) \mapsto g \exp(x_1 X_1 + y_1 JX_1 + ... + x_n X_n + y_n JX_n) \ (g \in G)$$

and

$$G \times G \ni (x, y) \mapsto xy^{-1} \in G$$

is holomorphic. We call a Lie group whose Lie algebra is $\mathfrak{g}_0$, a real form of a complex Lie group G.

Proof of that G is a complex manifold. By von-Neumann Cartan Theorem, there is $\epsilon > 0$ such that G is a smooth real manifold, which has a local coordinate system

$$\phi_g : B(0, \epsilon) \ni (x_1 + y_1 i, ..., x_n + y_n i) \mapsto gExp(x_1 X_1 + y_1 JX_1 + ... + x_n X_n + y_n JX_n) \ (g \in G)$$

Let us fix any $g, g' \in G$ such that $\phi_g(B(0, \epsilon)) \cap \phi_{g'}(B(0, \epsilon)) \neq \phi$. From the definition of Lie group, there is $\delta > 0$ such that $\eta : exp(B(0, \delta)) \to G$ be a locally isomorphism as topological groups. Since $\epsilon > 0$ is sufficiently small, there is $A \in exp(B(0, \delta))$ such that

$$\eta(A) = (g')^{-1} g$$

Therefore, for each $(z_1, ..., z_n) \in B(0, \epsilon)$, there exists unique $(z'_1, ..., z'_n) \in B(0, \epsilon)$

$$Aexp(z_1 X_1 + ... + z_n X_n) = exp(z'_1 X_1 + ... + z'_n X_n)$$

This means

$$log(Aexp(z_1 X_1 + ... + z_n X_n)) = z'_1 X_1 + ... + z'_n X_n$$

Let us fix $Y_1, ..., Y_m \in M(N, \mathbb{C})$ such that $\{X_i\}_{i=1}^n \cup \{JX_i\}_{i=1}^n \cup \{Y_i\}_{i=1}^m$ is a basis of $M(N, \mathbb{C})$. And

$$\mathbb{C}^n \ni (z_1, ..., z_n) \mapsto z_1 X_1 + ... + z_n X_n \in M(N, \mathbb{C})$$

and

$$M(N,\mathbb{C}) \ni X \mapsto \log(A\exp(X)) \in M(N,\mathbb{C})$$

and

$$M(N,\mathbb{C}) \ni Y = z_1'X_1 + ... + z_n'X_n + y_1Y_1 + ... + y_mY_m \mapsto (z_1', ..., z_n') \in \mathbb{C}^n$$

are holomorphic mapping, respectively. So,

$$\mathbb{C}^n \ni (z_1, ..., z_n) \mapsto (z_1', ..., z_n') \in \mathbb{C}^n$$

is holomorphic. □

Proof of that inverse operation and multiplicity operation of G is holomorphic. From the proof of von Neumann Cartan Theorem, it is enough to show that for any $g \in G$ $Ad(g)$ is $\mathbb{C}$ linear. Because G is connected, there are $X_1, ..., X_m \in \mathfrak{g}$ such that $Exp(X_1)...Exp(X_m) = g$. So, $Ad(g) = ad(X_1)...ad(X_m)$. From the definition of complexication, $ad(X_1), ..., ad(X_m)$ are $\mathbb{C}$ linear. □

Definition 3.13.11 (Holomorphic Representation)**.** *Let G be a complex Lie group and V be a finite dimensional $\mathbb{C}$-vector space. We say (π, V) is a holomorphic representation of G if (π, V) is a continuous holomorphic representation of G and $\pi : G \to GL(n,\mathbb{C})$ is a holomorphic homomorphism.*

Proposition 3.13.12. *The followings are settings and assumptions.*

- *(S1) G_1 is a connected complex Lie group.*
- *(S2) G_2 is a complex Lie group.*
- *(S3) $\phi : G_1 \to G_2$ is a continuous homomorphic map from G_1 to G_2.*

Then ϕ is homomorphic if and only if $d\phi_{1_{G_1}} : T_{1_{G_1}}(G_1) \to T_{1_{G_2}}(G_2)$ is a $\mathbb{C}$ linear.

Proof. It is clear from the proof of Theorem3.4.14. □

Proposition 3.13.13. *The followings are settings and assumptions.*

- *(S1) G is a connected complex Lie group.*
- *(S2) (π, V) is a finite dimensional continuous representation of G.*

Then (π, V) is a holomorphic representation of G if and only if the differential representation of π is $\mathbb{C}$ linear.

Proof. It is followed from the definition of differential representation and Proposition3.13.12. □

Definition 3.13.14 (Complexification and Real Form of Lie Group)**.** *The followings are settings and assumptions.*

- *(S1) G is a complex Lie group. We set $\mathfrak{g} := Lie(G)$.*
- *(S2) H is a closed subgroup of G. We set $\mathfrak{h} := Lie(H)$.*
- *(A1) $\mathfrak{g}$ is a compexification of $\mathfrak{h}$.*

Then we say H is a real form of G and G is a complexification of H.

3.14 Simply Connected Lie Group

3.14.1 Universal covering group of Lie group

Proposition 3.14.1 (Universal covering group)**.** *The followings are settings and assumptions.*

- *(S1) G_1 is a Lie group which is locally isomorphic to a Lie subgroup of $GL(n,\mathbb{C})$ G_2.*
- *(A1) G_1 is path-connected.*

Let

$$\hat{G_1} := [([0,1],\{0\}),(G_1,\{1_{G_1}\})]$$

and for each $c_1, c_2 \in \hat{G_1}$ $c_1 \sim c_2$ if there is a homotop Φ from c_1 to c_2 such that

$$\Phi(s,0) = e,\ \Phi(s,1) = c_1(1) = c_2(2)\ (\forall s)$$

and

$$\tilde{G_1} := \hat{G_1}/\sim$$

and

$$p : \hat{G_1} \ni c \mapsto [c] \in \tilde{G_1}$$

and

$$q : \tilde{G_1} \ni [c] \mapsto c(1) \in G_1$$

and

$$[c_1] \cdot [c_2] := [c_1 c_2] \ (for\ c_1, c_2 \in \hat{G_1})$$

Then

(i) *There is a Lie group structure of $\tilde{G_1}$ such that $p : \tilde{G_1} \to G_1$ is locally isomorphism of Lie groups.*

(ii) *$Lie(G_1) = Lie(\tilde{G_1})$*

STEP1. Showing $\sim$ is equivalent relationship on $\hat{G_1}$. It is easy to show by the fact homotop is equivalent relationship. □

STEP2. Showing the multiple operation of $\tilde{G}$ is well-defined. Let us fix any $c_1, d_1, c_2, d_2 \in \hat{G}$ such that $c_1 \sim c_2$ and $d_1 \sim d_2$. Then there is Φ_c, Φ_d such that Φ_c is a homotopy from c_1 to c_2 and Φ_d is a homotopy from d_1 to d_2. Because $\Phi_c \cdot \Phi_d$ is a homotopy from $c_1 \cdot d_1$ to $c_2 \cdot d_2$, $c_1 \cdot d_1 \sim c_2 \cdot d_2$. So, the multiple operation of $\tilde{G}$ is well-defined.

□

STEP3. Showing q is surjective. This is from (A1). □

STEP4. Showing $\tilde{G_1}$ is group. This is from the group structure on G_1. □

STEP5. Constructing the topology of $\tilde{G_1}$. There is $\epsilon > 0$ such that

$$Exp : Lie(G_1) \cap B(O, \epsilon) \to Exp(B(O, \epsilon)) \cap G_1$$

is C^ω-class homeomorphism and

$$\sup_{X \in B(O,\epsilon)} ||exp(X) - E|| < 1$$

For each $s \in [0, 1]$, we set

$$W_{e,s} := \{[[0, 1] \ni t \to Exp(tsX)] | X \in Lie(G_1) \cap sB(O, \epsilon)\}$$

and for each $\tilde{g} \in \tilde{G_1}$

$$W_{\tilde{g},s} := \tilde{g} W_{e,s}$$

We will show $\{W_{\tilde{g},s}\}_{\tilde{g} \in \tilde{G_1}, s \in [0,1]}$ satisfies the axiom of system of fundamental neighborhoods.

Let us fix any $[c][d] \in [c]W_{e,s}$, $[d] \in W_{e,s}$. Clearly, there is $s_1 \in [0, 1]$ such that for any $t \in [0, 1]$

$$d(t)Exp(s_1 B(O, \epsilon)) \subset Exp(sB(O, \epsilon))$$

Let us fix any $X \in s_1 B(O, \epsilon)$. We set $Z := d(1)Exp(X)$. Because $Exp(sB(O, \epsilon))$ is simply connected, $d(\cdot)Exp(\cdot X) \sim Exp(\cdot Z)$. This implies that

$$c(\cdot)d(\cdot)Exp(\cdot X) \sim c(\cdot)Exp(\cdot Z)$$

So,

$$[cd]W_{e,s_1} \subset [c]W_{e,s}$$

Let us fix any $[c_1][d_1] = [c_2][d_2] \in [c_1]W_{e,s_1} \cap [c_2]W_{e,s_2}$, $[d_1] \in W_{e,s_1}$ and $[d_2] \in W_{e,s_2}$. By the argument in the previous paragraph, there is $s_3 \in [0, 1]$ such that

$$[c_1 d_1]W_{e,s_3} \subset [c_1]W_{e,s_1},\ [c_2 d_2]W_{e,s_3} \subset [c_2]W_{e,s_2}$$

So,

$$[c_1 d_1]W_{e,s_3} \subset [c_1]W_{e,s_1} \cap [c_2]W_{e,s_2}$$

□

STEP6. Showing that $\tilde{G}$ is a topological group. Firstly, we will show $\tilde{G}$ is Housdorff space. Let $[c]\tilde{G} \setminus \{e\}$. Because G is Housdorff space, there is $s \in (0,1]$ such that

$$e \notin c(1)Exp(B_m(O, s\epsilon))$$

So,

$$[e] \notin [c]W_{e,s}$$

Consequently, $\tilde{G}$ is Housdorff space.

□

STEP7. Showing that q is a local isomorphism. Because $ExpB_m(O,\epsilon)$ is simply connected,

$$q|_{W_{e,1}} : W_{e,1} \ni [c] \to c(1) \in Exp(B_m(O,\epsilon))$$

is injective. And clearly $q|_{W_{e,1}}$ is surjective. Because $ExpB_m(O,\epsilon)$ is simply connected, for any $s \in [0,1]$ and $[c] \in W_{e,1}$ such that $[c]W_{e,s} \in W_{e,1}$,

$$q([c]W_{e,s}) = c(1)ExpB_m(O, s\epsilon)$$

So, $q|_{W_{e,1}}$ is continuous and open map. Because Exp is continuous, there is $s_0 \in [0,1]$ such that

$$Exp(B_m(O, s_0\epsilon)Exp(B_m(O, s_0\epsilon) \subset Exp(B_m(O, s_0\epsilon))$$

Because $ExpB_m(O,\epsilon)$ is simply connected,

$$[c_1][c_2] \in W_{e,s_0} \iff c_1(1)c_2(1) \in Exp(B_m(O, s_0\epsilon))$$

Consequently, q is a local isomorphism. □

Showing that $\tilde{G}$ is path-connected. Let us fix any $[c] \in \tilde{G}$. We set, for each $s \in [0,1]$,

$$C(s) := [c(s\cdot)]$$

Then, clearly, C is a continuous path from $[\{e\}]$ to c. □

Proposition 3.14.2. *Let G be a path-connected topological group and $\tilde{G}$ be a universal covering group of G. Let us assume $*$ be the operation of $\pi(G)$. Then for any $c_1 \in C([0,1], G)$ such that $c(0) = e$ and $c_2 \in \pi(G)$,*

$$[c_1] \cdot [c_2] = [c_1] * [c_2] = [c_2] \cdot [c_1]$$

Proof. We set

$$\Phi_1(s,t) := c_1(L(s(2t-1)) + (1-s)t)c_2(L(2st) + (1-s)t)$$

and

$$\Phi_2(s,t) := c_2(L(s(2t-1)) + (1-s)t)c_1(L(2st) + (1-s)t)$$

Here,

$$L(u) := \begin{cases} 0 & (u \le 0) \\ u & (0 \le u < 1) \\ 1 & (u \ge 1) \end{cases}$$

Clearly, Φ_1 is a homotop from $c_1 \cdot c_2$ to $c_1 * c_2$ and Φ_2 is a homotop from $c_2 \cdot c_1$ to $c_1 * c_2$. □

By Proposition3.14.2, the following holds. We will show another proof using adjoint representation of Lie group.

Proposition 3.14.3. *Let G be a path-connected Lie group and $\tilde{G}$ be a universal covering group of G. Then $q^{-1}(e)$ is contained in the center of $\tilde{G}$. In special, $\pi(G)$ is commutative group.*

STEP1. Showing that $Ad(g) = id$ $(\forall g \in q^{-1}(e))$. Let us fix any $g_0 \in q^{-1}(e)$ and $Y \in Lie(\tilde{G})$. By the definition of Ad,

$$g_0 Exp(tY)g_0^{-1} = Exp(tAd(g_0)Y) \ (|t| \ll 1)$$

So,

$$Exp(t\iota(Y)) = q(Exp(tY)) = q(g_0Exp(tY)g_0^{-1}) = q(Exp(tAd(g_0)Y)) = Exp(t\iota(Ad(g_0)Y))$$

This implies

$$\iota(Y) = \iota(Ad(g_0)Y)$$

Because q is a local isomorphism, ι is an isomorphism. So, $Y = Ad(g_0)Y$. □

STEP2. Showing that $q^{-1}(e)$ is contained in the center of $\tilde{G}$. Because $\tilde(G)$ is path-connected, it is enough to show g_0 is commutative with $Exp(B(O,\epsilon))$ for sufficient small $\epsilon > 0$.

$$g_0 Exp(Y) = g_0 Exp(Y) g_0^{-1} g_0 = Exp(Ad(g_0)Y) g_0 = Exp(Y) g_0$$

□

Theorem 3.14.4. *The followings are settings and assumptions.*

- *(S1) $G_{i,1}$ is a Lie group which is locally isomorphic to a Lie subgroup of $GL(n,\mathbb{C})$ $G_{i,2}$ $(i = 1, 2)$.*
- *(A1) $Lie(G_{1,1})$ and $Lie(G_{2,1})$ are isomorphic as Lie algebras.*

then $G_{1,1}$ and $G_{2,1}$ are isomorphic as Lie groups.

3.14.2 Lifting a Homomorphism of Lie Algebras⋆

Lemma 3.14.5. *The followings are settings and assumptions.*

- *(S1) G_1, G_2 are connected Lie group.*
- *(S2) $\phi : Lie(G_1) \to Lie(G_2)$ is a homomorphism.*
- *(S3) U is an open neighborhood of the zero vector in $Lie(G_1)$ such that $Exp : U \to G_1$ is a diffeomorphism to an open neighborhood of e and*

$$\text{For each } X, Y, Z \in U \exp(X)\exp(Y) = \exp(Z) \implies \exp(\phi(X))\exp(\phi(Y)) = \exp(\phi(Z))$$

- *(S4) V is an open neighborhood of e in G_1 such that*

$$V^{-1} = V,\ VV \subset Exp(U)$$

- *(S5) We set*

$$\Phi_o(Exp(X)) := Exp(\phi(X))\ (Exp(X) \in V)$$

- *(S6) $c_1 \in C([0,1], G_1)$ such that $c_1(0) = e$.*

then

- *(i) There uniquely exists $c_2 \in C([0,1], G_2)$ such that for each $0 \leq s_0 \leq s_1 \leq 1$, if*

$$c_1(s)^{-1}c_1(t) \in V\ \ (\forall s, t \in [s_0, s_1])$$

then

$$c_2(s)^{-1}c_2(t) \in V\ \ (\forall s, t \in [s_0, s_1])$$

- *(ii) If c_1 and c_1' are homotopic preserving the start point and the end point, then c_2 and c_2' are homotopic preserving the start point and the end point.*

Proof of the uniqueness in (i). Let us fix c_2 that satisfies the condition in (i). Let us fix $N \in \mathbb{N}$ such that for any $i \in \{1, 2, ..., N\}$ and $s, t \in [t_{i-1}, t_i]$ $c_1(s)^{-1}c_1(t) \in V$. Here, $t_i := \dfrac{i}{N}$.

Let us fix any $t \in [0.1]$. There is i such that $t \in [t_{i-1}, t_i]$. Then from the condition in (i)

$$c_2(t_{i-1})^{-1}c'(t) = \Phi_o(c_1(t_{i-1})^{-1}c_1(t)), c_2(t_{\nu-1})^{-1}c_2(t_\nu) = \Phi_o(c_1(t_{\nu-1})^{-1}c_1(t_\nu))\ (\nu = 1, 2, ..., i-1)$$

Therefore,

$$\begin{aligned} & c_2(t) = ec_2(t) = c_2(t_0)^{-1}c_2(t) = c_2(t_0)^{-1}c_2(t_1)^{-1}...c_2(t_{i-2})^{-1}c_2(t_{i-1})c_2(t_{i-1})^{-1}c_2(t) \\ = \ & \Phi_o(c_1(t_0)^{-1}c_1(t_1)^{-1})...\Phi_o(c_1(t_{i-2})^{-1}c_1(t_{i-1}))\Phi_o(c_1(t_{i-1})^{-1}c_1(t)) \end{aligned}$$

I mean,

$$c_2(t) = \Phi_o(c_1(t_0)^{-1}c_1(t_1))...\Phi_o(c_1(t_{i-2})^{-1}c_1(t_{i-1}))\Phi_o(c_1(t_{i-1})^{-1}c_1(t)) \tag{3.14.1}$$

Consequently, c_2 is uniquely defined by c_1. □

Proof of the existence in (i). We take over the notations in the proof of the uniqueness in (i). We define c_2 by (3.14.1). Let us fix any $s_0, s_1 \in [0,1]$ such that $s_0 \leq s_1$ and

$$c(s)^{-1}c(t) \in V \ (\forall s,t \in [s_0,s_1])$$

Let us fix any $[s_0,s_1]$. Then there exist i,j such that

$$s \in [t_{i-1},t_i], t \in [t_{j-1},t_j]$$

First, let us assume that $i \leq j$. Then

$$\begin{aligned} & c_2(s)^{-1}c_2(t) = (\Phi_o(c_1(t_0)^{-1}c_1(t_1))...\Phi_o(c_1(t_{i-2})^{-1}c_1(t_{i-1}))\Phi_o(c_1(t_{i-1})^{-1}c_1(s)))^{-1} \\ & (\Phi_o(c_1(t_0)^{-1}c_1(t_1)^{-1})...\Phi_o(c_1(t_{j-2})^{-1}c_1(t_{j-1}))\Phi_o(c_1(t_{j-1})^{-1}c_1(t))) \\ = & \ \Phi_o(c_1(s)^{-1}c_1(t_{i-1}))\Phi_o(c_1(t_{i-1})^{-1}c_1(t_i))...\Phi_o(c_1(t_{j-2})^{-1}c_1(t_{j-1}))\Phi_o(c_1(t_{j-1})^{-1}c_1(t)) \\ = & \ \Phi_o(c_1(s)^{-1}c_1(t)) \end{aligned}$$

In the case when $i > j$, by the same argument as the above,

$$c_2(s)^{-1}c_2(t) = \Phi_o(c_1(s)^{-1}c_1(t))$$

□

Proof of (ii). Let us fix $F : [0,1] \times [0,1] \to G_1$ such that F is a homotopy from c_1 to c_1'. We set

$$c_1^s(t) := c_1^s(t) \ ((s,t) \in [0,1] \times [0,1])$$

For each $s \in [0,1]$, let c_2^s denote the curve in (i) regarding

$$[0,1] \ni t \mapsto F(s,t) \in G_2$$

And we set

$$F'(s,t) := c_2^s(t) \ ((s,t) \in [0,1] \times [0,1])$$

Clearly $F'(0,\cdot) = c_2(\cdot)$ and $F'(0,\cdot) = c_2'(\cdot)$. Let us fix $N \in \mathbb{N}_{\geq 1}$ such that

$$F(s,t)^{-1}F(s',t') \in V \ (\forall s,s',t,t' \in [0,1] \text{ s.t } |s-s'| \leq \frac{1}{N}, |t-t'| \leq \frac{1}{N})$$

It is enough to show

$$F'(s,t)^{-1}F'(s',t') = \Phi_o(F(s,t)^{-1}F(s',t')) \ (\forall s,s' \in [t_i,t_{i+1}], \forall t,t' \in [t_j,t_{j+1}], \forall i,j) \tag{3.14.2}$$

If the above equation holds, F' is continuous and $F'(s,1) = F'(s',1)$ $(\forall s,s')$. Then, F' is a homotopy from c_2 to c_2' preserving the start point and the end point.

We will show (3.14.2) by mathematical induction. We set

$$g_{s,t,t'} := c_1^s(t)^{-1}c_1^s(t'), g'_{t,s,s'} := c_1^s(t)^{-1}c_1^{s'}(t)$$

Then

$$c_2^s(t) = c_2^s(t_j)\Phi_o(g_{s,t_j,t}), c_2^{s'}(t) = c_2^s(t_j)\Phi_o(g'_{s',t_j,t'})$$

If $j=0$, $g'_{t,s,s'} = c_2^s(0) = c_2^{s'}(0) = e$. So, if $j=0$,

$$c_2^{s'}(t_j) = c_2^s(t_j)\Phi_o(g'_{t,s,s'}) \tag{3.14.3}$$

If we assume that (3.14.2) holds for any fixed $(j-1)$, this equation holds. Then

$$g^{-1}_{s,t_j,t}g'_{t_j,s,s'} = c_1^s(t)^{-1}c_1^{s'}(t) \in V$$

and

$$g^{-1}_{s,t_j,t}g'_{t_j,s,s'}g_{s',t_j,t} = c_1^s(t')^{-1}c_1^{s'}(t)c_1^{s'}(t)^{-1}c_1^s(t') = c_1^s(t)^{-1}c_1^s(t') \in V$$

That implies

$$\begin{aligned} c_2^{s'}(t') &= c_2^s(t)(c_2^s(t)c_2^s(t_j))(c_2^s(t_j)^{-1}c_2^{s'}(t_j))(c_2^{s'}(t_j)^{-1}c^{s'}(t')) \\ &\quad \text{(By (3.14.3))} \\ &= c_2^s(t)\Phi_o(g_{s,t_j,t})^{-1}\Phi_o(g'_{t,s,s'})\Phi_o(g_{s',t_j,t'}) \\ &\quad (g_{s,t_j,t}^{-1}g'_{s',t_j,t'} \in V) \\ &= c_2^s(t)\Phi_o(g_{s,t_j,t}^{-1}g'_{t,s,s'})\Phi_o(g_{s',t_j,t'}) \\ &\quad ((g_{s,t_j,t}^{-1}g'_{s',t_j,t'})g_{s',t_j,t'} \in V) \\ &= c_2^s(t)\Phi_o(g_{s,t_j,t}^{-1}g'_{s',t_j,t'}g_{s',t_j,t'}) = c_2^s(t)\Phi_o(c_1^s(t)^{-1}c_1^s(t')) \end{aligned}$$

Therefore, (3.14.2) holds for $(j+1)$. □

Theorem 3.14.6. *The followings are settings and assumptions.*

(S1) G_1, G_2 *are connected Lie group.*

(S2) $\tau : G_1 \to G_2$ *is an isomorphism of Lie groups.*

(S3) $c_2 \in C([0,1], G_2)$.

then

(i) There uniquely exists $c_1 \in C([0,1], G_1)$ *such that*

$$\tau \circ c_1 = c_2$$

We call it the lifting of c_2.

(ii) Let us fix any $c_2' \in C([0,1], G_2)$ *which is homotopic to* c_2*. Then the lifting of* c_2' c_1' *is homotopic to* c_1.

Proof of (i). By applying Lemma3.14.5 to $d\tau^{-1}$, there exists c_1 that satisfies the condition (i) in Lemma3.14.5. There exists $n \in \mathbb{N}$ such that

$$c_2(u)^{-1}c_2(v) \in V \ (\forall u, v \in [0,1] \text{ s.t } |u-v| < \frac{1}{n})$$

We set $t_j := \frac{j}{n}$ $(j = 0, 1, ..., n)$. Let us fix any $t \in [0,1]$. There exists j such that $t \in [t_j, t_{j+1}]$. Then

$$c_1(t_0)^{-1}c_1(t_1) = \Phi_o(c_2(t_0)^{-1}c_2(t_1)), ..., c_1(t_{j-1})^{-1}c_1(t_j) = \Phi_o(c_2(t_{j-1})^{-1}c_2(t_j)), c_1(t_j)^{-1}c_1(t) = \Phi_o(c_2(t_j)^{-1}c_2(t))$$

Since $c_1(t_0) = e$ and $c_2(t_0) = e$ and $\tau \circ \Phi_o = id$,

$$\tau(c_1(t_1)) = c_2(t_1), ..., \tau(c_1(t_{j-1})^{-1}c_1(t_j)) = c_2(t_{j-1})^{-1}c_2(t_j), \tau(c_1(t_j)^{-1}c_1(t)) = c_2(t_j)^{-1}c_2(t)$$

Then

$$\tau(c_1(t)) = c_2(c_2(t)$$

We found that c_1 in (i) can be induced by the curve in Lemma3.14.5. Therefore, from the uniqueness in (i) of Lemma3.14.5, the uniqueness holds. □

Proof of (ii). We found that c_1 in (i) can be induced by the curve in Lemma3.14.5. Therefore, from (ii) in Lemma3.14.5, (ii) holds. □

Theorem 3.14.7. *The universal covering group of any Lie group is simply connected.*

Proof. Let us fix any Lie group G. Let $\tilde{G}$ denote the universal covering of G and $q : \tilde{G} \to G$ denote the natural projection. Let us fix any curve $\tilde{c} : [0,1] \to \tilde{G}$. Then there is $n \in \mathbb{N}$ and $\epsilon > 0$ and $g_1, ..., g_n \in G$ and $0 = t_0 < s_1 < t_1 < ... < t_{n-1} < s_n < t_n = 1 \subset [0,1]$ such that

$$exp : Lie(G) \cap B(O, \epsilon) \to U$$

is diffeoomorphic and

$$q(\tilde{c}([0,t_1])) \subset g_1U, q(\tilde{c}([s_i, t_{i+1}])) \subset g_{i+1}U \ (i = 1, 2, ..., n)$$

and

$$q(\tilde{c}([s_i, t_i])) \subset g_iU \cap g_{i+1}U \ (i = 1, 2, ..., n-1)$$

From the definition of the system of neighborhood in $\tilde{G}$, for each $i \in \{1, 2, ..., n\}$, there is a continuous map $\Phi^i : [t_{i-1}, t_i] \times [0, 1] \to G$ such that

$$\tilde{c}(s) = \Phi^i(s, \cdot) \ (t_{i-1} \leq \forall s \leq t_i)$$

From the definition of the system of neighborhood in $\tilde{G}$, for each i, there is a continuous map $\Psi^i : [s_i, t_i] \times [0, 1] \times [0, 1] \to G$ such that for each $v \in [s_i, t_i]$ $\Psi^i(v, \cdot_1, \cdot_2)$ is a homotopy from $\Phi^i(v, \cdot)$ to $\Phi^{i+1}(v, \cdot)$.

We set

$$F(s,t) := \begin{cases} \Phi^i(t, 1-s) & t_{i-1} \leq t < s_i \\ \Psi^i(t, \dfrac{(t-s_i)}{t_i - s_i}, 1-s) & s_i \leq t < t_i \end{cases}$$

Clearly F is continuous and a homotopy from $q \circ \tilde{c}$ to $\{e\}$. From Theorem3.14.8, the lifting $\tilde{c}$ is homotopic to $\{e\}$. □

Theorem 3.14.8. *The followings are settings and assumptions.*

(S1) G_1 *is a connected and simply connected Lie group.*

(A1) G_2 *is a connected Lie group.*

(S2) $\phi : Lie(G_1) \to Lie(G_2)$ *is a homomorphism of Lie algebras.*

then

(i) There uniquely exists the continuous homomorphism Φ *of Lie groups such that*

$$d\Phi_e = \phi$$

(ii) In addition, let us assume ϕ *is surjective isomorphism. Then* $Ker\Phi$ *is isomorphic to the fundamental group of* G_2*.*

Proof of the existence in (i). Let us take over the notations in Lemma3.14.5. Let us fix $g \in G$. Let us fix any $c_1 : [0, 1] \to G$ such that $c_1(0) = e$ and $c_1(1) = g$. Let c_2 denote the curve in Lemma3.14.5. We set $\Phi(g) = c_2(1)$. From (ii) in Lemma3.14.5, Φ is well-defined. And, from Lemma3.14.5, Φ is locally homomorphism and continuous and $d\Phi_e = \phi$. From (3.14.1), if $g = g_1 g_2 ... g_n, g_i \in V (i = 1, 2, ..., n)$ then

$$\Phi(g) = \Phi(g_1)\Phi(g_2)...\Phi(g_n) \tag{3.14.4}$$

Since Φ is locally homomorphism, from the above equation, Φ is a homomorphism. □

Proof of the uniqueness in (i). From Theorem3.5.2, Φ is locally uniquely identified. From (3.14.4), Φ is uniquely identified. □

Proof of (ii). Remark that the natural projection $q : \tilde{G_2} \to G_2$ is a continuous homomorphism and dq_e is an isomorphism as Lie algebras and $Ker(q)$ is isomorphic to the fundamental group of G_2. From (i), Corollary3.14.9 holds. Let Ψ denote an isomorphism from G_1 to $\tilde{G_2}$. $q \circ \Phi$ is a continuous homomorphism and $dq_e \circ d\Phi_e$ is an isomorphism as Lie algebras. From the same argument as the proof of (i), $Ker(\Phi)$ is isomorphic to $Ker(q)$. □

Corollary 3.14.9. *The followings are settings and assumptions.*

(S1) G_1, G_2 *are simply connected Lie group.*

then G_1 *and* G_2 *are isomorphic with each other if and only if* $Lie(G_1)$ *and* $Lie(G_2)$ *are isomorphic with each other.*

Proof. It is clear from Theorem(i). □

Corollary 3.14.10. *The followings are settings and assumptions.*

(S1) G *is a simply connected Lie group.*

(S2) (π, V) *is a finite dimensional representation of* G*.*

then there uniquely exists the finite dimensional continuous representation of G (τ, V) *such that*

$$d\pi = \tau$$

Proof. It is clear from Theorem(i). □

Proposition 3.14.11. *The followings are settings and assumptions.*

(S1) G is a complex Lie group.

(S2) $\tilde{G}$ is the universal covering of G.

(S3) φ is the covering homomorphism from $\tilde{G}$ to G.

Then

(i) $\tilde{G}$ is a complex Lie group.

(ii) φ is holomorphic.

(iii) φ is locally biholomorphic.

Proof of (i). (i) is from Proposition3.13.10. □

Proof of (ii). Since $d\phi = id_{Lie(G)}$ is $\mathbb{C}$-linear, from Proposition3.13.12, ϕ is holomorphic. □

Proof of (iii). From Proposition3.14.1, φ is locally diffeomorphic. From (ii) and the inverse function theorem for several variable holomorphic mapping(See [17]), φ is locally biholomorphic. □

3.15 Compact Lie Group

T.B.D

Chapter 4

Irreducible Decomposition of Unitary Representation

4.1 Some Facts Admitted without Proofs

In this subsection, We will present some facts that we will use without proof in the pages that follow. For the following Proposition, see [4].

Proposition 4.1.1 (Shur LemmaII). *Let G be a topological group and (π, V) be an continuous irreducible representation of G and $A : V \to V$ be a continuous intertwining operator with respect to G such that $A \neq 0$. Then there is $\lambda \in \mathbb{C}$ such that $A = \lambda I$.*

Definition 4.1.2 (Extreme point). *Let*

(S1) V is a vector space on $\mathbb{C}$.

(S2) A is a convex set of V.

(S3) $x \in A$.

We say x is an extreme point of A if for any $y, z \in A$ and $\lambda \in [0, 1]$ such that $x = \lambda y + (1 - \lambda)z$ $x = y = z$. We denote the set of all extreme points of A by $Ex(A)$.

Definition 4.1.3 (Extreme set). *Let*

(S1) V is a vector space on $\mathbb{C}$.

(S2) A is a convex set of V.

(S3) $B \in A$.

We say B is an extreme set of A if for any $y, z \in A$ and $\lambda \in [0, 1]$ such that $x = \lambda y + (1 - \lambda)z \in B$ then $y, z \in B$.

For the following three Propositions, see [13].

Theorem 4.1.4 (S.Mazur Theorem). *Let*

(S1) $(V, \{p_n\}_{n\in\mathbb{N}})$ is a semi-normed space on $\mathbb{R}$.

(S2) $x_0 \in V$.

(S3) $A \subset V$ is a closed convex subset with $x_0 \notin A$.

Then there is real-valued continuous linear function f such that $f(x_0) = 1$ and $|f(x)| < 1$ $(\forall x \in A)$.

Proposition 4.1.5. *Let*

(S1) $(V, \{p_n\}_{n\in\mathbb{N}})$ is a semi-normed space.

(S2) f is a real-valued continuous linear functional on V.

(S3) K is a compact convex subset of V.

Then $\{x \in K | f(x) = \max\{f(x) | x \in K\}\}$ is an extreme set of K.

Proposition 4.1.6 (Krein-Millman Theorem). *Let*

(S1) $(V, \{p_n\}_{n\in\mathbb{N}})$ *is a semi-normed space.*

(S2) K *is a compact convex subset of* V*.*

(S3) $Ex(K)$ *is the set of all extreme ompact convex subset of* V*.*

Then

(i) $Ex(K)$ *is not empty.*

(ii) K *is the closure of the convex full of* $Ex(K)$*.*

Theorem 4.1.7 (Stone Weierstrass Theorem, lattice version)**.** *Let*

(S1) X *is a compact metric space.*

(S2) V *is a* $\mathbb{R}$*-vector subspace of* $C(X, \mathbb{R})$*.*

(A1) $\vee$ *means the pointwise maximum. Then* $f \vee g \in V$ $(\forall f, g \in V)$*.*

(A2) For any $x, y \in X$ *such that* $x \neq y$*, there is* $f \in V$ *such that* $f(x) \neq f(y)$*.*

Then V *is dense in* $C(X, \mathbb{R})$*.*

4.2 Continuity of representation

4.2.1 Uniform boundedness principle

Theorem 4.2.1 (Uniform boundedness principle)**.** *Let*

(S1) X *is a banach space.*

(S2) Y *is a normed space.*

(S3) $\{T_\lambda\}_{\lambda\in\Lambda} \subset B(X, Y)$*.*

(A1) For any $v \in X$*,* $\{||T_\lambda v||\}_{\lambda\in\Lambda}$ *is bounded.*

Then $\{||T_\lambda||\}_{\lambda\in\Lambda}$ *is bounded.*

Proof. We set $A_n := \{v \in X| \; ||T_\lambda v|| \leq n \; (\forall\lambda \in \Lambda)\}$ $(n \in \mathbb{N})$. $\{A_n\}_{n=1}^{\infty}$ satisfies the assumptions in Baire category thereom. By Baire category thereom, there is $n \in \mathbb{N}$ such that $A_n^{\circ} \neq \phi$. So there is $v_0 \in X$ and $\epsilon > 0$ such that $B(v_0, 2\epsilon) \subset A_n$. For any $\lambda \in \Lambda$ and $w \in X$ such that $||w|| = 1$,

$$||T_\lambda w|| = ||\frac{1}{\epsilon}T_\lambda(\epsilon w + v_0) - \frac{1}{\epsilon}T_\lambda v_0||$$

because $v_0, w + v_0 \in B(v_0, \epsilon)$

$$= ||\frac{1}{\epsilon}T_\lambda(\epsilon w + v_0) - \frac{1}{\epsilon}T_\lambda v_0|| \leq ||\frac{1}{\epsilon}T_\lambda(\epsilon w + v_0)|| + ||\frac{1}{\epsilon}T_\lambda v_0|| \leq \frac{n}{\epsilon} + \frac{n}{\epsilon} = \frac{2n}{\epsilon}$$

So, $||T_\lambda|| \leq \frac{2n}{\epsilon}$ $(\forall\lambda \in \Lambda)$ □

4.2.2 Weakly continuity of representation

Theorem 4.2.2. *Let*

(S1) G *is a local compact topological group.*

(S2) (π, V) *is a representation of* G*.*

(A1) For any $u \in V$*,* $G \ni g \mapsto \pi(g)u \in \mathbb{C}$ *is continuous.*

Then (π, V) *is continuous.*

Proof. Let us fix U_0 which is a local compact neighborhood of e. By (A1) and uniform boundedness principle,

$$\sup_{g\in U_0} ||\pi(g)|| < \infty$$

Let us fix any $\epsilon > 0$ and $g_0 \in G$ and $u_0 \in V$. By (A1), there is U_1 which is an open neighborhood of e such that $U_1 \subset U_0$

$$||\pi(g_0U_1)u_0 - u_0|| < \frac{\epsilon}{2}$$

So, for any $x \in U_1$ and $u \in B(u_0, \frac{\epsilon}{2(\sup_{g \in U_0} ||\pi(g)|| + 1)})(||\pi(g_0)|| + 1)$,

$$||\pi(g_0 x)u - \pi(g_0)u_0|| \leq ||\pi(g_0 x)u - \pi(g_0 x)u_0|| + ||\pi(g_0 x)u_0 - \pi(g_0)u_0|| < ||\pi(g_0)||_{op}||\pi(x)||_{op}||u - u_0|| + \frac{\epsilon}{2} < \epsilon$$

□

In speciality, the following holds. However, this theorem can be proved without using Theorem4.2.2. The proof is given below.

Theorem 4.2.3. *Let*

(S1) G is a topological group.

(S2) (π, V) is a unitary representation of G.

(A1) For any $u, v \in V$, $G \ni g \mapsto (\pi(g)u, v) \in \mathbb{C}$ is continuous.

Then (π, V) is continuous.

Proof. Let us fix any $u \in V$ and $g \in G$. Let us fix any $v \in B(u, \frac{\epsilon}{12(2||u|| + 1)})$. There is U which is an open neighborhood of e such that

$$|(\pi(g^{-1}h)u, u) - ||u||^2| \leq \frac{\epsilon}{2}$$

By (S2), for any $h \in gU$ and $v \in B(u, \frac{\epsilon}{2(||u|| + 1)})$,

$$\begin{aligned}
&||\pi(h)u - \pi(g)v||^2 = ||u||^2 - 2Re(\pi(g^{-1}h)u, v) + ||v||^2 = ||u||^2 - 2Re(u, v) + ||v||^2 + 2Re(u, v) - 2Re(\pi(g^{-1}h)u, v) \\
&= ||u - v||^2 + 2Re(u - \pi(g^{-1}h)u, v) = ||u - v||^2 + 2Re(u - \pi(g^{-1}h)u, u) + 2Re(u - \pi(g^{-1}h)u, v - u) \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + 2||u - \pi(g^{-1}h)u||||v - u|| \leq \frac{2\epsilon}{3} + 2(||u|| + ||\pi(g^{-1}h)u||)||u - v|| = \frac{2\epsilon}{3} + 2(||u|| + ||u||)||u - v|| \\
&= \frac{2\epsilon}{3} + 4||u||||u - v|| \leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

So, (π, V) is continuous. □

4.3 Cyclic representation and Unitary dual

4.3.1 Definition and Basic Properties

Definition 4.3.1 (Cyclic representation)**.** *Let G be a topological group and (π, V) be a continuous representation of G. We say (π, V) is a cyclic representation of G if there is $v \in V$ such that*

$$\overline{\{\sum_{i=1}^{N} \pi(g_i)v | g_1, ..., g_N \in G\}} = V$$

Clearly the following holds.

Proposition 4.3.2. *Let G be a topological group. Any continuous irreducible representation of G is a cyclic representation.*

By Proposition3.4.6, the following holds.

Proposition 4.3.3. *Let G be a Lie group and (π, V) be a continuous cyclic representation of G. Then V is countable. In speaciality, if π is unitary representation and $dim\pi = \infty$, then $V \simeq l^2$ as Hilbert space.*

By Proposition4.3.3, we can set of all continuous irreducible unitary representations of a Lie group.

Notation 4.3.4. *Let G be a Lie group. We set*

$$\Omega_c := \{(\pi, V) |\ V \text{ is closed subspace of } l^2 \text{ and } (\pi, V) \text{ is a continuous cyclic representation of } G\}$$

Definition 4.3.5 (Unitary dual)**.** *Let G be a Lie group. We set*

$$\hat{G} := \{(\pi, V) |\ V \text{ is closed subspace of } l^2 \text{ and } (\pi, V) \text{ is a continuous irreducible representation of } G\} / \simeq$$

Here, $\simeq$ is the isomorphic relation as continuous representations. We call $\hat{G}$ the unitary dual of G.

The following Proposition says that in the definition of unitary dual, the isomorphicness as continuous representation is equal to the isomorphicness as continuous unitary representation.

Proposition 4.3.6. *The followings are settings and assumptions.*

(S1) G is a Lie group.

(S2) (π, V) is a continuous irreducible representation of G.

(S3) P_1, P_2 are inner products on V such that (π, V) is a continuous unitary representation with P_i $(i = 1, 2)$.

Then there is $\lambda > 0$ such that

$$P_1 = \lambda P_2$$

Proof. From Riez Representation Theorem, for each $v \in V$, there uniquely exists $\phi(v) \in V$ such that $P_1(\cdot, v) = P_2(\cdot, \phi(v))$. For any $v_1, v_2 \in V$ and $g \in G$ and $c_1, c_2 \in \mathbb{C}$,

$$\begin{aligned} & P_1(\cdot, \pi(g)(c_1v_1 + c_2v_2)) = \bar{c_1}P_1(\pi(g^{-1})\cdot, v_1) + \bar{c_2}P_1(\pi(g^{-1})\cdot, v_2) = \bar{c_1}P_2(\pi(g^{-1})\cdot, \phi(v_1)) + \bar{c_2}P_2(\pi(g^{-1})\cdot, \phi(v_2)) \\ = \; & P_2(\cdot, \pi(g)(c_1\phi(v_1) + c_2\phi(v_2))) \end{aligned}$$

This means that

$$\phi(\pi(g)(c_1v_1 + c_2v_2)) = \pi(g)(c_1\phi(v_1) + c_2\phi(v_2))$$

Therefore, ϕ is G-linear.

We will show that ϕ is a closed operator. Let us fix $\{u_n\}_{n=1}^{\infty} \subset V$ such that $\{(u_n, \phi(u_n))\}_{n=1}^{\infty}$ is a cauchy sequence. Then there are $u, v \in V$ such that

$$\lim_{n\to\infty} u_n = u, \lim_{n\to\infty} \phi(u_n) = v$$

Then for any $w \in V$ and $n \in \mathbb{N}$,

$$P_1(w, u_n) = P_2(w, \phi(u_n))$$

That implies that

$$P_1(w, u) = P_2(w, v)$$

Therefore, $\phi(u) = v$. Consequently, ϕ is a closed operator. From the Closed Map Theorem, ϕ is continuous.

From the Shur Lemma II, there is $\lambda \in \mathbb{C}$ such that $\phi = \lambda id_V$. Since P_1 and P_2 are positive definite, $\lambda > 0$. □

Proposition 4.3.7. *The followings are settings and assumptions.*

(S1) G is a Lie group.

(S2) (π_i, V_i) is a continuous unitary cyclic representation of G with cyclic vector v_i such that $||v_i|| = 1$ $(i = 1, 2)$.

(A1) $(\pi_1(g)v_1, v_1) = (\pi_2(g)v_2, v_2)$ $(\forall g \in G)$.

Then (π_1, V_1) and (π_2, V_2) are isomorphic as continuous unitary representation of G.

STEP1. Construction of orthonormal basis of V_1. Let $\{g_i\}_{i=1}^{\infty}$ is a dense subset of G. We set $\{h_i\}_{i=1}^{\infty}$ is a subgroup of G generated by $\{g_i\}_{i=1}^{\infty}$. There is a $\{f_i\}_{i=1}^{\infty} \subset \{h_i\}_{i=1}^{\infty}$ such that $\{\pi_1(f_i)v_1\}_{i=1}^{\infty}$ is a basis of the vector space W_1 which is generated by $\{\pi_1(h_i)v_1\}_{i=1}^{\infty}$. We take $\{w_i\}_{i=1}^{\infty}$ which is the orthonormal basis of W_1 by Gram-Schmit orthogonalization.

At the end of this step, we will show $\{\pi_2(f_i)v_2\}_{i=1}^{\infty}$ is a basis of the vector space W_2 which is generated by $\{\pi_2(h_i)v_2\}_{i=1}^{\infty}$. For showing this proposition, it is enough to show for each $a_1, ..., a_N \in \mathbb{C}$

$$\sum_{i=1}^{N} a_i\pi_1(f_i)v_1 = 0 \iff \sum_{i=1}^{N} a_i\pi_2(f_i)v_2 = 0 \tag{4.3.1}$$

Because of (S_2) and (A_1),

$$\begin{aligned} & \sum_{i=1}^{N} a_i\pi_1(f_i)v_1 = 0 \iff (\sum_{i=1}^{N} a_i\pi_1(f_i)v_1, \pi_1(g)v_1) = 0 \ (\forall g \in G) \iff \sum_{i=1}^{N} a_i(\pi_1(g^{-1}f_i)v_1, v_1) = 0 \ (\forall g \in G) \\ \iff & \sum_{i=1}^{N} a_i(\pi_2(g^{-1}f_i)v_2, v_2) = 0 \ (\forall g \in G) \iff (\sum_{i=1}^{N} a_i\pi_2(f_i)v_1, \pi_2(g)v_1) = 0 \ (\forall g \in G) \iff \sum_{i=1}^{N} a_i\pi_2(f_i)v_2 = 0 \end{aligned}$$

So, (4.3.1) holds. □

STEP2. Construction of orthonormal basis of V_2. By (A1), clearly

$$||\sum_{i=1}^{N} a_i\pi_1(f_i)v_1||_{V_1} = ||\sum_{i=1}^{N} a_i\pi_2(f_i)v_2||_{V_2} \ (\forall a_1, ..., a_N \in \mathbb{C}) \tag{4.3.2}$$

We set, for each $w_i = \sum_{j=1}^{N_i} a_{i,j}\pi_1(f_j)v_1$,

$$w_i' := \sum_{j=1}^{N_i} a_{i,j}\pi_2(f_j)v_2$$

We will show $\{w_i'\}_{i=1}^{\infty}$ is an orthonormal basis of V_2. By (A_1), $\{w_i'\}_{i=1}^{\infty}$ is clearly orthonormal. Let us fix any $k \in \mathbb{N}$. Then there are $a_1, ..., a_N \in \mathbb{C}$ such that

$$\pi_1(f_k)v_1 = \sum_{i=1}^{N} a_i w_i$$

Because $w_i \in W_1$, by (4.3.1),

$$\pi_2(f_k)v_2 = \sum_{i=1}^{N} a_i w_i'$$

So, $\{w_i'\}_{i=1}^{\infty}$ is an orthonormal basis of V_2. □

STEP3. Construction of isomorphism. We set

$$\Phi(\sum_{i=1}^{N} a_i w_i) := \sum_{i=1}^{N} a_i w_i' \ (a_1, ..., a_N \in \mathbb{C})$$

Clearly Φ is an unitary isomorphism between Hilbert spaces. We will show Φ is G-linear. Because $w_1 = v_1$ and $w_1' = v_2$,

$$\Phi(v_1) = v_2$$

Let us fix any $i \in \mathbb{N}$. Then there are $a_1, ..., a_n \in \mathbb{N}$ such that

$$\pi_1(g_i)v_1 = \sum_{j=1}^{n} a_j w_j$$

Because $w_i \in W_1$, by (4.3.1),

$$\pi_2(g_i)v_2 = \sum_{j=1}^{n} a_j w_j'$$

So,

$$\Phi(\pi_1(g_i)v_1) = \pi_2(g_i)\Phi(v_1)$$

Because W_1 is dense in V_1 and Φ is unitary, Φ is G-linear. □

Proposition 4.3.8. *Let (π, V) be a continuous unitary representation of a topological groupG. Then there is a subset of G-invariant cyclic subspaces D such that*

$$V = \overline{\bigoplus_{W \in D} W}$$

Proof. We denote the all of nonzero invariant closed cyclic subspaces by $\mathfrak{D}$. Clearly $\mathfrak{D} \neq \phi$. We set

$$\mathfrak{T} := \{D \subset \mathfrak{D} | v_i \in W_i (i = 1, 2, ..., N), \{W_i\}_{i=1}^{N} \text{ is a distinct subset of } D, \sum_{i=1}^{N} v_i = 0 \implies v_i = 0 \ (\forall i)\}$$

Let us fix any every totally ordered subset of $\mathfrak{T}$, T. Clearly $\cup_{D\in T} D \in \mathfrak{T}$. So, by Zorn's lemma, $\mathfrak{T}$ has a maximum element D. We set $V_0 := \bigoplus_{W\in D} W$. Let us assume $V_0^{\perp}$ is nonzero. Then $V_0^{\perp}$ has a nonzero invariant closed cyclic subspace W. Clearly, $D \cup \{W\} \in \mathfrak{T}$. This contradicts that D is a maximum element. So, $V_0^{\perp} = \{0\}$ and $V = \bar{V_0}$. □

4.3.2 Cyclic Representation and Jordan Normal Form

The content of this section is independent of the content that follows. In this subsection, we pointed out that for each square matrix A a Jordan block corresponds to a cyclic representation $\langle \pi_A(\mathbb{Z})z \rangle$ $(\exists z)$ when A is regular. Here, for each $m \in \mathbb{Z}$ $\pi_A(m)z := A^m z$. When A is not regular, a Jordan block corresponds to a space generated by 'orbit' $\langle \pi_A(\mathbb{N})z \rangle$ $(\exists z)$. And we show we can get the Jordan normal form of A by repeating that we get a good space generated by an 'orbit' and add it to an existing direct sums of those spaces, by the same argument as the proof of Proposition4.3.8.

Notation 4.3.9. *The followings are settings and assumptions.*

(S1) A is a squared matrix of order n.

Then

$$\pi_A(m)z := A^m z (m \in \mathbb{N}, z \in \mathbb{C}^n)$$

And if A is regular, we define

$$\pi_A(m)z := A^m z (m \in \mathbb{Z}, z \in \mathbb{C}^n)$$

When A is regular, $(\pi_A, \mathbb{C}^n)$ is a continuous representation of $\mathbb{Z}$. Since $\mathbb{Z}_{\geq 0}$ is a monoid, in general(not assuming that A is regular), $(\pi_A, \mathbb{C}^n)$ is a continuous representation of the monoid $\mathbb{Z}_{\geq 0}$.

Proposition 4.3.10. *The followings are settings and assumptions.*

(A1) A is a jordan block $J_n(\alpha)$.

Then for $w_1 := e_n$

$$\mathbb{C}^n = \langle \pi_A(\mathbb{Z}_{\geq 0})w_1 \rangle$$

In speacialty, when $\alpha \neq 0$, $\pi_{J_n(\alpha)}$ is a cyclic representation of $\mathbb{Z}$ with a cyclic vector e_n.

From Proposition4.3.10, we can see that ***Jordan normal form of a regular matrix is a decomposition of the representation of the group $\mathbb{Z}$ π_A into cyclic representations of $\mathbb{Z}$***. In general(not assuming that A is regular), ***Jordan normal form of a regular matrix is a decomposition of the representation of the monoid $\mathbb{Z}_{\geq 0}$ π_A into representations $\{(\pi_i, V_i)\}_{i=1}^m$ of $\mathbb{Z}_{\geq 0}$ that statisfies the following condition***.

$$\exists z \in V_i \text{ s.t } V_i = \langle \mathbb{Z}_{\geq 0} \cdot z \rangle$$

From the point of view, we give another proof of exisense of Jordan normal form.

Definition 4.3.11. *The followings are settings and assumptions.*

(S1) V is a $\mathbb{C}$ vector space.

(S2) $f \in End_{\mathbb{C}}(V)$.

$$d_f(z) := \dim \langle \pi_A(\mathbb{Z}_{\geq 0})z \rangle \ (z \in V)$$

and

$$d_f := \max\{d_f(z) | z \in V\}$$

Proposition 4.3.12. *The followings are settings and assumptions.*

(S1) V is a $\mathbb{C}$ vector space.

(S2) $f \in End_{\mathbb{C}}(V)$.

(S3) $z \in V$.

Then $z, Az, ..., A^{d_f(z)-1}z$ are linear independent.

Proof. We set $m := d_f(z)$. And aiming contradiction, let us assume $z, Az, ..., A^{m-1}z$ are linear dependent. Then there are $m' \leq m-1$ and $a_0, ..., a_{m'-1}$ such that

$$A^{m'}z = a_0 z + a_1 Az + ... + a_{m'-1}A^{m'-1}z$$

That implies that $\dim\langle a_0 z, ..., a_{m'-1}A^{m'-1}z \rangle \leq m-1$. This is contradiction. □

Proposition 4.3.13. *The followings are settings and assumptions.*

(S1) $A \in M(n, \mathbb{C})$.

(S2) $P \in GL(n, \mathbb{C})$.

Then

$$d_A = d_{PAP^{-1}}$$

Proof. There is $z \in \mathbb{C}^n$ such that $d_A = \dim\langle \pi_A(\mathbb{Z}_{\geq 0})z\rangle$. From Proposition4.3.12, $z, Az, ..., A^{m-1}z$ are linear independent. So, $Pz, PAz = PAP^{-1}Pz, ..., PA^{m-1} = (PAP^{-1})^{m-1}Pz$ are linear independent. Therefore, $d_{PAP^{-1}} \leq d_A$. □

Theorem 4.3.14. *The followings are settings and assumptions.*

(S1) $A \in M(n, \mathbb{C})$

then the followings hold.

(i) There is $P \in GL(n, \mathbb{C})$ *and* $\alpha_1, ..., \alpha_K \in \mathbb{C}$ *such that*

$$P^{-1}AP = \begin{pmatrix} J(\alpha_1) & O & ... & O \\ O & J(\alpha_2) & ... & O \\ ... & ... & ... & ... \\ O & & ... & J(\alpha_K) \end{pmatrix}$$

Here, for each i*, there are* $j_1, ..., j_{n_i}$ *such that*

$$J(\alpha_i) = \begin{pmatrix} J_1(\alpha_i) & O & ... & O \\ O & J_2(\alpha_i) & ... & O \\ ... & ... & ... & ... \\ O & & ... & J_{n_i}(\alpha_i) \end{pmatrix}$$

and $J_k(\alpha_i)$ *is a* j_k*-th square matrix*

$$J_k(\alpha_i) = \begin{pmatrix} \alpha_i & 1 & & & O \\ O & \alpha_i & 1 & & O \\ ... & ... & ... & ... & ... \\ O & & & \alpha_i & 1 \\ O & & & & \alpha_i \end{pmatrix}$$

We call $J_k(\alpha_i)$ *is a Jordan block.*

(ii) For any W_1 *and* W_2 *such that* W_1 *and* W_2 *are* $J_k(\alpha_i)$*-invariant subspaces and* $\mathbb{C}^\nu = W_1 \oplus W_2$*,* $W_1 = \mathbb{C}^\nu$ *or* $W_2 = \mathbb{C}^\nu$*.*

Proof of (i). We set $V := \mathbb{C}^n$. Let $\{\alpha_1, ..., \alpha_K\}$ be the set of all eigenvalues of A. We set for each $i \in \{1, 2, ..., K\}$

$$d_A(z) := \dim\langle \pi_A(\mathbb{Z}_{\geq 0})z\rangle \ (z \in V)$$

and

$$V_i := \{z \in V | \langle \pi_A(\mathbb{Z}_{\geq 0})z\rangle \text{ has just one eigenvalue } \alpha_i\}$$

and

$$M_i := \max\{d_A(z) | z \in V_i\}$$

and

$$V_{i,M_i} := \{z \in V_i | \dim\langle \pi_A(\mathbb{Z}_{\geq 0})z\rangle = M_i\}$$

Remark $V_i \neq \phi$ and $M_i < \infty$ for any $i \in \mathbb{N}$ since each eigenvalue space is not empty.

First, we will show

$$V_i = W_{i,M_i} := \{z \in V | (A - \alpha E)^{M_i} z = 0\} \tag{4.3.3}$$

Let us fix any $z \in V_i$. By Proposition4.3.12, Then there are $a_0, ..., a_{d_A(z)-1}$ such that

$$f(A)z = 0, f(x) := a_0 + a_1 x + ... + a_{d_A(z)-1} x^{d_A(z)-1} + x^{d_A(z)}$$

If f has a root which is not equal to α, that is contradict that $z \in V_i$. So, $f(x) = (x-\alpha)^m$. This means that $(A-\alpha E)^m z = 0$.

Next, for aiming contradiction, let us assume that there exists $z \in W_{i,M_i} \setminus V_i$. So there exists polynomial f such that $f(A)z \neq 0$ and $f(A)z$ has an eigenvalue $\alpha_j \neq \alpha_i$. From Proposition2.1.3, there are polynomial ϕ_1, ϕ_2 such that

$$\phi_1(x)(x - \alpha_j) + \phi_2(x)(x - \alpha_i)^{M_i} = 1$$

So,

$$0 \neq f(A)z = \phi_1(A)(x-\alpha_j)f(A)z + \phi_2(x)(x-\alpha_i)^{M_i}f(A)z = \phi_2(A)f(A)(x-\alpha_i)^{M_i}z = 0$$

That is contradiction. Therefore, (4.3.3) holds. From the same argument,

$$\cup_{k\leq m}V_{i,k} = W_{i,m} \ (m = 1, 2, ..., M_i) \tag{4.3.4}$$

So,

$$V_{i,m} = W_{i,m} \setminus W_{i,m-1} \ (m = 1, 2, ..., M_i) \tag{4.3.5}$$

We set

$$\mathcal{V}_{i,M_i} := \{\{z_1, ...z_m\} \subset V_{i,M_i} | \text{ For any } a \in \mathbb{C}^m \setminus \{0\}, \sum_{i=1}^{m} a_i z_i \in V_{i,M_i}\}, K_i := \max\{m|\{z_1, ...z_m\} \in \mathcal{V}_{i,M_i}\}$$

and we take

$$\{z_{M_i,1}, ..., z_{M_i,K_i}\} \in \mathcal{V}_{i,M_i}$$

By (4.3.4),

$$\{(A-\alpha_i)z_{M_i,1}, ..., (A-\alpha_i)z_{M_i,K_i}\} \in \mathcal{V}_{i,M_i-1}$$

Next, we will show

$$V = \oplus_{i=1}^{K} V_i$$

Let us fix any $z \in V$. Then $\dim\langle \mathbb{Z}_{\geq 0} z\rangle \leq n$. Therefore, there are $\alpha_{\varphi(1)}, ..., \alpha_{\varphi(L)} \subset \{\alpha_1, ..., \alpha_K\}$ such that $\Pi_{i=1}^{L}(A - \alpha_{\varphi(i)})^{m_i} z = 0$. From Proposition2.1.3, there are polynomial $f_1, ..., f_L$ such that

$$\sum_i (z - \alpha_{\varphi(i)})^{m_i} f_i = E$$

This implies

$$z = \sum_i \Pi_{j\neq i}(A - \alpha_{\varphi(j)})^{m_j} f_i(A) z$$

Since $\Pi_{j\neq i}(A-\alpha_{\varphi(j)})^{m_j} f_i(A)z \in W_{\varphi(j),m_j} = V_{\varphi(j)}$, $V =$
$sum_i V_i$ holds. By the same argument as Section2, we can show $\sum_i V_i = \oplus_i V_i$.

Hereafter, by the same argument as Section2, the Theorem holds. □

Proof of (ii). For aiming contradiction, there are $J_k(\alpha)$ invarinat subspace W_1 and W_2 such that

$$\mathbb{C}^\nu = W_1 \oplus W_2$$

By decomposing $J_k(\alpha)|W_1$ and $J_k(\alpha)|W_2$ into Jordan normal forms, $d(J_k(\alpha)) < k$. It is contradict with Proposition4.3.13. □

4.4 $*$-weak topology of $L^1(G)$

Definition 4.4.1 ($*$-weak topology)**.** *Let V be a normed space. We denote the weakest topology in which for any $x \in V$ $V^* \ni f \mapsto f(x) \in \mathbb{C}$ is continuous by $\mathcal{O}_w(V^*)$. We call this topology $*$-weak topology of V^*.*

Clearly the following two propositions holds.

Proposition 4.4.2. *Let V be a normed space. $\mathcal{O}_w(V^*)$ is induced by the family of seminorms $\{\cdot(x)\}_{x\in V}$.*

Proposition 4.4.3. *Let V be a separable normed space and $\{x_n\}_{n\in\mathbb{N}}$ be a dense subset of V. Then*

$$d : V^* \times V^* \ni (f, g) \mapsto \sum_{n=1}^{\infty} \frac{|f(x_n) - g(x_n)|}{1 + |f(x_n) - g(x_n)|} \in [0, \infty)$$

is a metric on V^* and $\mathcal{O}_w(V^*)$ is induced by d.

Theorem 4.4.4 (Banach-Alaoglu theorem)**.** *Let V be a separable normed space and $\{x_n\}_{n\in\mathbb{N}}$ be a dense subset of V. Then $B := \{f \in V^* | ||f|| \leq 1\}$ is a compact subset in $\mathcal{O}_w(V^*)$.*

Proof. Because $(V^*, \mathcal{O}_w)$ is metrizable, it is enough to show $(V^*, \mathcal{O}_w)$ is sequencial compact. Let us fix any $\{f_n\}_{n\in\mathbb{N}} \subset B$. By the same argument as the proof of Proposition2.5.20, there is a subsequence $\{g_n\}_{n\mathbb{N}} = \{f_\varphi(n)\}_{n\in\mathbb{N}}$ such that for any $i \in \mathbb{N}$ $\lim_{n\to\infty} g_n(x_i)$ exists.

Let us fix $x \in V$ and $\epsilon > 0$. Let us fix x_i such that $||x - x_i|| < \frac{\epsilon}{3}$. Because $\{g_n(x_i)\}_{n\in\mathbb{N}}$ is a cauchy sequence, there is $n_0 \in \mathbb{N}$ such that $|g_n(x_i) - g_m(x_i)| < \frac{\epsilon}{3}$ $(\forall m, n \geq n_0)$. Then for any $m, n \geq n_0$

$$|g_m(x) - g_n(x)| \leq |g_m(x) - g_m(x_i)| + |g_m(x_i) - g_n(x_i)| + |g_n(x) - g_n(x_i)| \leq 2||x -_x i|| + \frac{\epsilon}{3} \leq \epsilon$$

So $\{g_n(x)\}_{n\in\mathbb{N}}$ is a cauchy sequence. This implies $\lim_{n\to\infty} g_n(x)$ exists. We set

$$g(x) := \lim_{n\to\infty} g_n(x) \ (x \in V)$$

Cearly $||g|| \leq 1$ and $w - \lim_{n\to\infty} g_n = g$. □

4.5 Positive definite function on a group

4.5.1 Definition and Basic properties

Definition 4.5.1 (Positive definite function on a group)**.** *Let G be a group and $\varphi \in C(G, \mathbb{C})$. We say φ is positive definite if for any $n \in \mathbb{C}$ and $g_1, g_2, ..., g_n \in G$ and $c_1, c_2, ..., c_n \in \mathbb{C}$*

$$\sum_{j,k} c_j \bar{c_k} \varphi(g_j^{-1} g_k) \geq 0 \tag{4.5.1}$$

Example 4.5.2. *Let G be a group and (π, V) be a unitary representation of G and $v \in V$. Then the following is a positive definite function.*

$$(\pi(\cdot)v, v) \tag{4.5.2}$$

Proof. For any $n \in \mathbb{C}$ and $g_1, g_2, ..., g_n \in G$ and $c_1, c_2, ..., c_n \in \mathbb{C}$

$$\sum_{j,k} c_j \bar{c_k} (\pi(g_j^{-1} g_k)v, v) = \sum_{j,k} c_j \bar{c_k} (\pi(g_k)v, \pi(g_j)v) = (\sum_k \bar{c_k}\pi(g_k)v, \sum_j \bar{c_j}\pi(g_j)v) = ||\sum_k \bar{c_k}\pi(g_k)v||^2 \geq 0$$

□

Proposition 4.5.3. *Let G be a group and φ is a positive definite function on G. Then*

(i) $\varphi(e) \geq 0$

(ii) $\varphi(g^{-1}) = \overline{\varphi(g)}$

(iii) $|\varphi(g)| \leq \varphi(e)$

(iv) $|\varphi(g_1) - \varphi(g_2)|^2 \leq \frac{1}{2}\varphi(e)|\varphi(e) - Re\varphi(g_1^{-1}g_2)|$

Proof of (i). We succeed in the notation of Definition4.5.2. By setting $n = 1$ and $g_1 = e$ and $c_1 =$, (i) holds. □

Proof of (ii). By setting $n = 2$ and $g_1 = e$ and $g_2 = g$ and $c_1 = 1$ and $c_2 = a$,

$$(1 + |a|^2)\varphi(e) + a\varphi(g) + \bar{a}\varphi(g^{-1}) \geq 0$$

By setting $a = 1$,

$$Im\varphi(g) = -Im\varphi(g^{-1})$$

By setting $a = i$,

$$Re\varphi(g) = Re\varphi(g^{-1})$$

So, (ii) holds. □

Proof of (iii). By the above proof of (ii),

$$(1 + |a|^2)\varphi(e) \geq -2Re(a\varphi(g))$$

By setting $a = -exp(-iarg(a))$,

$$2\varphi(e) \geq 2|\varphi(g)|$$

So, (iii) holds. □

Proof of (iv). We set $n = 3$, $c_3 = 1$, $g_3 = 3$ in (). Then we get

$$0 \leq c_1\bar{c_2}\varphi(g_1g_2^{-1}) + c_2\bar{c_1}\varphi(g_2g_1^{-1}) + c_1\varphi(g_1) + c_2\varphi(g_2) + \bar{c_1}\varphi(g_1^{-1}) + \bar{c_2}\varphi(g_2^{-1}) + \varphi(e) + |c_1|^2\varphi(e) + |c_2|^2\varphi(e)$$

By (ii),

$$0 \leq 2Re(c_1\bar{c_2}\varphi(g_1g_2^{-1})) + 2Re(c_1\varphi(g_1) + c_2\varphi(g_2)) + \varphi(e) + |c_1|^2\varphi(e) + |c_2|^2\varphi(e)$$

Moreover, we set $c_1 = -c_2 = \alpha$. Then

$$\begin{aligned} 0 &\leq -2|\alpha|^2 Re(\varphi(g_1g_2^{-1})) + 2Re(\alpha(\varphi(g_1) - \varphi(g_2)) + \varphi(e) + 2|\alpha|^2\varphi(e) \\ &= 2|\alpha|^2(\varphi(e) - Re(\varphi(g_1g_2^{-1}))) + 2Re(\alpha(\varphi(g_1) - \varphi(g_2)) + \varphi(e) \end{aligned}$$

We can assume $\varphi(g_1) \neq \varphi(g_2)$. We set $\alpha = -\varphi(e)\dfrac{\overline{\varphi(g_1) - \varphi(g_2)}}{2|\varphi(g_1) - \varphi(g_2)|^2}$. Then $2Re(\alpha(\varphi(g_1) - \varphi(g_2)) + \varphi(e) = 0$ and $2|\alpha|^2(\varphi(e) - Re(\varphi(g_1g_2^{-1}))) = \dfrac{\varphi(e)(\varphi(e) - Re(\varphi(g_1g_2^{-1})))}{2|\varphi(g_1) - \varphi(g_2)|^2}$. So, we get (iv). □

The following is clear.

Proposition 4.5.4. *Let G be a group and φ is a positive definite function on G. Then*

(i) φ_1, φ_2 are positive definite functions on G and α_1, α_2 are positive numbers. Then $\alpha_1\varphi_1 + \alpha_2\varphi_2$ is a positive definite function on G.

(ii) We set

$$\mathbb{P}_1 := \{\varphi|\varphi \text{ is a continuous positive definite function on } G \text{ such that } \varphi(e) = 1\}$$

and

$$\mathbb{P}_0 := \{\varphi|\varphi \text{ is a continuous positive definite function on } G \text{ such that } \varphi(e) \leq 1\}$$

and

$$\mathbb{P} := \{\varphi|\varphi \text{ is a continuous positive definite function on } G\ \}$$

Then $\mathbb{P}_1$ and $\mathbb{P}_2$ and $\mathbb{P}$ are convex.

Theorem 4.5.5 (Schur product theorem)**.** *Let $M := \{m_{i,j}\}_{i,j}$ and $N := \{n_{i,j}\}_{i,j}$ be nonnegative definite m-th Hermitian matrices. Then $M \circ N := \{m_{i,j}n_{i,j}\}_{i,j}$ is nonnegative definite. We call $M \circ N$ the Hadamard product of M and N.*

Proof. There are $A := \{a_{i,j}\}_{i,j}$ and $A := \{b_{i,j}\}_{i,j}$ such that

$$M = A^*A,\ N = B^*B$$

This means

$$m_{i,j} = \sum_{i=1}^{m} \bar{a_{i,j}}a_{i,k}, n_{i,j} = \sum_{l=1}^{m} \bar{b_{l,j}}b_{l,k}$$

So,

$$m_{i,j}n_{i,j} = \sum_{i,l=1}^{m} a_{i,k}b_{l,k}\bar{a_{i,j}}\bar{b_{l,j}}$$

For each i, l, we set the $(m, 1)$-matrix $v_{i,l}$ by

$$v_{i,l} = {}^t(a_{i,1}b_{i,1}, ..., a_{i,m}b_{i,m})$$

Then $v_{i,l}v_{i,l}^*$ is a m-th nonnegative definite Hermite matrix and

$$M \circ N = \sum_{i,l} v_{i,l}v_{i,l}^*$$

So, $M \circ N$ is nonnegative definite. □

Proposition 4.5.6. *Let φ_1, φ_2 are positive definite functions on a group G. Then $\varphi_1\varphi_2$ is a positive definite function on a group G.*

Proof. Let us fix any $g_1, ..., g_m \in G$. By Proposition4.5.3, $\{(\varphi_1\varphi_2)(g_i^{-1}g_j)\}_{i,j}$ is an Hermite matrix. By Theorem4.5.5, $\{(\varphi_1\varphi_2)(g_i^{-1}g_j)\}_{i,j}$ is nonnegative definite. So, $\varphi_1\varphi_2$ is a positive definite function on a group G. □

4.5.2 GNS construction for unitary representation

We introduce the following notation.

Notation 4.5.7. *Let G be a Lie group and $f \in C(G)$. Then*

$$f^*(x) := \Delta_R(x)\overline{f(x^{-1})}\ (x \in G)$$

Clearly the following holds.

Proposition 4.5.8. *Let G be a Lie group and $f \in C(G)$.*

(i) $f^* \in C(G)$.

(ii) $f^{**} = f$.

Theorem 4.5.9 (GNS construction)**.** *Let G is a Lie group.*

(S1) G *is a Lie group.*

(S2) φ *is a continuous positive definite function on* G.

(S3) We set $(f,g) := \varphi * f * g^*(e)\ f,g \in C_c(G)$.

(S4) We set $\mathcal{H}_0 := C_c(G) \setminus N$. *Here,* $N := \{f \in C_c(G) | ||f|| = 0\}$.

(S5) $T_g[f] := [f(\cdot g)]\ ([f] \in \mathcal{H}_0, g \in G)$

Then

(i) $(f,g) = \int_G \varphi(x^{-1}y)\overline{f^*(y)}g^*(x)dx_R dy_R = \int_G \varphi(xy^{-1})f(y)\overline{g(x)}dx_R dy_R = \int_G \overline{\varphi(xy^{-1})}f(x)\overline{g(y)}dx_R dy_R$

(ii) $\mathcal{H}_0$ *is a pre-Hilbert space.*

(iii) T *is well-defined continuous unitary representation on* $\mathcal{H}_0$ *of* G.

(iv) We set $\mathcal{H}$ *be the completion of* $\mathcal{H}_0$. *Then* T *is well-defined continuous unitary representation on* $\mathcal{H}$ *of* G.

(iv) $\mathcal{H}$ *is separable.*

(v) Let us assume $\{f_n\}_{n\in\mathbb{N}} \subset C_c(G)$ *and* $f \in C_c(G)$ *and* $sup_{n\in\mathbb{N}}||f_n||_\infty < \infty$ *and* $\lim_{n\to\infty} f_n = f$ *(pointwise convergense). Then* $\lim_{n\to\infty} ||f_n - f|| = 0$.

(vi) $||f|| \leq \sup_{x,y\in supp(f)} |\varphi(xy^{-1})|^{\frac{1}{2}} ||f||_{L^1(G)}\ (\forall f \in C_c(G))$

(vii) $(\mathcal{H}, T)$ *is cyclic.*

(viii) $\varphi(g) = (T_g v, v)\ (\forall g \in G)$.

(ix) If $\varphi(\cdot) = (\pi(\cdot)u, u)$ *for* (π, V) *which is a continuous cyclic unitary representation of* G *with cyclic vector* u. *Then* (π, V) *and* $(T, \mathcal{H})$ *are isomorphic as continuous unitary representations.*

STEP1. Proof of (i).

$$(f,g) = (\varphi * f^{**}) * g^*(e) = \int_G \varphi * f^{**}(x^{-1})g^*(x)dx_R = \int_G\int_G \varphi(x^{-1}y^{-1})f^{**}(y)dy_R g^*(x)dx_R$$
$$= \int_G\int_G \varphi(x^{-1}y^{-1})\overline{f^*(y^{-1})}\Delta(y)dy_R g^*(x)dx_R$$

By Proposition3.6.15,

$$= \int_G\int_G \varphi(x^{-1}y)\overline{f^*(y)}g^*(x)dy_R dx_R = \int_G\int_G \varphi(x^{-1}y)f(y^{-1})\overline{g(x^{-1})}\Delta(y)\Delta(x)dy_R dx_R$$
$$= \int_G\int_G \varphi(xy^{-1})f(y)\overline{g(x)}dy_R dx_R$$

□

STEP2. Proof of $(f,f) \leq 0\ (\forall f \in C_c(G))$. By the same argument as in the proof of Proposition5.2.3, there is $\{E_{n,i}\}_{n\in\mathbb{N},1\leq i\leq\varphi(n)}$ and $\{x_{n,i}\}_{n\in\mathbb{N},1\leq i\leq\varphi(n)}$ such that

$$\{E_{n,i}\}_{n\in\mathbb{N},1\leq i\leq\varphi(n)} \subset \mathcal{B}(G)\text{:disjoint } (\forall n \in \mathbb{N})$$

and

$$x_{n,i} \in E_{n,i}\ (\forall n \in \mathbb{N}, 1 \leq \forall i \leq \varphi(n))$$

and

$$||f(x) - f(x_{n,i})|| \leq \frac{1}{n} \ (\forall x \in E_{n,i}, \forall n \in \mathbb{N}, , 1 \leq \forall i \leq \varphi(n))$$

and

$$||\varphi(x^{-1}y) - \varphi(x_{n,i}^{-1}x_{n,y})|| \leq \frac{1}{n} \ (\forall x \in E_{n,i}, \forall y \in E_{n,j}, \forall n \in \mathbb{N}, , 1 \leq \forall i \leq \varphi(n))$$

We set

$$F_n(x,y) := \sum_{i,j} \varphi(x_{n,i}^{-1}x_{n,y}) f(x_{n,i}) \overline{f(x_{n,j})} \chi_{E_n,i}(x) \chi_{E_n,i}(y) \ (x, y \in G, n \in \mathbb{N})$$

and

$$F(x,y) := \varphi(x^{-1}y) f(x) \overline{f(y)} \ (x, y \in G)$$

Then clearly

$$\lim_{n \to \infty} F_n(x,y) = F(x,y) \ (\forall x, y \in G)$$

and

$$||F||_\infty \leq ||\varphi||_\infty ||f||_\infty^2$$

So, by Lebesugue convergence theorem,

$$\lim_{n \to \infty} \int_G \int_G F_n(x,y) dx_R dy_R = \int_G \int_G F(x,y) dx_R dy_R = ||f||^2$$

Because φ is positive definite,

$$\int_G \int_G F_n(x,y) dx_R dy_R = \sum_{i,j} \varphi(x_{n,i}^{-1}x_{n,j}) f(x_{n,i}) \overline{f(x_{n,j})} \geq 0$$

□

STEP3. Proof of $(g,f) = \overline{(f,g)}$ $(\forall f \in C_c(G))$. By Proposition4.5.3, $\varphi(yx^{-1}) = \overline{\varphi(xy^{-1})}$ $(\forall x, y \in G)$. So, by (i), $(g,f) = \overline{(f,g)}$ $(\forall f \in C_c(G))$ □

STEP4. Proof of (ii). By STEP2,

$$|(f,g)| \leq ||f|| ||g|| \ (\forall f, g \in C_c(G))$$

So, $(\cdot_1, \cdot_2)$ is well-defined on $\mathcal{H}_0$ by this inequality. Consequently, (ii) holds. □

STEP5. Proof of that $(T_z f, T_z g) = (f,g)$ $(\forall f, g \in C_c(G), \forall z \in G)$.

$$(T_z f, T_z g) = \int_G \int_G \varphi(xy^{-1}) T_z f(x) \overline{T_z f(y)} dx_R dy_R = \int_G \int_G \varphi(xy^{-1}) f(xz) \overline{f(yz)} dx_R dy_R$$
$$= \int_G \int_G \varphi(xz(yz)^{-1}) f(xz) \overline{f(yz)} dx_R dy_R = \int_G \int_G \varphi(xy^{-1}) f(x) \overline{f(y)} dx_R dy_R = (f,g)$$

□

STEP6. Proof of that T is well-defined and unitary. It is clear from STEP5. □

STEP7. Proof of (iii). By STEP6, it is enough to show T is continuous. Let us fix any $f, g \in C_c(G)$. By Theorem4.2.3, it is enough to show $G \ni z \to (T_z f, g) \in \mathbb{C}$ is continuous. Let us fix any $\epsilon > 0$ and fix any $z \in G$. Let us fix U such that U is a compact neighborhood of e and $U^{-1} = U$. For $x \in supp(f)U$, there is V_x and U_x such that V_x is an open neighborhood of x and U_x is a compact neighborhood of e and $U_x \subset U$ and $U_x^{-1} = U_x$

$$|f(yz) - f(y)| \leq \frac{\epsilon}{(\int_G \int_{supp(f)U_0} |\varphi(xy^{-1}) T_{z^{-1}} g(x)| dx_R dy_R + 1)} \ (\forall y \in V_x, \forall z \in U_x)$$

Because $supp(f)U$ is compact, there is $V_{x_1}, ..., V_{x_n}$ which is a covering of $supp(f)U$. $U_0 := U_{x_1} \cap ... \cap U_{x_n}$. For any $w \in zU_0$,

$$|(T_w f, g) - (T_z f, g)| = |(T_{z^{-1}w} f, T_{z^{-1}} g) - (f, T_{z^{-1}} g)| \leq \int_G \int_{supp(f)U_0} |\varphi(x^{-1}y) g(x)| |f(yz) - f(y)| dy_R dx_R \leq \epsilon$$

□

STEP8. Proof of (iv). By Proposition6.4.16, $\mathcal{H}\prime$ is clearly separable. Because $\mathcal{H}\prime$ is dense in $\mathcal{H}$, $\mathcal{H}$ is separable. □

STEP9. Proof of (v). (v) is proved by Lebesgue convergence theorem. □

STEP10. Proof of (vi). This is followed by

$$||f||^2 \leq \sup_{x,y \in supp(f)} |\varphi(xy^{-1})|(\int_G |f(g)|dg)^2 \ (\forall f \in C_c(G))$$

□

STEP11. Constructing a cyclic vector. There is $\{f_n\}_{n=1}^{\infty} \subset C_c(G)$ such that $supp(f_n) \subset exp(B(O, \frac{1}{n})$ and $f_n \geq 0$ and $\int_G f_n dg = 1$ $(\forall n \in \mathbb{N})$. Then for any $n \in \mathbb{N}$

$$||f_n||^2 \leq ||\varphi||_\infty \int_G f(x)f(y)dxdy = ||\varphi||_\infty$$

So, there is subsequence $\{f_{\alpha(n)}\}_{n=1}^{\infty}$ and $v \in \mathcal{H}$ such that

$$w - \lim_{n\to\infty} f_{\alpha(n)} = v$$

Then for any $f \in C_c(G)$

$$(f, v) = \lim_{n\to\infty}(f, f_n) = \lim_{n\to\infty} \int_{supp(f)} \int_{supp(f_n)} \varphi(xy^{-1})f(y)f_n(x)dxdy$$

By the same argument as in the proof of STEP7,

$$\begin{aligned}
&\lim_{n\to\infty} |\int_{supp(f)} \int_{supp(f_n)} \varphi(xy^{-1})f(y)f_n(x)dxdy - \int_{supp(f)} \varphi(y^{-1})f(y)dy| \\
&= \lim_{n\to\infty} |\int_{supp(f)} \int_{supp(f_n)} \varphi(y^{-1})f(yx)f_n(x)dxdy - \int_{supp(f)} \varphi(y^{-1})f(y)dy| \\
&= \lim_{n\to\infty} \int_{supp(f)} \int_{supp(f_n)} \varphi(e)|f(yx) - f(y)|f_n(x)dxdy \\
&\leq \int_{supp(f)} \sup_{z \in supp(f_n)} \varphi(e)|f(yz) - f(y)|dy = 0
\end{aligned}$$

So,

$$(f, v) = \varphi * f(e)$$

□

STEP12. Calculas of $f * k^*$. Let us fix any $f, k \in C_c(G)$. By Proposition5.2.3, $\int_G T_{y^{-1}} f k^*(y)dy$ exists. By the same argument as in the proof of STEP2 and STEP7, there is $\{E_{n,i}\}_{n\in\mathbb{N}, 1\leq i\leq\varphi(n)}$ and $\{x_{n,i}\}_{n\in\mathbb{N}, 1\leq i\leq\varphi(n)}$ such that

$$\{E_{n,i}\}_{n\in\mathbb{N}, 1\leq i\leq\varphi(n)} \subset \mathcal{B}(G)\text{:disjoint } (\forall n \in \mathbb{N})$$

and

$$y_{n,i} \in E_{n,i} \ (\forall n \in \mathbb{N}, 1 \leq \forall i \leq \alpha(n))$$

and

$$||k^*(y) - k^*(y_{n,i})|| \leq \frac{1}{n} \ (\forall y \in E_{n,i}, \forall n \in \mathbb{N}, , 1 \leq \forall i \leq \alpha(n))$$

and

$$||f(xy^{-1}) - f(xy_{n,i}^{-1})|| \leq \frac{1}{n} \ (\forall x \in supp(f)supp(k), \forall y \in E_{n,j}, \forall n \in \mathbb{N}, , 1 \leq \forall i \leq \alpha(n))$$

We set for $n \in \mathbb{N}$

$$F_n(x) := \int_G \sum_{i=1}^{\alpha(n)} f(xy_{n,i}^{-1})k^*(y_{n,i})\chi_{E_{n,i}}(y)dy \ (x \in G)$$

Then

$$\lim_{n\to\infty} F_n = f * k^* \text{ (pointwise convergence)}$$

and

$$||F_n||_\infty \leq ||f||_\infty ||k^*||_\infty dg(supp(f)supp(k))dg(supp(k^*)) \ (\forall n \in \mathbb{N})$$

So, by (v),

$$\lim_{n\to\infty} F_n = f * k^* \text{ (in } \mathcal{H})$$

Also,

$$F_n = \sum_{i=1}^{\alpha(n)} T_{y_{n,i}^{-1}} f k^*(y_{n,i})$$

By Proposition5.2.3 and (vi),

$$\lim_{n\to\infty} F_n = \int_G T_{y^{-1}} f k^*(y) dy \text{ (in } \mathcal{H})$$

So,

$$\int_G T_{y^{-1}} f k^*(y) dy = f * k^*$$

□

STEP13. Proof of (vii). Let us fix any $f, k \in C_c(G)$.

$$(f,k) = \varphi * (f * k^*)(e) = (f * k^*, v) = (\int_G T_{y^{-1}} f k^*(y) dy, v) = \int_G (T_{y^{-1}} f k^*(y), v) dy = \int_G (f k^*(y), T_y v) dy$$
$$= \int_G (f, k(y^{-1}) T_y v \delta_R(y) dy = (f, \int_G k(y^{-1}) T_y v \delta_R(y) dy)$$

So,

$$k = \int_G k(y^{-1}) T_y v \Delta_R(y) dy$$

By the same argument as in the proof of Proposition5.2.3, $k \in \overline{\{\sum_{i=1}^m c_i \pi(g_i) v | c_i \in \mathbb{C}, g_i \in G, i = 1, 2, ..., m, m \in \mathbb{N}\}}$ So, v is a cyclic vector of $\mathcal{H}$. □

STEP14 Proof of (viii). For any $f \in C_c(G)$,

$$\int_G \varphi(g^{-1} f(g) dg = \varphi * f(e) = (f, v) = (\int_G f(y^{-1}) T_y v \Delta_R(y) dy, v) = \int_G f(y^{-1})(T_y v, v) \Delta_R(y) dy$$
$$= \int_G f(y)(T_{y^{-1}} v, v) dy$$

So, for any $y \in G$,

$$\varphi(g^{-1}) = (T_{y^{-1}} v, v)$$

□

STEP15 Proof of (ix). This is clearly followed by Proposition4.3.7. □

By the proof of Theorem4.5.9, the following holds.

Proposition 4.5.10. *Let G is a Lie group. We will succeed in notations of Theorem4.5.9.*

(S1) G is a Lie group.

(S2) φ is a bounded borel measurable function on G.

(A1) $(f, f) := \varphi * f * f^*(e) \geq 0$ $(\forall f \in C_c(G))$.

Then by the same method to Theorem4.5.9, we can construct a cyclic continuous unitary representation $(T, \mathcal{H})$ with a cyclic vector v and $\varphi(g) = (T_g v, v)$ (a.e. $g \in G$).

4.5.3 The topology of positive definite functions

Definition 4.5.11 (The topology of $\mathbb{P}_1$). *Let G be a Lie group. We denote the minimal topology of $\mathbb{P}_1$ in which*

$$\mathbb{P}_1 \ni \varphi \mapsto \int_G \varphi(g)f(g)dg_r \in \mathbb{C} \text{ is continuous for every } f \in L^1(G) \tag{4.5.3}$$

by τ_1.

By Proposition3.4.6, there are $\{U_n\}_{n=1}^{\infty} \subset \mathcal{O}(G)$ such that U_n is relative compact and $U_n \subset U_{n+1}$ $(\forall n \in \mathbb{N})$ and $G = \cup_{n=1}^{\infty} U_n$.

$$d(f_1, f_2) := \sum_{i=1}^{\infty} \frac{||f_1 - f_2||_{L^\infty(\bar{U}_i)}}{2^i(1 + ||f_1 - f_2||_{L^\infty(\bar{U}_i)})} \ (f_1, f_2 \in \mathbb{P}_1)$$

By Proposition4.5.3, d is a metric on $\mathbb{P}_1$. We call this topology the pontryagin topology of $\mathbb{P}_1$ and denote this by τ_2.

The following is clear.

Proposition 4.5.12. *Let G be a Lie group and $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathbb{P}$ and φ be a complex-value function on G and $\{\varphi_n\}_{n\in\mathbb{N}}$ compact converges to φ. Then $\varphi \in \mathbb{P}$.*

Proposition 4.5.13. *Let G be a Lie group. Then there is $\{f_n\}_{n\in\mathbb{N}} \subset C_c(G)$ such that for every $f \in C_c(G)$ and $\epsilon > 0$ there is $n \in \mathbb{N}$ such that $||f - f_n||_\infty < \epsilon$.*

Proof. By Proposition3.4.6, there is a sequence of compact subsets of G $\{K_n\}_{n\in\mathbb{N}}$ such that $K_n \subset K_{n+1}^\circ$ $(\forall n \in \mathbb{N})$ and $G = \cup_{n\in\mathbb{N}} K_n$. Then there is $\{g_n\}_{n\in\mathbb{N}} \subset C_c(G)$ such that

$$g_n|K_n \equiv 1 \text{ and } supp(g_n) \subset K_{n+1}^\circ \ (\forall n \in \mathbb{N})$$

Because $C(K_n)$ is separable for every $n \in \mathbb{N}$(see [**?**]), for each $n \in \mathbb{N}$ there is $\{h_{n,m}\}_{m\in\mathbb{N}}$ which is a dense subset of $C(K_n)$. We set $f_{n+1,m} := g_n h_{n+1,m}$ $(m, n \in \mathbb{N})$. Clearly $\{f_{n,m}\}_{n,m\in\mathbb{N}} \subset C_c(G)$.

Let us fix any $f \in C_c(G)$ and $\epsilon > 0$. Then there is $n\mathbb{N}$ such that $supp(f) \subset K_n$. Because $f \in C(K_{n+1})$, there is $m \in \mathbb{N}$ such that $||f|K_{n+1} - h_{n+1,m}|K_{n+1}||_\infty < \epsilon$. Because $g|K_n \equiv 1$ and $supp(f) \subset K_n$, $||f - f_{n+1,m}|K_{n+1}||_\infty = ||gf|K_{n+1} - gh_{n+1,m}|K_{n+1}||_\infty = ||f|K_{n+1} - h_{n+1,m}|K_{n+1}||_\infty < \epsilon$. □

Proposition 4.5.14. *Let G be a Lie group. Then τ_1 satisfies the first countable axiom.*

Proof. Let us assume $\{f_n\}_{n\in\mathbb{N}}$ be in Proposition. Let us fix any $\varphi_0 \in \mathbb{P}_1$. We set

$$V(\varphi_0, f_n, \frac{1}{m}) := \{\varphi \in \mathbb{P}_1 || \int_G (\varphi - \varphi_0) f_n dg_r| < \frac{1}{m}\} \ (n, m \in \mathbb{N})$$

Let us fix any $\epsilon > 0$ and $f \in L^1(G)$. Because $C_c(G)$ is dense in $L^1(G)$(Proposition6.4.16), by Proposition, there is $n, l \in \mathbb{N}$ such that $||f - f_n||_{L^1(G)} < \frac{\epsilon}{4}$. Let us fix $m \in \mathbb{N}$ such that $\frac{1}{m} < \frac{\epsilon}{4}$. Let us fix any $\varphi \in V(\varphi_0, f_n, \frac{1}{m})$.

$$|\int_G (\varphi(g) - \varphi_0(g)) f(g) dg_r| \le |\int_G (\varphi(g) - \varphi_0(g)) f_n(g) dg_r| + \int_G |\varphi(g) - \varphi_0(g)||f(g) - f_n(g)| dg_r \le \frac{\epsilon}{4} + 2\int_G |f(g) - f_n(g)| dg_r < \epsilon$$

So, $V(\varphi, f_n, \frac{1}{m}) \subset V(\varphi, f, \epsilon)$. Because $\{V(\varphi_0, f, \epsilon)\}_{f\in L^1(G), \epsilon>0}$ is a neighborhood basis at φ_0, $\{V(\varphi_0, f_n, \frac{1}{m})\}_{m,n\in\mathbb{N}}$ is also a neighborhood basis at φ_0. □

Proposition 4.5.15. *Let*

(i) X_1 and X_2 are topological spaces.

(ii) $f : X_1 \to X_2$ satisfies

If $\{x_n\}_{n\in\mathbb{N}}$ converges x in X_1 then $\{f(x_n)\}_{n\in\mathbb{N}}$ converges $f(x)$ in X_2

(iii) X_1 satisfies the first countable axiom.

then f is continuous.

Proof. Let us assume f is not continuous. Then there is an open set of X_2 O such that $f^{-1}(O)$ is not open set of X_1. Then there is $x \in f^{-1}(O)$ such that for any neighborhood of x N, $N \not\subseteq f^{-1}(O)$. By (iii), we can take $\{V_{x,n}\}_{n\in\mathbb{N}}$ which is a countable neighborhood basis at x. Then there is $\{x_n\}_{n\in\mathbb{N}} \subset X_1$ such that $x_n \in V_{x,n} \setminus f^{-1}(O)$ $(\forall n \in \mathbb{N})$. Because $\{x_n\}_{n\in\mathbb{N}}$ converges x, by (ii), $\{f(x_n)\}_{n\in\mathbb{N}}$ converges $f(x) \in O$. Because $f(x_n) \in O^c$ $(\forall n \in \mathbb{N})$, $f(x) \in \bar{O^c} = O^c$. This is contradiction. □

Notation 4.5.16. *Let G be a topological group. We denote the set of all continuous positive definite functions by $\mathbb{P}$. And we set*

$$\mathbb{P}_1 := \{\varphi \in \mathbb{P} | \varphi(e) = 1\}$$

Example 4.5.17. *Let G be a group and (π, V) is a unitary representation of G. Then $\Phi_\pi(v \otimes v)$ is a positive definite function.*

Proof. For any $n \in \mathbb{C}$ and $g_1, g_2, ..., g_n \in G$ and $c_1, c_2, ..., c_n \in \mathbb{C}$

$$\sum_{j,k} c_j \bar{c_k} \Phi_\pi(v \otimes v)(g_j^{-1} g_k) = \sum_{j,k} c_j \bar{c_k} (\pi(g_k)v, \pi(g_j)v) = (\sum_k c_k \pi(g_k)v, \sum_j c_j \pi(g_j)v) \geq 0$$

□

Lemma 4.5.18. *Let*

(i) G be a Lie group.

(ii) $f \in C_c(G)$.

(iii) $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathbb{P}_0$.

(iv) $\phi \in \mathbb{P}_0$.

(v) $\{\phi_n\}_{n\in\mathbb{N}}$ converges to ϕ in τ_1.

Then $\{\phi_n * f\}_{n\in\mathbb{N}}$ compact converges to $\phi * f$.

STEP1. Showing that $\{\phi_n * f\}_{n\in\mathbb{N}}$ pointwise converges to $\phi * f$. Let us fix any $g \in G$. Then

$$\phi_n * f(g) = \int_G \phi_n(gh^{-1}) f(h) dg_r(h) = \int_G \phi_n((hg^{-1})^{-1}) f((hg^{-1})g) dg_r(h) = \int_G \phi_n(h^{-1}) f(hg) dg_r(h)$$
$$= \int_G \phi_n(h) f(h^{-1}g) \Delta_r(h) dg_r(h)$$

by (v)

$$\to \int_G \phi(h) f(h^{-1}g) \Delta_r(h) dg_r(h) = \phi * f(g) \ (n \to \infty)$$

□

STEP2. Showing that $\{\phi_n * f\}_{n\in\mathbb{N}}$ are equicontinuous. We will show that for each $g_0 \in G$ and $\epsilon > 0$ there is a neighborhood of e V such that

$$|\phi_n * f(g) - \phi_n * f(g_0)| < \epsilon \ (\forall g \in g_0 V, \forall n \in \mathbb{N})$$

Let us fix any $g_0 \in G$ and $\epsilon > 0$. Because $f \in C_c(G)$, $f\Delta_r$ is uniformly continuous. So, there is a neighborhood of e V such that

$$|f(g) - f(h)| < \frac{\epsilon}{2(dg_r(supp(f)) + 1)(||\Delta_r(g)||_{L^\infty(supp(f))} + 1)} \ (\forall g, h \in G \ s.t \ g^{-1}h \in V)$$

Then, for any $g \in g_0 V$,

$$|\phi_n * f(g) - \phi_n * f(g_0)| = |\int_G \phi_n(h^{-1})(f(hg) - f(hg_0) dg_r(h)| \leq \int_G |f(hg) - f(hg_0)| dg_r(h) < \epsilon$$

□

STEP3. Showing that $\{\phi_n * f\}_{n\in\mathbb{N}}$ compact converges to φ. Let us fix any K is a compact subset of G and $\epsilon > 0$. Because φ is uniformly continuous on K, there is V which is a neighborhood of e such that

$$|\varphi(g_1) - \varphi(g_2)| < \frac{\epsilon}{3} \ (\forall g_1, g_2 \in K \ s.t \ g_1^{-1} g_2 \in V)$$

By STEP2, for each $g \in K$, there is $V_g \subset V$ which is a neighborhood of e such that

$$|\varphi_n(g) - \varphi_n(h)| < \frac{\epsilon}{3} \ (\forall h \in gV_g, n \in \mathbb{N})$$

Because $K \subset \cup_{g\in K} gV$ and K is compact, there is $g_1, g_2, ..., g_n$ such that $K \subset \cup_{i=1}^n g_i V_{g_i}$.

By STEP1, for each $i \in \{1, 2, ..., n\}$, there is k_i such that

$$|\varphi_m(g_i) - \varphi(g_i)| < \frac{\epsilon}{3} \ (\forall m \geq k_i)$$

We set $K := \max_{i \in \{1,2,\dots,n\}} k_i$. Let us fix any $g \in G$ and $m \geq K$. There is i such that $g \in g_i V_{g_i}$.

$$|\varphi_m(g) - \varphi(g)| \leq |\varphi_m(g) - \varphi_m(g_i)| + |\varphi_m(g_i) - \varphi(g_i)| + |\varphi(g_i) - \varphi(g)| < \epsilon$$

□

Theorem 4.5.19 (D.A.Raikov-R.Godement-H.Yoshizawa Theorem)**.** *Let G be a Lie group and τ_1, τ_2 be topologies which are defined in Definition4.5.11. Then $\tau_1 = \tau_2$.*

Strategy for our proof. Clearly $\tau_1 \subset \tau_2$. Let us fix any $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathbb{P}_0$ and $\phi \in \mathbb{P}_0$ such that $\phi_n \to \phi$ in τ_1. By Proposition4.5.15, it is enough to show $\phi_n \to \phi$ in τ_2.

Let us fix any $\epsilon > 0$ and K which is a compact subset of G. By Proposition4.5.3, there is V which is a neighborhood of e such that

$$|\varphi(g_1) - \varphi(g_2)| < \frac{\epsilon}{3} \ (\forall g_1, g_2 \in K \ s.t \ g_1^{-1} g_2 \in V)$$

Then there is $f \in C_c(G)$ such that $supp(f) \subset V$ and $f \leq 0$ on G and $\int_G f dg_r = 1$. □

STEP1. Evaluation of $\varphi_n * f - f$. For any $n \in \mathbb{N}$ and $g \in G$

$$|\varphi_n * f(g) - \varphi_n(g)| \leq |\int_G (\varphi_n(gh^{-1}) - \varphi_n(g)) f(h) dg_r(h)| \leq \int_G |\varphi_n(gh^{-1}) - \varphi_n(g)| f(h) dg_r(h)$$

By Proposition4.5.15

$$\leq \int_G \frac{1}{\sqrt{2}} \int_G (\varphi_n(e) - Re\varphi_n(h))^{\frac{1}{2}} f(h)^{\frac{1}{2}} f(h)^{\frac{1}{2}} dg_r(h) \leq \frac{1}{\sqrt{2}} (\int_G (\varphi_n(e) - Re\varphi_n(h)) f(h) dg_r(h))^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}} (\int_G (Re\varphi(e) - Re\varphi_n(h)) f(h) dg_r(h))^{\frac{1}{2}}$$

Because $\phi_n \to \phi$ in τ_1, there is $n_0 \in \mathbb{N}$ such that

$$\int_G |Re\varphi_n(h) f(h) - Re\varphi(h) f(h)| dg_r(h) < \frac{\epsilon^2}{9}$$

So,

$$|\varphi_n * f(g) - \varphi_n(g)| \leq \frac{\epsilon}{3} \int_G |\varphi(e) - \varphi(h)| f(h) dg_r(h) < \frac{\epsilon}{3} \ (\forall g \in G, n \geq n_0)$$

Similarily,

$$|\varphi * f(g) - \varphi(g)| < \frac{\epsilon}{3} \ (\forall g \in G, n \geq n_0)$$

□

STEP2. Showing this theorem. By Lemma4.5.18, there is $n_1 \in \mathbb{N}$ such that

$$|\varphi_n * f(g) - \varphi * f(g)| < \frac{\epsilon}{3} \ (\forall g \in K, n \geq n_1)$$

So, by STEP1,

$$|\varphi_n(g) - \varphi(g)| < |\varphi_n(g) - \varphi_n * f(g)| + |\varphi_n * f(g) - \varphi * f(g)| + |\varphi * f(g) - f(g)| < \epsilon \ (\forall g \in K, n \geq \max n_0, n_1)$$

□

Proposition 4.5.20. *Let G be a Lie group. Then $\mathbb{P}_1$ is compact.*

Proof. Let us fix any $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathbb{P}_1$. By Banach-Alaoglu Theorem, there is a cauchy subsequence $\{\phi_{\alpha(n)}\}_{n\in\mathbb{N}}$ in $*$-weak topology. Because $L^1(G)^* = L^\infty(G)$ (see [15]), there is a bounded borel function φ such that $\{\phi_{\alpha(n)}\}_{n\in\mathbb{N}}$ converges to φ in weak-$*$ topology. So, φ satisfies assumptions in Proposition4.5.10. By Proposition4.5.10, we can assume φ is continuous. □

4.5.4 Extreme points

Proposition 4.5.21. *Let*

(S1) G *is a Lie group.*

(S2) φ_1, φ_2 *are continuous functions on* G.

(A1) $\varphi_1 * f = \varphi_2 * f$ $(\forall f \in C_c(G))$.

Then $\varphi_1 = \varphi_2$.

Proof. Let us fix any $g \in G$. There is a sequence $\{f_n\}_{n\in\mathbb{N}} \subset C_c^+(G)$ such that $\int_G f_n dg_r = 1$ $(\forall n \in \mathbb{N})$. By the same argument as the proof of Theorem4.5.9, $\varphi_1(g) = \varphi_2(g)$. □

Proposition 4.5.22. *We will succeed in notations of Theorem4.5.9. Let*

(S1) G *is a Lie group.*

(S2) φ_1, φ_2 *are continuous positive definite functions on* G.

(A1) $(\cdot_1, \cdot_2)_{\varphi_1} = (\cdot_1, \cdot_2)_{\varphi_2}$

Then $\varphi_1 = \varphi_2$.

Proof. By Theorem4.5.9, $\varphi_1 * f = \varphi_2 * f$ $(\forall f \in C_c(G))$. By Proposition4.5.21, $\varphi_1 = \varphi_2$. □

Proposition 4.5.23. *Let*

(S1) G *is a Lie group.*

Then $Ex(\mathbb{P}_0) \setminus 0 = Ex(\mathbb{P}_1)$.

Proof of $\subset$. Let us fix any $\varphi \in Ex(\mathbb{P}_0) \setminus 0$. If $\varphi(e) < 1$, then $\varphi = \varphi(e)\dfrac{\varphi}{\varphi(e)} + (1 - \varphi(e))0$. This means $\varphi \notin Ex(\mathbb{P}_0)$. So, $\varphi(e) = 1$. □

Proof of $\supset$. Let us fix any $\varphi \in Ex(\mathbb{P}_1)$. Let us fix any $\varphi_1, \varphi_2 \in Ex(\mathbb{P}_0)$ and $\alpha_1, \alpha_2 \in [0, 1]$ such that $\varphi = \alpha_1\varphi_1 + \alpha_2\varphi_2$. Then $1 = \varphi(e) = \alpha_1\varphi_1(e) + \alpha_2\varphi_2(e)$. Then $\varphi_1(e) = \varphi_2(e) = 1$. So, $\varphi = \varphi_1 = \varphi_2$. □

Proposition 4.5.24. *Let*

(S1) G *is a Lie group.*

(S2) By GNS construction we set

$$\Phi : \mathbb{P}_1 \ni \varphi \mapsto (T, \mathcal{H}_\varphi) \in \Omega_c$$

Then $Ex(\mathbb{P}_1) = \mathbb{P}_1 \cap \Phi^{-1}(\hat{G})$.

Proof of $\subset$. Let us fix any $\varphi \in Ex(\mathbb{P}_1)$. Let us fix any closed G-invariant subspaces of $\mathcal{H}_\varphi$ V_1, V_2 such that $\mathcal{H}_\varphi = V_1 + V_2$ and $V_1 \neq 0$. Let us set P_i be the orthogonal projection of V_i $(i = 1, 2)$. Let us fix $v \in \mathcal{H}_\varphi$ such that $\varphi(g) = (T_g v, v)$ $(\forall g \in G)$. Because $V_1 \perp V_2$ and P_i is commutative with T_g $(\forall i, g \in G)$ and $1 = ||v||^2 = ||P_1v||^2 + ||P_2v||^2$, $\varphi(g) = ||P_1v||^2\dfrac{(T_gP_1v, P_1v)}{||P_1v||^2} + ||P_2v||^2\dfrac{(T_gP_2v, P_2v)}{||P_2v||^2}$. Because $\varphi \in Ex(\mathbb{P}_1)$, $(T_gv, v) = (T_gP_1v, P_1v) = (T_gP_1v, v)$ $(\forall g \in G)$. So, $(v, T_{g^{-1}}v) = (P_1v, T_{g^{-1}}v)$ $(\forall g \in G)$. Because $(T, \mathcal{H}_\varphi)$ is cyclic, $v = P_1v$. So, $V_1 = \mathcal{H}_\varphi$. □

Proof of $\supset$. Let us fix any $\varphi \in \mathbb{P}_1 \cap \Phi^{-1}(\hat{G})$. Let us fix $\varphi_1, \varphi_2 \in \mathbb{P}_1$ and $\alpha_1, \alpha_2 \in [0, 1]$ such that $\varphi = \alpha_1\varphi_1 + \alpha_2\varphi_2$. We set for $f + \{f \in C_c(G) | ||f||_\varphi = 0\}) \in C_c(G)/\{f \in C_c(G) | ||f||_\varphi = 0\}$

$$\pi_i(f + \{f \in C_c(G) | ||f||_\varphi = 0\}) := f + \{f \in C_c(G) | ||f||_{\varphi_i} = 0\} \ (i = 1, 2)$$

Because $\{f \in C_c(G) | ||f||_\varphi = 0\} \subset \{f \in C_c(G) | ||f||_{\varphi_i} = 0\}$ $(i = 1, 2)$, π_1, π_2 are well defined and surjective.

Let us fix any $w \in \mathcal{H}_{\varphi_1}$. Because $|(\pi_1(u), \pi_1(w))_{\mathcal{H}_{\varphi_1}}| \leq \dfrac{1}{\alpha_1}|(u, w)| \leq \dfrac{1}{\alpha_1}||u||||w||$. So, by Riez representation theorem, there is $Aw \in \mathcal{H}_\varphi$ such that $(\pi_1(u), \pi_1(w))_{\mathcal{H}_{\varphi_1}} = (u, Aw)$ $(\forall u \in \mathcal{H}_\varphi)$. Clearly A is continuous and linear. If $A = 0$, then $\varphi_1 = 0$. This is contradiction. So, $A \neq 0$. Because $(T, \mathcal{H}_\varphi)$ is irreducible, by Shur Lemma(see Proposition4.1.1), there is $\lambda_1 \in \mathbb{C}$ such that $T = \lambda_1 I$. There is $w_1 \in \mathcal{H}_{\varphi_1}$ such that $\pi_1(w_1) \neq 0$. Then $0 < ||\pi_1(w_1)||^2_{\varphi_1} = \bar{\lambda}||w_1||^2$. So, $\lambda_1 > 0$. And, $(\cdot_1, \cdot_2)_{\varphi_1} = \lambda_1(\cdot_1, \cdot_2)_\varphi$. By Proposition4.5.22, $\varphi_1 = \lambda_1\varphi$. $1 = \varphi_1(e) = \lambda_1\varphi(e) = \lambda_1$. So, $\varphi_1 = \lambda$. □

By Proposition4.5.24, Krein Millman Theorem(Theorem4.1.6), Raikov-Godement-Yoshizawa Theorem(Theorem4.5.19), the following hold.

Theorem 4.5.25 (I.M. Gelfand-D.A. Raikov Theorem). *Let*

(S1) G *is a Lie group.*

(S2) K *is a compact subset of* G.

(S3) $\epsilon > 0$.

(S4) φ *is a continuous positive definite function on* G.

Then $\alpha_1, ..., \alpha_m > 0$ *and* $\varphi_1, ..., \varphi_m \in Ex(\mathbb{P}_1)$ *such that*

$$||\varphi - \sum_{i=1}^{m} \alpha_i \varphi_i||_{L^\infty(K)} < \epsilon$$

Theorem 4.5.26 (I.M. Gelfand-D.A. Raikov Theorem). *Let*

(S1) G *is a Lie group.*

(S2) $g_1, g_2 \in G$.

(A1) $T_{g_1} = T_{g_2}$ $(\forall (T, V) \in \hat{G})$.

Then $g_1 = g_2$.

Proof. Let us fix $g_1, g_2 \in G$ such that $g_1 \neq g_2$. We set $g_0 := g_1 g_2^{-1}$. There is $f \in C_c^+(G)$ s.t $g_0 \notin supp(f)^{-1} supp(f)$ and $||f||_2 = 1$. We set

$$\varphi(g) := (R_g f, f) \ (g \in G)$$

Because the right regular representation R is continuous on $L^2(G)$, φ is continuous positive definite function on G.

$$\varphi(g_0) = \int_G f(gg_0) f(g) dg_r(g) = 0$$

Because $1 = \varphi(e) = \varphi(e) - \varphi(g_0)$, by Theorem4.5.25, Then $\alpha_1, ..., \alpha_m > 0$ and $\varphi_1, ..., \varphi_m \in Ex(\mathbb{P}_1)$ such that

$$\sum_{i=1}^{m} \alpha_i (\varphi_i(e) - \varphi_i(g_0)) \neq 0$$

So, there is i such that $\varphi_i(g_0) \neq 1$. Because $\varphi_i \in \mathbb{P}_1$, by Proposition4.5.24, $(T, \mathcal{H}_{\varphi_i}) \in \hat{G}$ and there is $v \in \mathcal{H}_{\varphi_i}$ such that $||v||_{\varphi_i} = 1$ and $\varphi_i(g_0) = (T_{g_0} v, v)_{\varphi_i}$. So, $T_{g_0} \neq I$. This implies that $T_{g_1} \neq T_{g_2}$. □

4.6 Topology of unitary dual

Definition 4.6.1 (Fell topology). *By GNS construction we set*

$$\Phi : \mathbb{P}_1 \ni \varphi \mapsto (T, \mathcal{H}_\varphi) \in \Omega_c$$

Here, we assume the topology of $\mathbb{P}_1$ *is the pontryagin topology and* Ω_c *is the set of all separable cyclic unitary representation of* G. *We set the toplogy of* Ω_c *by* $\{O \subset \Omega_c | \Phi^{-1}(O)$ *is open set*$\}$. *We call this topology Fell topology of* Ω_c.

4.7 Direct Integral of Hilbert spaces

Definition 4.7.1. *Let*

(S1) $(X, \mathfrak{B}, \mu)$ *is a measurable space.*

We say X *is localizable if there is* $N \subset X$ *and* $\{X_i\}_{i=1}^{\infty} \subset \mathfrak{B}$ *such that*

(i) $\{X_i\}_{i=1}^{\infty}$ *is disjoint.*

(ii) $N \cap \cup_{i=1}^{\infty} X_i = \phi$.

(iii) $X = N \cup \cup_{i=1}^{\infty} X_i$.

(iv) $\mu(X_i) < \infty$ $(\forall i \in \mathbb{N})$.

(v) $\mu(F) = \sum_{i=1}^{\infty} \mu(F \cap X_i)$ $\forall F \in \mathfrak{B}$.

Because Lie group is σ-compact, the following holds.

Proposition 4.7.2. *Let*

(S1) G *is a Lie group.*

(S2) μ *is a left invariant measure.*

Then $(G, \mathfrak{B}, \mu)$ *is localizable.*

Notation 4.7.3 (Locally almost everywhere)**.** *Let*

(S1) $(X, \mathfrak{B}, \mu)$ *is a meaurable space.*

(S2) For each $x \in X$*, the proposition* $P(x)$ *is given.*

We denote P *holds loc. a.e* $x \in X$ *if for any* $Y\mathfrak{B}$ *such that* $\mu(Y) < \infty$ P *holds loc. a.e* $x \in Y$*.*

Proposition 4.7.4 (Direct Integral of Hilbelt spaces)**.** *Let*

(S1) $(X, \mathfrak{B}, \mu)$ *is a meaurable space.*

(S2) $\{H(x)\}_{x \in X}$ *is a family of Hilbert spaces.*

(S3) $\Pi := \Pi_{x \in X} H(x)$.

(S4) $\mathfrak{G} \subset \Pi$.

(S5) $\mathfrak{R} := \{f \in \mathfrak{G} | f = 0 \text{ loc-a.e. } x \in X\}$

We say $\mathfrak{G}$ *is a Direct Integral of* $\{H(x)\}_{x \in X}$ *if*

(i) If $v_1, v_2 \in \mathfrak{G}$ *and* $a, b \in \mathbb{C}$ *then* $av_1 + bv_2 := \{av_1(x) + bv_2(x)\}_{x \in X} \in \mathfrak{G}$.

(ii) If $v \in \mathfrak{G}$ *then* $X \in x \mapsto ||v(x)||_{H(x)} \in \mathbb{R}$ *is measurable.*

(iii) If $v \in \mathfrak{G}$ *then* $\int_X ||v(x)||^2_{H(x)} \mu(x) < \infty$.

(iv) Let us fix any $f \in \Pi$ *such that*

(a) There is $\varphi \in L^2(X)$ *such that* $||f||_{H(x)} \leq \varphi(x)$ $(\forall x \in X)$

(b) For any $g \in \mathfrak{G}$*,* $X \ni x \mapsto (f(x), g(x))_{H(x)} \in \mathbb{C}$ *is measurable.*

Then there is $h \in \mathfrak{G}$ *such that for any* $g \in \mathfrak{G}$

$$(f(x) - h(x), g(x)) = 0 \text{ for loc-a.e } x \in X \tag{4.7.1}$$

(v) Let us fix any $f \in \Pi$ *such that*

(a) $||f(\cdot)||_{H(\cdot)} \in L^2(X)$

(b) There is $h \in \mathfrak{G}$ *such that* $f(x) = h(x)$ *for loc-a.e* $x \in X$*.*

Then $f \in \mathfrak{G}$.

Then $\mathfrak{G}/\mathfrak{R}$ *is a Hilbert space. We call this Wils Direct Integral of* $(X, \mu, \{H(x)\}_{x \in X})$ *with respect to* $\mathfrak{G}$ *and denote this by* $\int_X^{\mathfrak{G}} H(x) d\mu(x)$

Proof. It is enough to show that any cauchy sequence of $\mathfrak{G}$ has a convergent subsequence. Let us fix any cauchy sequence of $\mathfrak{G}$, $\{v_n\}_{n=1}^{\infty}$. Then there is subsequence $\{v_{\varphi(i)}\}_{i=1}^{\infty}$ such that

$$\sum_{i=1}^{\infty} ||v_{\varphi(i+1)} - v_{\varphi(i)}||^2 < \infty$$

and

$$\sum_{i=1}^{\infty} ||v_{\varphi(i+1)} - v_{\varphi(i)}|| < \infty$$

So,

$$\int_X \sum_{i=1}^{\infty} ||v_{\varphi(i+1)}(x) - v_{\varphi(i)}(x)||^2_{H(x)} d\mu(x) = \sum_{i=1}^{\infty} \int_X ||v_{\varphi(i+1)}(x) - v_{\varphi(i)}(x)||^2_{H(x)} d\mu(x) = \sum_{i=1}^{\infty} ||v_{\varphi(i+1)} - v_{\varphi(i)}||^2 < \infty$$

So,

$$\sum_{i=1}^{\infty}||v_{\varphi(i+1)}(x)-v_{\varphi(i)}(x)||^2_{H(x)}<\infty \text{ loc-a.e } x\in X$$

So, $\{v_{\varphi(i)}(x)\}_{i=1}^{\infty}$ is cauchy sequence for loc-a.e $x\in X$. Because for any $x\in X$ $H(x)$ is Hilbert space, $\{v_{\varphi(i)}(x)\}_{i=1}^{\infty}$ converges to some $v(x)\in H(x)$ for loc-a.e $x\in X$. Because $||v(x)||^2_{H(x)}=\lim_{n\to\infty}(v_n(x),v_n(x))$ for loc-a.e $x\in X$, $||v(\cdot)||_{H(\cdot)}$ is measurable. For loc-a.e $x\in X$,

$$||v_n(x)||\leq||v_n(x)-v_1(x)||+||v_1(x)||\leq\sum_{i=2}^{n}||v_i(x)-v_{i-1}(x)||+||v_1(x)||$$

So, for loc-a.e $x\in X$,

$$||v(x)||\leq\sum_{i=2}^{\infty}||v_i(x)-v_{i-1}(x)||+||v_1(x)||$$

Here,

$$\int_X(\sum_{i=2}^{\infty}||v_i(x)-v_{i-1}(x)||+||v_1(x)||)^2d\mu(x)\leq\lim_{n\to\infty}\int_X(\sum_{i=2}^{n}||v_i(x)-v_{i-1}(x)||+||v_1(x)||)^2d\mu(x)$$
$$\leq\lim_{n\to\infty}(\sum_{i=1}^{n}||v_{i+1}-v_i||^2+||v_1||^2+||v_1||\sum_{i=1}^{n}||v_{i+1}-v_i||+(\sum_{i=1}^{n}||v_{i+1}-v_i||)^2)<\infty$$

So,

$$\sum_{i=2}^{\infty}||v_i(\cdot)-v_{i-1}(\cdot)||+||v_1(\cdot)||\in L^2(X,\mu)$$

Let us fix any $u\in\mathfrak{G}$ and $n\in\mathbb{N}$.

$$(v_n(x),u(x))=(\frac{1}{2}||v_n(x)+u(x)||^2-\frac{1}{2}||v_n(x)||^2-\frac{1}{2}||u(x)||^2)+i(\frac{1}{2}||v_n(x)+iu(x)||^2-\frac{1}{2}||v_n(x)||^2-\frac{1}{2}||iu(x)||^2)$$

So, $(v_n(\cdot),u(\cdot))$ is measurable. This implies that $(v(\cdot),u(\cdot))$ is measurable. By (iv), there is $v_0\in\mathfrak{G}$ such that for $u\in\mathfrak{G}$ and for loc-a.e $x\in X$

$$(v(x)-v_0(x),u(x))=0$$

So, for any $n\in\mathbb{N}$, $(v(x)-v_0(x),v_n(x)-v_0(x))=0$. This implies that for loc-a.e $x\in X$ $(v(x)-v_0(x),v(x)-v_0(x))=0$. So,

$$v(x)=v_0(x)\ \text{ loc-a.e } x\in X$$

By (v), $v\in\mathfrak{G}$.

For loc-a.e $x\in X$ and $n\mathbb{N}$,

$$||v(x)-v_n(x)||\leq 2(\sum_{i=2}^{\infty}||v_i(x)-v_{i-1}(x)||+||v_1(x)||)$$

and $\sum_{i=2}^{\infty}||v_i(\cdot)-v_{i-1}(\cdot)||+||v_1(\cdot)||\in L^2(X)$. So, by Lebesgue convergence theorem,

$$\lim_{n\to\infty}||v-v_n||^2=\lim_{n\to\infty}\int_X||v(x)-v_n(x)||^2d\mu(x)=0$$

□

FWils=mL^2A‘xNglS By Theorem4.2.3, the following holds.

Proposition 4.7.5 (Direct Integral of Unitary representations). *Let*

(S1) $(X,\mathfrak{B},\mu)$ *is a meaurable space.*

(S2) $\{H(x)\}_{x\in X}$ *is a family of Hilbert spaces.*

(S3) $\Pi:=\Pi_{x\in X}H(x)$.

(S4) $\mathfrak{G}\subset\Pi$.

(S5) $\int_X^{\mathfrak{G}}H(x)d\mu(x)$ *is the direct integral of* $(X,\mu,\{H(x)\}_{x\in X})$ *with respects to* $\mathfrak{G}$.

(S6) G *is a topological group.*

(S7) π_x *is a continuous unitary representation on* $H(x)$ $(x \in X)$.

(A1) *For any* $g \in G$ *and* $v := \{v(x)\}_{x \in X} \in \mathfrak{G}$, $\pi(g)v := \{\pi_x(g)v(x)\}_{x \in X} \in \mathfrak{G}$

(A2) *For any* $v := \{v(x)\}_{x \in X} \in \mathfrak{G}$, $G \ni g \mapsto \pi(g)v\mathfrak{G}$ *is continuous.*

Then $(\pi, \int_X^{\mathfrak{G}} H(x)d\mu(x))$ *is continuous unitary representation. We call this direct integral representation of* $(X, \mu, \{\pi(x), H(x)\}_{x \in X})$ *and denote this by* $\int_X^{\mathfrak{G}} \pi(x)d\mu(x)$.

4.8 Decomposition of an affine type function

Definition 4.8.1 (Baire Set)**.** *Let* X *be a locally compact topological space. We denote the minimal borel family in which any element of* $C_c(X)$ *is measurable by* $\mathfrak{B}_0$. *We call the elemant of* $\mathfrak{B}_0$ *Baire set.*

Definition 4.8.2 (Support of measure)**.** *Let*

(S1) X *is a locally compact topological space.*

(S2) $\mathfrak{B}$ *is the minimal borel set family containing all relative compact open sets.*

(S3) μ *is a nonnegative measure on* $\mathfrak{B}$.

(S4) $F \subset X$.

We say F *supports* μ *if for any* $A\mathfrak{B}$ *such that* $A \cap F = \phi$, $\mu(A) = 0$.

Definition 4.8.3 (Regular borel measure)**.** *Let*

(S1) X *is a locally compact hausdorff topological space.*

(S2) $\mathfrak{B}$ *is the minimal borel set family containing all relative compact open sets.*

(S3) μ *is a nonnegative measure on* $\mathfrak{B}$.

(A1) *For any compact set* A, $\mu(A) < \infty$.

(A2) $\mu(A) = sup\{\mu(C) | C \in \mathfrak{B},\ C \subset A \text{ and } C \text{ is compact.}\}$.

(A3) $\mu(B) = sup\{\mu(C) | C \in \mathfrak{B},\ A \subset C \text{ and } C \text{ is an open set.}\}$.

Then we say μ *is regular borel measure on* X.

Definition 4.8.4 (Upper semicontinuous function)**.** *Let*

(S1) X *is a topological space.*

We say $f \in Map(X, \mathbb{R})$ *is upper continuous for any* $c \in \mathbb{R}$ $f^{-1}((-\infty, c))$ *is an open set.*

Definition 4.8.5 (Affine type function)**.** *Let* $\mathcal{D}$ *be a vector space and* X *be a convex subset of* $\mathcal{D}$ *and* f *be a real valued function on* $\mathcal{D}$. *We say* f *is affine type if*

$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y)\ (\forall \lambda \in [0,1], \forall x, y \in X)$$

We denote the set of all continuous affine type function on $\mathcal{D}$ *by* $A(X)$.

Notation 4.8.6. *Let*

(S1) $(\mathcal{D}, \{||\cdot||_n\}_{n \in \mathcal{N}}$ *is a seminormed vector space.*

(S2) X *is a compact convex subset of* $\mathcal{D}$.

We set

$$B(X) := \{f \in Map(X, \mathbb{R} | f \text{ is an upper semicontinuous and convex on } X\}$$

and

$$CB(X) := B(X) \cap C(X)$$

and

$$CB_0(X) := CB_0(X) - CB_0(X)$$

Definition 4.8.7 (Vector lattice)**.** *Let*

(S1) $(V, \leq)$ *is a partialy ordered vector space.*

(S2) $\vee$ *is a binary operation on* V.

We say $(V, \leq, \vee)$ *is vector lattice if for any* $x, y, z \in V$

(i) If $x \leq y$ *then* $x + z \leq y + z$.

(ii) If $x \leq y$ *then* $\alpha x \leq \alpha y$ $(\forall \alpha \geq 0)$.

(iii) $x \vee y$ *is a least upper bound.*

Proposition 4.8.8. *Let*

(S1) $(\mathcal{D}, \{||\cdot||_n\}_{n\in\mathcal{N}}$ *is a seminormed vector space.*

(S2) X *is a compact convex subset of* $\mathcal{D}$.

Then

(i) If $f, g \in CB(X)$ *then* $\max(f, g) \in CB(X)$.

(ii) $CB_0(X)$ *is a vector lattice with the pointwise order and pointwise maximum.*

(iii) $CB_0(X)$ *is dense in* $C(X)$.

Proof of (i). Let us fix any $x, y \in X$ and $\lambda \in [0, 1]$. Then

$$\begin{aligned}\max(f(\lambda x + (1-\lambda)y), g(\lambda x + (1-\lambda)y)) &\leq \max(\lambda f(x) + (1-\lambda)f(y), \lambda g(x) + (1-\lambda)g(y)) \\ &\leq \lambda \max(f(x), g(x)) + (1-\lambda)max(f(x), g(x))\end{aligned}$$

So, $\max(f, g) \in CB(X)$ ☐

Proof of (ii). Let us fix any $f_1, f_2, g_1, g_2 \in CB(X)$. For each $x \in X$

$$f_1(x) - g_1(x) \leq f_2(x) - g_2(x) \iff f_1(x) + g_2(x) \leq f_2(x) + g_1(x)$$

So,

$$\max(f_1 - g_1, f_2 - g_2) = \max(f_1 + g_2, f_2 + g_1) - (g_1 + g_2)$$

So, by (i), $\max(f_1 - g_1, f_2 - g_2) \in CB_0(X)$. ☐

Proof of (iii). By Hahn-Banach Theorem, for any $x, y \in X$ such that $x \neq y$, there is $h \in CB_0(X)$ such tat $h(x) \neq h(y)$. So, by Stone-Weierstrass Theorem in Vector Lattice(Theorem4.1.7), (iii) holds. ☐

Definition 4.8.9 (Order of Regular Borel measures)**.** *Let*

(S1) X *is a locally compact hausdorff topological space.*

(S2) $\mathfrak{B}$ *is the minimal borel set family containing all relative compact open sets.*

(S3) μ_1, μ_2 *are regular borel measures on* X.

We denote $\mu_1 \prec \mu_2$ *if*

$$\mu_1(f) \leq \mu_2(f) \ (\forall f \in CB(X))$$

Proposition 4.8.10. *Let*

(S1) $(\mathcal{D}, \{||\cdot||_n\}_{n\in\mathcal{N}}$ *is a seminormed vector space.*

(S2) X *is a compact convex subset of* $\mathcal{D}$.

(S3) μ_1, μ_2 *are regular borel measure on* X.

(A1) $\mu_1 \prec \mu_2$ *and* $\mu_2 \prec \mu_1$.

Then $\mu_1 = \mu_2$.

Proof. This is from Proposition4.8.8. ☐

Proposition 4.8.11. *Let*

(S1) $(\mathcal{D}, \{||\cdot||_n\}_{n\in\mathcal{N}}$ *is a seminormed vector space.*

(S2) X *is a compact convex subset of* $\mathcal{D}$.

(S3) μ_1, μ_2 *are regular borel measure on* X.

(A1) $\mu_1 \prec \mu_2$.

(S4) $f \in A(X)$.

Then $\mu_1(f) = \mu_2(f)$.

Proof. Because $f \in CB(X) \cap (-CB(X))$, $\mu_1(f) = \mu_2(f)$. □

Definition 4.8.12 (Upper envelope function)**.** *Let*

(S1) $(\mathcal{D}, \{||\cdot||_n\}_{n\in\mathcal{N}}$ *is a seminormed vector space.*

(S2) X *is a compact convex subset of* $\mathcal{D}$.

(S3) $f \in C(X, \mathbb{R})$.

We set

$$\tilde{f}(x) := \inf\{h(x) | h \in A(X), h \geq f\} \ (x \in X)$$

Proof of $\{h \in A(X) | h \geq f\} \neq \phi$. Because X is compact and $f \in C(X, \mathbb{R})$, $||f||_{L^\infty(X)} < \infty$. Constant function with $||f||_{L^\infty(X)}$ is continuous affine type function. So, $\{h \in A(X) | h \geq f\} \neq \phi$. □

Proposition 4.8.13. *Let*

(S1) $(\mathcal{D}, \{||\cdot||_n\}_{n\in\mathcal{N}}$ *is a seminormed vector space.*

(S2) X *is a compact convex subset of* $\mathcal{D}$.

Then

(i) For any $f \in C(X, \mathbb{R})$, $\tilde{f}$ *is bounded and upper semicontinuous.*

(ii) For any $f \in C(X, \mathbb{R})$, $f \leq \tilde{f}$.

(iii) For any $f \in CB(X)$, $f = \tilde{f}$.

(iv) For any $f, g \in CB(X)$, $\widetilde{f+g} \leq \tilde{f} + \tilde{g}$.

(v) For any $f, g \in CB(X)$, $|\tilde{f} - \tilde{g}| \leq ||f - g||_{L^\infty(X)}$.

(vi) For any $f \in CB(X)$ *and* $r \in (0, \infty)$, $\tilde{rf} = r\tilde{f}$.

Proof of (i). Because $\tilde{f} \leq ||f||_{L^\infty(X)}$, $\tilde{f}$ is bounded. Let us fix any $c \in \mathbb{R}$ and $x \in \tilde{f}^{-1}((-\infty, c))$. Then there is $h \in A(X)$ such that $h(x) < c$. Because h is continuous, there is V which is a neighborhood of 0 such that $h(x+y) < c \ (\forall y \in V \cap X)$. So, $\tilde{f}(x+y) < c \ (\forall y \in V \cap X)$. This means that $x + V \subset \tilde{f}^{-1}((-\infty, c))$. So, $\tilde{f}$ is upper semicontinuous. □

Proof of (ii). (ii) is clear from the definition of upper envelope functions. □

Proof of (iii). We set $K := \{(x, r) \in X \times \mathbb{R} | 0 \leq r \leq f(x)\}$. Because X is compact and f is continuous concave, K is compact convex subset of $X \times \mathbb{R}$. Aiming contradiction, let us assume $f(x_0) < \tilde{f}(x_0)$ for some $x_0 \in X$. $(x_0, \tilde{f}(x_0)) \notin K$. By Theorem4.1.4, there is L which is a continuous $\mathbb{R}$-linear functional on $\mathcal{D} \times \mathbb{R}$ such that

$$L(x_0, \tilde{f}(x_0)) > 1 > L(x, f(x)) \ (\forall x \in X)$$

This implies $(\tilde{f}(x_0) - f(x_0))L(0, 1) > 0$. So,

$$L(0, 1) > 0$$

We set

$$h(x) := \frac{1 - L(x, 0)}{L(0, 1} \ (x \in \mathfrak{D})$$

Then $h \in A(X)$ and

$$L(x, h(x)) = 1 \ (\forall x \in \mathfrak{D})$$

So,

$$L(x_0, \tilde{f}(x_0)) > L(x, h(x)) > L(x, f(x)) \ (\forall x \in X)$$

This implies

$$0 < L(x, h(x)) - L(x, f(x)) = L(0, h(x) - f(x)) = (h(x) - f(x))L(0, 1) \ (\forall x \in X)$$

So,

$$f(x) < h(x)\ (\forall x \in X)$$

Similarly,

$$h(x_0) < \tilde{f}(x_0)$$

These two equation contradict with each other. □

Proof of (iv). Let us fix any $x \in X$ and $\epsilon > 0$. Then there is $h_1, h_2 \in A(X)$ such that $f \leq h_1$ and $g \leq h_2$ and $h_1(x) \leq \tilde{f}(x) + \epsilon$ and $h_2(x) \leq \tilde{g}(x) + \epsilon$. Because $h_1 + h_2 \in A(X)$ and $f + g \leq h_1 + h_2$. $\widetilde{f+g}(x) \leq h_1(x) + h_2(x)$. So, $\widetilde{f+g}(x) \leq \tilde{f}(x) + \tilde{g}(x) + 2\epsilon$. □

Proof of (v). By (iv), for any $x \in X$.

$$\tilde{f}(x) - \tilde{g}(x) \leq f \widetilde{- g +} g(x) - \tilde{g}(x) \leq \widetilde{f - g}(x)$$

Because $||f - g|| \in A(X)$, $\widetilde{f - g} \leq ||f - g||$. So, (v) holds. □

Proof of (vi). This is clear from the definition of upper envelope functions. □

Definition 4.8.14 (Convex cone)**.** *Let*

(S1) $\mathcal{D}$ is a $\mathbb{R}$-vector space.

(S2) $v_1, v_2, ..., v_m \in \mathcal{D}$.

Then

$$cc(v_1, v_2, ..., v_m) := \{\sum_{i=1} a_i v_i | a_i \geq 0\ (\forall i)\}$$

Proposition 4.8.15. *Let*

(S1) $\mathcal{D}$ is a $\mathbb{R}$-vector space.

(S2) $v_1, v_2, ..., v_m \in \mathcal{D}$.

(A1) $0 \in ex(cc(v_1, v_2, ..., v_m))$.

Then there is $w_1, ..., w_n \in\in \mathcal{D}$ such that $w_1, ..., w_n$ are linear independent and

$$cc(v_1, v_2, ..., v_m) \subset cc(w_1, w_2, ..., w_n)$$

Proof. We set $n_0 := dim\{v_1, ..., v_m\}$. Using mathematical induction on $m - n_0$, we prove this proposition. Let us fix any $d \in \mathbb{N}$. Let us assume this proposition holds for $m - n_0 \leq d$ and $m - n_0 = d + 1$. Then we can assume

$$v_m = -\sum_{i=1}^{k} a_i v_i + \sum_{j=1}^{l} b_j v_{k+j}, k + l = m - 1$$

If $k = 0$ or $v_m \neq 0$, then $cc(v_1, v_2, ..., v_m) = cc(v_1, v_2, ..., v_{m-1})$. By the assumption of mathematical induction, this proposition holds. So, we can assume $k \neq 0$ and $v_m \neq 0$. If $l = 0$, $0 = \frac{1}{2}(v_m + \sum_{i=1}^{k} a_i v_i)$. This means $0 \notin ex(cc(v_1, ..., v_m))$. So, we can assume $l \neq 0$. Furthermore, we can assume

$$k := \min\{K \in \mathbb{N} | \exists \sigma : \{1, ..., m\} \to \{1, ..., m\}\text{:bijective}, \exists c_1, ..., c_K > 0, \exists d_1, ..., d_L \geq 0 (L := m - K) \text{ s.t.}$$
$$-\sum_{i=1}^{K} c_\sigma(i) v_{\sigma(i)} + \sum_{j=1}^{L} b_{\sigma(j)} v_{\sigma(k+j)} = 0\ \} - 1$$

We set

$$v'_{k+j} = \frac{-1}{l} \sum_{i=1}^{k} a_i v_i + b_j v_{k+j}\ (j = 1, ..., l)$$

Because of the minimalism of k, $0 \in ex(cc(v_1, ...v_k, v'_{k+1}, ..., , v'_{k+l}))$. Because $v_{k+j} = \frac{1}{b_j}(\sum_{i=1}^{k} a_i v_i + v'_{k+j})$ $(\forall j)$ and $\sum_{j=1}^{l} v'_{k+j} = v_m$,

$$cc(v_1, v_2, ..., v_m) \subset cc(v_1, ..., v'_{k+l}), k + l = m - 1$$

By the assumption of mathematical induction, this proposition holds. □

Proposition 4.8.16. *Let*

(S1) $(\mathcal{D}, \{||\cdot||_n\}_{n\in\mathcal{N}}$ *is a seminormed vector space.*

(S2) X *is a compact convex subset of* $\mathcal{D}$.

(S3) $x \in X$.

(A1) $f(x) = \tilde{f}(x)$ $(\forall f \in C(X, \mathbb{R}))$.

Then $x \in ex(X)$.

Proof. Aiming contradiction, let us assume $x \notin ex(X)$. Then there is $y, z \in X$ such that $y \neq z$ and $x = \dfrac{y+z}{2}$. We set $f(\cdot) := d(x, \cdot)$. By Proposition4.8.13,

$$0 = f(x) = \tilde{f}(x) \geq \frac{1}{2}(\tilde{f}(y) + \tilde{f}(z)) = \frac{1}{2}(f(y) + f(z)) > 0$$

This is contradiction. □

Proposition 4.8.17. *Let*

(S1) X *is a locally compact hausdorff topological space.*

(S2) $\mathfrak{B}$ *is the minimal borel set family containing all relative compact open sets.*

(S3) $\mathfrak{M}$ *is the set of all regular borel measures on* X.

(S4) $\mu \in \mathfrak{M}$.

Then $M_\mu := \{\nu \in \mathfrak{M} | \mu \geq 0, \mu \prec \nu\}$ *has a maximal element.*

STEP1. We set

$$\Phi := \{T \subset M_\mu | T \text{ is totally ordered with } \prec\}$$

Let us fix any $\mathfrak{N}$ which is totally ordered subset of M_μ with inclusion relationship. Clearly $\cup_{T\in\mathfrak{N}} T$ is totally ordered with $\prec$. So, by Zorn Lemma, Φ has a maximal element F. Because F is totally ordered with $\prec$, for any finite elements $\tau_1, ..., \tau_m \in F$, $\cap_{i=1}^m M_{\tau_i} \neq \phi$. □

STEP2. We set

$$S := \{\mu \in \mathfrak{M} | \mu(1) = \nu(1)\}$$

Because $S \subset \{F \in C(X)^* | ||F|| \leq |\nu(1)|\}$ and S is closed subset in $*$-weak topology, by Banach-Alaogrou Theorem, S is compact subset in $*$-weak topology. For any $\tau \in F$,

$$M_\tau = \cap_{f\in CB(X)} \{\mu \in S | \mu(f) \geq \nu(f)\} \cap \cap_{f\in C_c^+(X)} \{\mu \in S | \mu(f) \geq 0\}$$

So, $M_\tau \subset S$ is closed subset in $*$-weak topology, □

STEP3. By STEP1 and STEP2, $\cap_{\tau\in F} M_\tau \neq \phi$. Let us take a $\mu_0 \in \cap_{\tau\in F} M_\tau$. For aiming contradiction, let us assume there is $\mu \in M_\nu$ such that $\mu_0 \prec \mu$ and $\mu \neq \mu_0$. By Proposition, $\mu \notin F$. But $F \cap \{\mu\}$ is totally ordered. This is contradiction. So, μ_0 is a maximal element of M_ν. □

Proposition 4.8.18. *Let*

(S1) $(\mathcal{D}, \{||\cdot||_n\}_{n\in\mathcal{N}}$ *is a seminormed vector space.*

(S2) X *is a compact convex subset of* $\mathcal{D}$.

(S3) μ *is a maximal element in* $\mathfrak{M}$.

Then

$$\mu(f) = \mu(\tilde{f}) \ (\forall f \in C(X, \mathbb{R}))$$

Proof. We set

$$\rho(g) := \mu(\tilde{g}) \ (g \in C(X, \mathbb{R}))$$

Clearly ρ is a seminorm on $C(X, \mathbb{R})$. Let us fix any $f \in C(X, \mathbb{R})$.

$$L(rf) := r\mu(\tilde{f}) \ (r \in \mathbb{R})$$

By Hahn Banach Theorem, L has an extension L' which is a $\mathbb{R}$-linear functional on $C(X,\mathbb{R})$ such that $L' \leq \rho$. Let us fix any $g \in C(X,\mathbb{R})^+$. Because $-g \leq 0$, $\tilde{-g} \leq 0$. So,

$$L(-g) \leq \rho(-g) = \mu(\tilde{-g}) \leq \mu(0) \leq 0$$

This implies $0 \leq L(g)$. So, by Riez representation theorem, L is a regular borel measure.

Let us fix any $h \in CB(X)$. Because $-h$ is continuous and concave, by Proposition,

$$L(-h) \leq \rho(-h) = \mu(\tilde{-h}) = \mu(-h)$$

So, $\mu \prec L$. This implies $\mu = L$. So,

$$\mu(\tilde{f}) = L(f) = \mu(f)$$

□

Proposition 4.8.19. *Let*

- *(S1)* $(\mathcal{D}, \{||\cdot||_n\}_{n\in\mathcal{N}}$ *is a seminormed vector space.*
- *(S2)* X *is a compact convex subset of* $\mathcal{D}$.
- *(S3)* f *is continuous strictly convex function on* X.
- *(S4)* $z \notin ex(X)$.

Then $f(z) < \tilde{f}(z)$

Proof. By there are $x, y \in X$ such that $x \neq y$ and $z = \frac{1}{2}(x+y)$ Let us fix any $h \in A(X)$ such that $f \leq h$. Then

$$f(z) < \frac{1}{2}(f(x)+f(y)) \leq \frac{1}{2}h(x)+h(y)) = h(z)$$

So,

$$f(z) < \frac{1}{2}(f(x)+f(y)) \leq \tilde{f}(z)$$

□

Theorem 4.8.20 (Choquet Theorem)**.** *Let*

- *(S1)* $(\mathcal{D}, \{||\cdot||_n\}_{n\in\mathcal{N}}$ *is a seminormed vector space.*
- *(S2)* X *is a compact convex subset of* $\mathcal{D}$.
- *(S3)* $x_0 \in X$.

Then there are K *is a borel set and* μ *which is a regular borel probability measure on* X *such that* K *supports* μ *and* $X \setminus K \subset ex(X)$ *and*

$$\varphi(x_0) = \int_K \varphi(x) d\mu(x) \ (\forall \varphi \in A(X))$$

STEP1. Construction of continuous strictly convex function. We set $U := \{h \in A(X) | ||h||^\infty = 1\}$. Because X is compact metrizable, there is a countable set $\{h_n\}_{n\in\mathbb{N}} \subset U$ which is dense in U. We set

$$f := \sum_{n=1}^{\infty} \frac{h_n^2}{2^n}$$

We will show f is strictly convex. Let us fix any $x, y \in X$ such that $x \neq y$ and $\lambda \in (0,1)$. By Hahn-Banach Theorem, there is f which is a real-valued continuous linear functional on$\mathcal{D}$ and satisfies $f(x) > f(y)$. Because $\dfrac{f - \frac{f(x)+f(y)}{2}}{||f - \frac{f(x)+f(y)}{2}||_{L^\infty(\mathcal{D})}} \in U$, there is $n \in \mathbb{N}$ such that $h_n(x) > 0 > h_n(y)$.

$$\begin{aligned} h_n(\lambda x + (1-\lambda)y)^2 &= \lambda^2 h_n(x)^2 + (1-\lambda)^2 h_n(y)^2 + \lambda(1-\lambda)h_n(x)h_n(y) < \lambda^2 h_n(x)^2 + (1-\lambda)^2 h_n(y)^2 \\ &\leq \lambda h_n(x)^2 + (1-\lambda)h_n(y)^2 \end{aligned}$$

This implies that $f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y)$. So, f is strictly convex. □

STEP2. Construction of a regular borel measure. Because X is locally compact hausdorff space, by Riez-Markov-Kakutani Theorem, $\delta : C(X) \ni g \mapsto g(x) \in \mathbb{C}$ defines a regular borel measure. So, by Proposition4.8.17, there is a maximal element $\mu \in \mathfrak{M}$ such that $\delta \prec \mu$. By Proposition4.8.11, $\mu(g) = \delta(g)$ for any $g \in A(X)$. Because $1 \in A(X)$, $\mu(X) = 1$. □

STEP3. Construction of K. We set

$$K := \cup_{n\in\mathbb{N}} K_n, K_n := \{x \in X | \tilde{f}(x) - f(x) > \frac{1}{n}\}$$

Because $K_n = (\cap_{m\in\mathbb{N}}\{x \in X|\tilde{f}(x) - f(x) < \frac{1}{n} + \frac{1}{m}\})^c$ and $\tilde{f} - f$ is upper continuous, K_n is measurable for any $n \in \mathbb{N}$. So, K is borel measurable. By Proposition4.8.19, $X \setminus K \subset ex(X)$. By Proposition4.8.18, $\mu(f) = \mu(\tilde{f})$. So $\mu(K) = 0$. This implies $X \setminus K$ supports μ. □

4.9 Mautner-Teleman's theorem

Proposition 4.9.1. *Let*

(S1) G is a Lie group.

(S2) (π, V) is a continuous unitary cyclic representation of G with a cyclic vector ω.

Then there is a finite mesurable space $(X, \mathcal{M}, \mu)$ and a direct integral $\int_X^{\mathcal{G}} \omega(x)d\mu(x)$ which is isomorphic to (π, V) as continuous unitary representation.

STEP1. Decomposition of a matrix coefficient. We can assume

$$||\omega|| = 1$$

We set

$$\varphi(g) := (\pi(g)\omega, \omega)\ (g \in G)$$

Because $\mathbb{P}_1$ is a compact convex subset of $C(G)$ with compact convergence topology which is metrizable by countable seminorms. By Theorem4.8.20, there are μ which is a probability measure on $\mathbb{P}_1$ and X which is a borel mesurable set such that $X \subset ex(\mathbb{P}_1)$

$$F(\varphi) = \int_X F(\varphi_x)d\mu(x)\ (\forall F \in A(\mathbb{P}_1))$$

Here, $\varphi_x = x$. For any $g \in G$, $\mathbb{P}_1 \in \psi \mapsto Re\psi(g) \in \mathbb{R}$ and $\mathbb{P}_1 \in \psi \mapsto Im\psi(g) \in \mathbb{R}$ are continuous affine by Raikov-Godement-Yoshizawa Theorem(Theorem4.5.19). So,

$$\varphi(g) = \int_X \varphi_x(g)d\mu(x)\ (\forall g \in G)$$

□

STEP2. Construction of a family of irreducible representations. We set

$$(T(x), H(x)) : \text{The representation generated by the GNS construction } (x \in X)$$

and

$$\Pi := \Pi_{x\in X} H(x)$$

and

$$v(f, x) : \text{The projection of } f \text{ in } H(x)\ (f \in C_c(G), x \in X)$$

and

$$\mathfrak{D}_0 : \text{The vector space generated by } \{\lambda(\cdot)v(f, \cdot)|f \in C_c(G), \lambda \in L^\infty(X, \mu)\}$$

We set $\mathfrak{D}$ by the completion of $\mathfrak{D}_0$ with the inner product $(\cdot, \cdot) := \int_X (\cdot, \cdot)_{H(x)}d\mu(x)$. As we showed in the process of proving Proposition4.7.4, any cauchy sequence of $\mathfrak{D}_0$ has a subsequence which converges pointwise some element of Π. So, we can embedded $\mathfrak{D}$ in Π. Clearly $\mathfrak{D}$ is $\mathbb{C}$-linear subspace of Π. And, for each $\lambda \in L^\infty(X, \mu)$ and $f \in C_c(G)$, $X \ni x \mapsto ||\lambda(x)v(f, x)||_{H(x)}$ is measurable and L^2-integrable. So, forany $F \in \mathfrak{D}$, $X \ni x \mapsto ||F(x)||_{H(x)}$ is measurable and L^2-integrable. Clearly $\mathfrak{D}$ satisfies (v) in Proposition4.7.4. So, it is enough to show (iv) in Proposition4.7.4. Hereafter, let us fix any $u \in \Pi$ which satisfies (iv)(a) and (iv)(b) in Proposition4.7.4. There exists $\{v_n\}_{n\in\mathbb{N}} \subset \mathfrak{D}_0$ such that

$$\lim_{n\to\infty} ||v_n - u|| = \inf_{v\in\mathfrak{D}_0} ||v - u||$$

For each $u, v \in \Pi$,

$$P(u,v)(x) = \begin{cases} \dfrac{(u(x),v(x))}{||v(x)||^2} & (v(x) \neq 0) \\ 0 & (v(x) = 0) \end{cases} \quad (x \in X)$$

We will show

$$||u(x) - P(u,v)(x)|| \leq ||u(x) - v(x)|| \ (\forall v \in V, \forall x \in X) \tag{4.9.1}$$

Let us fix any $v \in V$ and $x \in X$. If $v(x) = 0$, (4.9.1) holds. So, we can assume $v(x) \neq 0$. Then

$$||u(x) - P(u,v)(x)||^2 = ||u(x)||^2 - \frac{|(u(x),v(x))|^2}{||v(x)||^2}$$

and

$$||u(x) - v(x)||^2 = ||u(x)||^2 - 2Re(u(x),v(x)) + ||v(x)||^2$$

So,

$$||v(x)||^2(||u(x) - v(x)||^2 - ||u(x) - P(u,v)(x)||^2) = |(u(x),v(x)) - ||v(x)||^2|^2 \geq 0$$

This implies (4.9.1). So, by (4.9.1) and Proposition2.5.14, $\{P(u,v_n)\}_{n\in\mathbb{N}}$ is a cauchy sequence. So, $u_0 := \lim_{n\to\infty} P(u,v_n) \in \mathfrak{D}$ exists. We will show u_0 $u \in \Pi$ which satisfies (iv)(4.7.1) in Proposition4.7.4. Aiming contradiction, let us assume that there are $u' \in \mathfrak{D}$ and a borel measurable set E such that $\mu(E) > 0$ and

$$(u(x) - u_0(x), u'(x)) \neq 0 \ (a.ex \in X)$$

As we showed in the process of proving Proposition4.7.4, any cauchy sequence of $\mathfrak{D}_0$ has a subsequence which converges pointwise some element of Π. So, we can assume $u' \in \mathfrak{D}_0$. We set

$$v := u' - P(u', u_0)$$

For any $x \in X$, we will show

$$(v(x), u_0(x)) = 0 \tag{4.9.2}$$

and

$$(u(x) - u_0(x), u_0(x)) = 0 \tag{4.9.3}$$

If $u_0(x) = 0$, the both clearly holds. So, we can assume $u_0(x) \neq 0$. Then,

$$(v(x), u_0(x)) = (u'(x), u_0(x)) - \frac{(u'(x), u_0(x))}{||u_0(x)||^2}|(u_0(x), u_0(x))|^2 = 0$$

This means (4.9.2) holds. Furthermore,

$$(u(x), u_0(x)) = (u(x), \frac{(u(x), v_\infty(x))}{||v_\infty(x)||^2} v_\infty(x)) = \frac{|(u(x), v_\infty(x))|^2}{||v_\infty(x)||^2} = (u_0(x), u_0(x))$$

This means (4.9.3) holds. For any $x \in E$,

$$\begin{aligned} &(u(x) - u_0(x), u'(x)) = (u(x), u'(x)) - (u_0(x), u'(x)) = (u(x), v(x)) + (u(x), P(u', u_0)(x)) - (u_0(x), u'(x)) \\ &\text{by } (4.9.2) \\ &= (u(x), v(x)) + (u(x), P(u', u_0)(x)) - (u_0(x), P(u', u_0)(x)) \\ &\text{by } (4.9.3) \\ &= (u(x), v(x)) \end{aligned}$$

So,

$$(u(x), v(x)) \neq 0 \ (\forall x \in E) \tag{4.9.4}$$

We will show

$$P(u', u_0) \in \mathfrak{D} \tag{4.9.5}$$

Clearly,

$$\lambda \in L^\infty(X), w \in \mathfrak{D} \implies \lambda w \in \mathfrak{D}$$

For $n \in \mathbb{N}$, we set

$$\lambda_n(x) := \begin{cases} \dfrac{(u'(x), u_0(x))}{||u_0(x)||^2} & (||v(x)|| \geq \dfrac{1}{n} \text{ and } ||u'(x)|| \leq n) \\ 0 & (otherwise) \end{cases} \quad (x \in X)$$

Because $\lambda_n \in L^\infty(X,\mu)$, $\lambda_n u_0 \in \mathfrak{D}$. Let us fix any $n_0 \in \mathbb{N}$. If $m,n \geq n_0$,

$$||\lambda_m u_0 - \lambda_n u_0|| \leq \int_{||u_0(x)|| \leq \frac{1}{n_0}, ||u'(x)|| \geq n_0} ||u'(x)||^2 d\mu(x)$$

The right side of this equation converges to 0 when $n \to \infty$. So, $\{\lambda_m u_0\}_{m\in\mathbb{N}}$ is a cauchy sequence. So, $P(u', u_0) = \lim_{m\to\infty} \lambda_m u_0$(pointwise convergence) is in $\mathfrak{D}$. We set

$$u_1 := u_0 + P(u,v)$$

By the way which is similar to the proof of (4.9.5), $P(u,v) \in \mathfrak{D}$. This implies $u_1 \in \mathfrak{D}$.

$$||u-u_1||^2 = ||u-u_0||^2 - 2Re(u-u_0, P(u,v)) + \frac{|(u,v)|^2}{||v||^2}$$

by Proposition4.9.2

$$= ||u-u_0||^2 - 2Re(u, P(u,v)) + \frac{|(u,v)|^2}{||v||^2} = ||u-u_0||^2 - \frac{|(u,v)|^2}{||v||^2} < \inf_{v\in\mathfrak{D}_0} ||v-u||^2$$

This is a contradiction. So, $(X, \mathcal{B}(X), \mu, \Pi, \mathfrak{D})$ is a direct integral of Hilbert spaces. □

STEP3. Construction of continuous unitary representation. We set

$$T_g v(f,x) := v(R_g f, x) \ (f \in C_c(G), x \in X)$$

Because

$$(v(R_g f, x), v(R_g g, x)) = (v(f,x), v(g,x)) \ (\forall f,g \in C_c(G), \forall x \in X)$$

T_g is a unitary operator on $\mathfrak{D}_0$. Because $\mathfrak{D}_0$ is dense in $\mathfrak{D}$, T_g has the unique extension on $\mathfrak{D}$. For any $f \in C_c(G)$ and $g_1, g_2 \in G$, $||T_{g_1} v(f,\cdot) - T_{g_2} v(f,\cdot)|| \leq \mu(X) ||R_{g_1} f - R_{g_2} f||_{L^\infty}$. So,

$$G \ni g \mapsto T_g v(f,\cdot) \in \mathfrak{D}$$

is continuous. Because T is unitary and $\mathfrak{D}_0$ is dense in $\mathfrak{D}$, T is weak continuous. So, T is strong continuous. Let us take $\{f_n\}_{n\in\mathbb{N}} \subset C_c^+(G)$ such that $\int_G f_n dg_r = 1$ and $supp(f_n) \subset exp(\{X \in M(n,\mathbb{C}|\ ||X|| \leq \frac{1}{n}\})$ $(\forall n \in \mathbb{N})$. Then $\{v(f_n,\cdot)\}_{n\in\mathbb{N}}$ has a subsequence which converges some $v \in \mathfrak{D}$. By the same way as the proof of Theorem4.5.9, we can show the following.

$$(v(f,\cdot), v(g,\cdot)) = (v(f,\cdot), \int_G g(y^{-1}) T_y^{-1} v(g,\cdot)) \Delta_r(y) dg_r(y)) \ (\forall f,g \in C_c(G))$$

$$v(g,\cdot) = \int_G g(y^{-1}) T_y^{-1} v(g,\cdot)) \Delta_r(y) dg_r(y) \ (\forall g \in C_c(G))$$

By the same way as the proof of Theorem4.5.9, g is in the closed subspace generated by $T(G)v$. Because $\mathfrak{D}_0$ is dense in $\mathfrak{D}$, T is cyclic with cyclic vector v. Clearly the following holds.

$$(T_g v)(x) = T_g^x v(x) \ (\forall x \in X)$$

Here, T^x is the representation by GNS construction for $x \in X$. So,

$$\varphi(g) = \int_X \varphi_x(g) d\mu(x) = \int_X (T_g^x v(x), v(x)) d\mu(x) = \int_X (T_g v(x), v(x)) d\mu(x) = (T_g v, v) \ (\forall g \in G)$$

By Proposition4.3.7, (π, V) and $(T, \int_X^{\mathfrak{D}} H(x) d\mu(x))$ are isomorphic as continuous unitary representations. □

By Proposition4.3.8 and Proposition4.9.1, the following holds.

Theorem 4.9.2 (mautner-Teleman's theorem)**.** *Let*

(S1) G is a Lie group.

(S2) (π, V) is a continuous unitary representation of G.

Then there is a a family of direct integral of continuous unitary representations $\{\int_{X_\lambda}^{\mathcal{D}_\lambda} \omega(x) d\mu_\lambda(x)\}_{\lambda\in\Lambda}$ such that

(i) (X_λ, μ_λ) is a finite measurable space $(\forall \lambda \in \Lambda)$.

(ii) $\int_{X_\lambda}^{\mathcal{D}_\lambda} \omega_\lambda(x) d\mu_\lambda(x)$ is a continuous cyclic unitary representation of G.

(iii) (π, V) and $\bigoplus_{\lambda\in\Lambda} \int_{X_\lambda}^{\mathcal{D}_\lambda} \omega_\lambda(x) d\mu_\lambda(x)$ are isomorphic as continuous unitary representations of G.

4.10 Review

Please note that the statements in this subsection are generally inaccurate. In this chapter, the following mautner-Teleman theorem is the main theorem(Theorem4.9.2).

Theorem (mautner-Teleman theorem)**.** *Let*

(S1) G is a Lie group.

(S2) (π, V) is a continuous unitary representation of G.

Then there is a a family of direct integral of continuous unitary representations $\{\int_{X_\lambda}^{\mathcal{D}_\lambda} \omega(x)d\mu_\lambda(x)\}_{\lambda\in\Lambda}$ such that

(i) (X_λ, μ_λ) is a finite measurable space $(\forall\lambda \in \Lambda)$.

(ii) $\int_{X_\lambda}^{\mathcal{D}_\lambda} \omega_\lambda(x)d\mu_\lambda(x)$ is a continuous cyclic unitary representation of G.

(iii) (π, V) and $\bigoplus_{\lambda\in\Lambda} \int_{X_\lambda}^{\mathcal{D}_\lambda} \omega_\lambda(x)d\mu_\lambda(x)$ are isomorphic as continuous unitary representations of G.

This theorem states that any continuous unitary representation of Lie group is decomposed into irreducible continuous unitary representations. The direct integral of continuous unitary representations $\{X, \mathfrak{D}, \mu, T_x, H(x)\}$ is a subset of $\Pi := \Pi_{x\in X} H(x)$ which satisfies the following main conditions.

(i) For any $u, v \in \mathfrak{D}$, $(u(\cdot), v(\cdot))$ is measurable and integrable.

(ii) $\{T_x\}_{x\in X}$ defines T which is a continuous and unitary action on $\mathfrak{D}$.

(iii) If $v \in \Pi$ and $||v(\cdot)||$ is measurable and bounded by a L^2 function and $(v(\cdot), u(\cdot))$ is measurable, v can be seen as the element of $\mathfrak{D}$ in a sense.

In special, $(T, \mathfrak{D})$ is a continuous unitary representation of G.

I also think that the following Gelfand-Raikov Theorem(Theorem4.5.26) obtained in the process of showing mautner-Teleman theorem is also a very significant theorem. This theorem states that we can distinguish any two element of Lie group G by the unitary dual $\hat{G}$ of G. The definition of a unitary dual is the set of all continuous irreducible unitary representation of G.

Theorem (I.M.Gelfand-D.A.Raikov Theorem)**.** *Let*

(S1) G is a Lie group.

(S2) $g_1, g_2 \in G$.

(A1) $T_{g_1} = T_{g_2}$ $(\forall(T, V) \in \hat{G})$.

Then $g_1 = g_2$.

Below, I would like to review the process of obtaining these two theorems with my personal opinions and impressions. We begin by examining the cyclic representation rather than directly examining the irreducible representation. The definition of the cyclic representation (π, V) with a cyclic vector v is the representation space is spanned by $\pi(G)v$. The definition of the cyclic representation is the representation whose any vector is a cyclic vector. One of the reasons for focusing on cyclic representations is to investigate the Jordan normal form with respect to matrices that cannot be diagonalized in matrix decomposition theory. Supposing (π, V) is a representation of $\mathbb{Z}$, $\pi(1)$ is similar a jordan block if and only if (π, V) is cyclic[**?**].

By Zorn lemma and the same argument as the diagonalization of unitary matrices, we can show that any continuous unitary representation of Lie group is decomposed into cyclic continuous unitary representations (Proposition4.3.8). So, the proof of mautner-Teleman theorem is attributed to the case for cyclic representations.

We focus on matrix coefficients whose form is $\varphi := (\pi(\cdot)v, v)$ from a continuous cyclic representation (π, V) with a cyclic vector v. φ satisfies the following condition.

$$\sum_{i=1}^{N} a_i\pi(g_i)v = 0 \iff \sum_{i=1} a_i\varphi(gg_i) = 0 \ (\forall g \in G)$$

This implies if $(\pi_1(\cdot)v_1, v_1) = (\pi_2(\cdot)v_2, v_2)$ then π_1 and π_2 are isomorphic as continuous unitary representations (Proposition4.3.7). So, this is the kicker to investigate $\varphi := (\pi(\cdot)v, v)$. This function satisfies the following conditions.

(i) $\varphi(e) \geq 0$

(ii) $\varphi(g^{-1}) = \overline{\varphi(g)}$

(iii) $|\varphi(g)| \leq \varphi(e)$

(iv) $|\varphi(g_1) - \varphi(g_2)|^2 \leq \frac{1}{2}\varphi(e)|\varphi(e) - Re\varphi(g_1^{-1}g_2)|$

(v) If $(f,g)_\varphi := \int_G \varphi(xy^{-1})f(y)\bar{g(x)}dg_r(x)dg_r(y)$ $(f,g \in C_c(G))$, then $(\cdot,\cdot)_\varphi$ satisfies a nonnegative Hermitian semibilinear form.

We call functions which satisfies these conditions positive definite functions even if they don't have a form $(\pi(\cdot)v,v)$. The right regular action R preserves this nonnegative Hermitian semibilinear form and continuos. So, we construct continuous unitary representation (T, H_φ). Taking a sequence of $C_c^+(G)$ $\{f_n\}_{n\in\mathbb{N}}$ such that $||f_n||_{L^1(G)} = 1$ $(\forall n)$, by Banach-Alaogrou Theorem (Theorem4.4.4), $\{f_n\}_{n\in\mathbb{N}}$ has a convergent subsequence which converges to some $v \in H_\varphi$ in $*$-weak topology. Banach-Alaoglu Theorem states the unit sphere on of dual of a separable normed space is sequencial compact in $*$-weak topology. v likes a dirac delta function whose support $\{e\}$. For any $g \in G$, T_gv likes a dirac delta function whose support $\{g^{-1}\}$. So, v is a cyclic vector of H_φ. Assigning $f = T_gv$ and $g = v$ in (v), we see $\varphi = (T.v, v)$. In special φ can been seen as a continuous positive definite function. This method of obtaining a continuous and cyclic unitary representation from a positive definite function is the GNS construction.

The GNS construction is a powerful technique that will be used with great success throughout this chapter. For example, if $g_1 \neq g_2$ in G, there is $f \in C_c^+(G)$ such that $g_1g_2^{-1} \notin supp(f)$ and $f(e) = 1$. So, the continuous cyclic unitary representation by GNS construction for $(R.f, f)$ separates e and $g_1g_2^{-1}$. So, by GNS construction, the claim is established with the 'irreducible' part in Gelfand-Raikov replaced by 'cyclic'.

We see GNS construction gives a map from the space of continuous positive definite functions to the set of all cyclic continuous unitary representations. So, we focus on $\mathbb{P}_1$ which is the set of all normalised continuous positive definite functions whose value at e is 1. There are two possible ways to set a topology in $\mathbb{P}_1$. One is the topology from compact convergence(Pontryagin topology). Another one is the $*$-weak topology. By the strong continuity (iv), these topology is the same. This is Raikov-Godement-Yoshizawa Theorem(Theorem4.5.19). A sketch of the proof of this theorem is shown below. Let us assume $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathbb{P}_1$ converges to $\varphi \in \mathbb{P}_1$ in $*$-weak topology. Then for any $f \in C_c(G)$, $\{f * \varphi_n\}_{n\in\mathbb{N}}$ converges to $f * \varphi \in \mathbb{P}_1$ pointwise. Because of (iv), $\{f * \varphi_n\}_{n\in\mathbb{N}}$ is equicontinuous on any compact subset. By the same argument of the proof of Ascoli?Arzel? theorem, $\{f * \varphi_n\}_{n\in\mathbb{N}}$ converges to $f * \varphi$. Because of (iv), taking f such that $supp(f)$ is sufficient small, $||\varphi_n - \{f * \varphi_n||_\infty\}_{n\in\mathbb{N}}$ and $||\varphi - \{f * \varphi||_\infty\}_{n\in\mathbb{N}}$ are uniformly small. So, $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathbb{P}_1$ compact converges to $\varphi \in \mathbb{P}_1$.

By this powerfull theorem, we can show important properties of the topology of $\mathbb{P}_1$. $*$-weak convergence preserves (iii) and (iv) and boundedness of positive definite functions. By GNS construnction, $*$-weak convergence preserves continuity of positive definite functions. So, $\mathbb{P}_1$ is closed subset of $*$-weak topology. By Banach-Alaoglu theorem and $L^1(G)^* = L^\infty(G)$, $\mathbb{P}_1$ is compact. Because $\mathbb{P}_1$ is convex, by Krein-Millman theorem, any $\varphi \in \mathbb{P}_1$ can be uniformly approximated by some convex combination of $\{\varphi_n\}_{n=1}^N \subset ex(\mathbb{P}_1)$ on any compact subset.

We see

$$ex(\mathbb{P}_1) = \mathbb{P}_1 \cap \Phi^{-1}(\hat{G})$$

Here, Φ is the map defined by GNS construction. Because by orthogonal projections we can get a convex combination decomposition of positive definite function from a decomposition of a representation space of GNS construction, the $\subset$ part is shown. By Shur Lemma, we can obtain a decomposition of a representation space of GNS construction from a decomposition of a element of $\mathbb{P}_1$. The above discussion show Gelfand-Raikov theorem.

Next step, we elaborate Krein-Millman theorem. I mean for each $\varphi \in \mathbb{P}_1$, there is a probability measure $\mu \in P(\mathbb{P}_1)$ such that there is $Y \subset ex(\mathbb{P}_1)$ which supports μ and

$$\varphi = \int_Y \varphi_x d\mu(x)$$

This is from Choquet Theorem(Theorem4.8.20).

I think our first step is to interpret the value $\varphi(g)$ in terms of inverted space and function. I mean for each $g \in G$, we interpret g as

$$f_g : \mathbb{P}_1 \ni \varphi \mapsto \varphi(g)$$

By Raikov-Godement-Yoshizawa Theorem, f_g is continuous. Because f_g is convex and concave, if we define

$$\mu_1 \prec \mu_2 :\iff \mu_1(f) \leq \mu_2(f) \text{ (for any } f \text{ which is a continuous convex function on } \mathbb{P}_1)$$

then

$$\varphi = \int_{\mathbb{P}_1} \varphi_x d\mu(x)$$

for any μ such that $\delta_\varphi \prec \mu$. As shown below, we find a mesurable subset of $\mathbb{P}_1$ which is defined by continuous strictly convex functions. If $f \in C(\mathbb{P}_1, \mathbb{R})$ is strictly convex, for any affine(convex and concave) function h which satisfies $f \leq h$,

$$\{x \in \mathbb{P}_1 | f(x) < h(x)\} \subset ex(\mathbb{P}_1)$$

It is rational to obtain the minimam function. So, we define the following upper envelope function $\tilde{f}$.

$$\tilde{f}(x) := \inf\{h(x)|f \leq h, h \in A(\mathbb{P}_1)\} \ (x \in \mathbb{P}_1)$$

Here, $A(\mathbb{P}_1)$ is the set of all continuous affine functions on $\mathbb{P}_1$. Becuase $\tilde{f}(x)$ is upper semicontinuous, $\{x \in \mathbb{P}_1|f(x) < \tilde{f}(x)\}$ is measurable. Because convex combination of countable dense subset of $\{h \in A(\mathbb{P}_1)|||h||_\infty = 1\}$ is continuous strictly convex by Hahn-Banach theorem, there is a continuous strictly convex function on $\mathbb{P}_1$. So, we find μ such that $\delta_\varphi \prec \mu$ and $\mu(f) = \mu(\tilde{f})$.

If $h \in C(\mathbb{P}_1, \mathbb{R})$ is convex, then $\tilde{-h} = -h$ by applying Hahn-Banach theorem to a convex set $\{(x, r) \in \mathbb{P}||0 \leq r \leq h(x)\}$. This can be inferred by drawing a graph of h in the 1-dimensional case. By this fact and Hahn-Banach extension theorem and Riez-Markov-Kakutani theorem, for any μ such that $\delta_\varphi \prec \mu$, there is a regular borel measure L such that $\mu \prec L$ and $L(f) = \mu(\tilde{f})$. So, if we take μ which is a maximal element of $\{\mu|\delta_\varphi \prec \mu\}$ by Zorn Lemma, $\mu(f) = \mu(\tilde{f})$.

We set $X := \{x \in \mathbb{P}_1|f(x) = \tilde{f}(x)\}$. By Theorem4.9.2, we can construct $\int_X^{\mathfrak{D}} H(x)d\mu(x)$ which is a direct integral unitary representations from $\{\Phi(x)\}_{x \in ex(X)}$. By the same way as GNS construction, we show $\int_X^{\mathfrak{D}} H(x)d\mu(x)$ is a continuous cyclic unitary with some cyclic vector v and $\varphi = (T.v, v)$. So, $\int_X^{\mathfrak{D}} H(x)d\mu(x)$ and π are isomorphic as continuous unitary repesentations.

Chapter 5

Irreducible Decomposition of Unitary Representation of Compact Group

5.1 Some facts admitted without proof

Theorem 5.1.1 (Stone Wierstrass Theorem)**.** *Let*

(S1) *X be a compact metric space.*

(S2) $A \subset C(X)$.

(A1) *A is a $\mathbb{C}$-vector subspace of $C(X)$.*

(A2) $1 \in A$.

(A3) *If $f \in A$, then $\bar{f} \in A$.*

(A4) *If $f, g \in A$, then $fg \in A$.*

(A5) *If $x \neq y \in X$, there is $f \in C(X)$ such that $f(x) \neq f(y)$.*

Then A is dense subset of $C(X)$ in uniformly convergence topology.

5.2 General topics on Bochner Integral

Definition 5.2.1 (Bochner Integral)**.** *Let*

(S1) *$(X, \mathcal{B}, \mu)$ is a measurable space.*

(S2) *Y is a Banach space.*

Then

(i) *We say $F : X \to Y$ is finite-value if there is $S \in \mathcal{B}$ such that $F(S)$ is a finite set and $F(X \setminus S) = \{0\}$ and $\mu(S) < \infty$. We set*

$$\int_X F(x) d\mu(x) = \sum_{\alpha \in F(S)} \alpha \mu(F^{-1}(\alpha))$$

(ii) *We say $F : X \to Y$ is a strong measurable if there are $\{F_n\}_{n=1}^{\infty}$ such that for each $n \in \mathbb{N}$ F_n is a finite valued and $\{F_n\}_{n=1}^{\infty}$ almost everywhere pointwise converges to F.*

(iii) *We say $F : X \to Y$ is Bochner integrable if F is strong measurable and there are $\{F_n\}_{n=1}^{\infty}$ such that for each $n \in \mathbb{N}$ F_n is a finite valued and $\{F_n\}_{n=1}^{\infty}$ almost everywhere pointwise converges to F and*

$$\int_X F(x) d\mu(x) := \lim_{n \to \infty} \int_X F_n(x) d\mu(x)$$

exists.

Because of the definition of Bochner integral, the following clearly holds.

Proposition 5.2.2. *Let*

(S1) $(X, \mathcal{B}, \mu)$ *is a measurable space.*

(S2) Y, Z *is a Banach space.*

(S3) $F : X \to Y$ *is Bochner integrable.*

(S3) $T : Y \to Z$ *is bounded linear.*

Then $T \circ F$ *is Bochner integrable and*

$$T \int_X F(x) d\mu(x) = \int_X T \circ F(x) d\mu(x)$$

Proposition 5.2.3. *Let*

(S1) X *is a compact space.*

(S2) B *is a banach space.*

(S3) $F \in C(X, B)$.

(S4) μ *is a finite borel measure on* X.

Then F *is bochner integrable and*

$$|| \int_X F(x) d\mu(x) || \leq \int_X ||F(x)|| d\mu(x)$$

Proof. By (S1) and (S3), $F(X)$ is compact. So, for each $n \in \mathbb{N}$, there is a finite open covering of $F(X)$ $O(F(x_{n,i})$ $(n = 1, 2, ..\alpha(n))$ such that $O(F(x_{n,i})$ is an open neighborhood of $F(x_{n,i})$ and $O(F(x_{n,i}) \subset B(F(x_{n,i}), \frac{1}{n})$. We can assume that for each $n \in \mathbb{N}$ and each $i \in \{1, ..., \alpha(n+1)\}$ there is $j \in \{1, ..., \alpha(n)\}$ such that $O(F(x_{n+1,i})) \subset O(F(x_{n,j}))$.

$$F_n(x) := \begin{cases} F(x_{n,1}) & x \in F(X) \cap B(F(x_{n,1}), \frac{1}{2^n}) \\ F(x_{n,i+1}) & x \in F(X) \cap (B(F(x_{n,i+1}), \frac{1}{2^n}) \setminus \cup_{j=1}^{i} B(F(x_{n,j}), \frac{1}{2^n})) \end{cases}$$

Clearly, for any $n \in \mathbb{N}$, F_n is finite valued and

$$||F_n(x) - F(x)|| < \frac{1}{2^n}$$

and

$$|| \int_X F_n(x) d\mu(x) - \int_X F_{n+1}(x) d\mu(x) || < \frac{1}{2^n} \mu(X)$$

So,

$$\lim_{n \to \infty} F_n(x) = F(x) \ (\forall x \in X)$$

and by (S2)

$$\lim_{n \to \infty} \int_X F_n(x) d\mu(x)$$

exists. □

5.3 General topics on Compact self-adjoint Operator

Definition 5.3.1 (Compact operator)**.** *Let*

(S1) W *is a normed linear space.*

(S2) V *is a Banach space.*

We say $T : W \to V$ *is a compact operator if* T *is linear and* $T(B(0, 1))$ *is a relative compact. We denote the set of all compact operator on* V *by* $B_0(W, V)$.

Proposition 5.3.2. *Let*

(S1) W *and* V *and* U *are normed linear space.*

Then

(i) If V is a Banach space, then $B_0(W,V)$ is a closed subspace of $B(W,V)$.

(ii) If $T \in B_0(W,V)$ and W_0 which is a linear subspace of W , then T_{W_0} is a compact operator.

(iii) If $T \in B(W,V)$ and $dim(ImT) < \infty$, T is a compact operator.

(iv) If $T \in B_0(W,V)$ and $S \in B(V,U)$, then $S \circ T$ is a compact operator.

(v) If $T \in B_(W,V)$ and $S \in B_0(V,U)$, then $S \circ T$ is a compact operator.

Proof of (i). Let us fix any $\{F_n\}_{n=1}^{\infty} \subset B_0(W,V)$ such that $F := \lim_{n\to\infty} F_n$ exists. Let us fix any $\{x_n\}_{n=1}^{\infty} \subset B(0,1)$. It is enough to show there is a subsequence $\{F(x_{\varphi(n)})\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} F(x_{\varphi(n)}$ exists. Because $\{F_n\}_{n=1}^{\infty} \subset B_0(W,V)$, there are subsequence $\{x_{\varphi_n(k)}\}_{k=1}^{\infty}$ $(n = 1,2,...)$ such that fo reach $n \in \mathbb{N}$ $\{x_{\varphi_{n+1}(k)}\}_{k=1}^{\infty}$ is a subsequence of $\{x_{\varphi_{n+1}(k)}\}_{k=1}^{\infty}$ and

$$||F_n(x_{\varphi_n(k)}) - F_n(x_{\varphi_n(l)})|| < \frac{1}{n} \ (\forall k,l \geq n)$$

We set

$$\psi(n) := \varphi_n(n) \ (n \in \mathbb{N})$$

Let us fix any $\epsilon > 0$. There is $n_0 \in \mathbb{N}$ such that

$$||F_k - F|| < \frac{\epsilon}{4} \ (\forall k \geq n_0)$$

and $\frac{1}{n_0} < \frac{\epsilon}{2}$. Let us fix any $k,l \geq n_0$. Then $\psi(k) = \varphi_k(k)$ and $\psi(l) = \varphi_l(l)$ and $k_0 \geq n_0$ and $l_0 \geq n_0$ and $\psi(k) = \varphi_{n_0}(k_0)$ and $\psi(l) = \varphi_{n_0}(l_0)$. So,

$$||F(x_{\psi(k)}) - F(x_{\psi(l)})|| \leq ||F_{n_0}(x_{\psi(k)}) - F_{n_0}(x_{\psi(l)})|| + \frac{\epsilon}{2} = ||F_{n_0}(x_{\varphi_{n_0}(k_0)}) - F_{n_0}(x_{\varphi_{n_0}(l_0)})|| + \frac{\epsilon}{2} \leq \epsilon$$

So, $\{F(x_{\psi(k)})\}_{k=1}^{\infty}$ is a cauchy sequence. Because V is Hilbert space, $\lim_{k\to\infty} F(x_{\psi(k)})$ exists. □

Proposition 5.3.3. *Let*

(S1) V is an inner product space.

(A1) $T \in B_0(V,V)$.

(A2) There is α which is a nonzero eigenvalue of T.

Any W which is eigenspace of α is finite dimensional.

Proof. Then there is a orthonormality $\{v_i\}_{i=1}^{\infty} \subset W$. Because $\frac{1}{\alpha}T$ is a compact operator, $\frac{1}{\alpha}TW = \{w \in W | ||w|| = 1\}$ is compact. By Proposition2.5.19, W has finite dimension. □

Lemma 5.3.4. *Let*

(S1) V is a Hilbert space.

(A1) T is a self adjoint operator from V to V.

(A2) $(Ku,u) = 0$ $(\forall u \in V)$.

Then $K = 0$

Proof. Let us fix any $v \in V$. We set $w := v + Kv$.

$$0 = (Kw,w) = (Kv + K^2v, v + Kv) = 2||Kv||^2$$

So, $||Kv|| = 0$. This implies $Kv = 0$. □

Lemma 5.3.5. *Let*

(S1) V is a Hilbert space.

(A1) T is a self adjoint compact operator from V to V.

(A2) $\lambda_+ := \sup_{v\in V, ||v||=1}(Kv,v) > 0$.

Then there is a $u_0 \in V$ such that

$$\lambda_+ = (Ku_0,u_0), Ku_0 = \lambda_+ u_0$$

Proof. Then there is $\{v_i\}_{i=1}^{\infty}\{v \in V| \ ||v|| = 1\}$ such that

$$\lim_{i\to\infty}(Kv_i, v_i) = \lambda_+$$

By Proposition2.5.20, we can assume there is $v_0, u_0 \in V$ such that

$$w - \lim_{i\to\infty} v_i = v_0$$

and

$$\lim_{i\to\infty} Kv_i = u_0$$

We will show $(Kv_0, v_0) = \lambda_+$.

$$\begin{aligned} & (Kv_0, v_0) = (Kv_i, v_i) + (Kv_i - u_0, v_0 - v_i) + (u_0, v_0 - v_i) + (Kv_0, v_0) - (Kv_i, v_0) \\ = \ & (Kv_0, v_0) = (Kv_i, v_i) + (Kv_i - u_0, v_0 - v_i) + (u_0, v_0 - v_i) + (v_0, Kv_0) - (v_i, Kv_0) \\ \to \ & \lambda_+ \ (i \to \infty) \end{aligned}$$

Let us fix $v \in V$ such that $||v|| = 1$. We set

$$f(t) := (Kv(t), v(t)), \ v(t) := \frac{v_0 + tv}{||v_0 + tv||} \ (|t| \ll 1)$$

then

$$f(t) = \frac{(Kv_0, v_0) + 2tRe(Kv_0, v) + t^2(Kv, v)}{||v_0||^2 + 2tRe(v_0, v) + t^2||v||^2}$$

So,

$$f(t)(||v_0||^2 + 2tRe(v_0, v) + t^2||v||^2) = (Kv_0, v_0) + 2tRe(Kv_0, v) + t^2(Kv, v)$$

Because $f(0) = \lambda_+$ and $f'(0) = 0$,

$$\lambda_+ Re(v_0, v) = Re(Kv_0, v)$$

And

$$\lambda_+ Re(v_0, iv) = Re(Kv_0, iv)$$

These imply

$$\lambda_+(v_0, v) = (Kv_0, v)$$

This means

$$Kv_0 = \lambda_+ v_0$$

□

The following Proposition clealy holds.

Proposition 5.3.6. *Let*

(S1) T is a self-adjoint continuous linear operator of Hilbert space V.

Then

(i) Any eigenvalue of P is a real number.

(ii) If $\alpha_1, \alpha_2 \in \mathbb{R}$ are different eigenvalues of P, $V_{\alpha_1} \perp V_{\alpha_2}$. Here V_{α_i} is the eigenvalue space of α_i $(i = 1, 2)$.

(iii) If (π, V) is a continuous representation of a topological group G and W is a G-invariant subspace of V, then $W^{\perp}$ is a G-invariant.

Lemma 5.3.7. *Let*

(S1) V is a Hilbert space.

(A1) T is a compact self adjoint operator from V to V.

(S2) $\sigma_+(T)$ is the set of all positive eigenvalues of G.

Any assumulation point of $\sigma_+(T)$ is zero.

Proof. If $\#\sigma_+(T) = \infty$, then there is no accumulation points of $\sigma_+(T)$. So, hereafter, we assume $\#\sigma_+(T) = \infty$. By Proposition5.3.5 and Proposition5.3.3, there is a sequence of positive eigenvalue $\lambda_1 > \lambda_2 > .... > 0$ and $\{v_i\}_{i=1}^{\infty} \subset V$ such that v_i is an eigenvector of λ_i $(i = 1, 2, ...)$ and $\lim_{i\to\infty} Kv_i$ exists.

$$\lambda_i^2 \leq \lambda_i^2 + \lambda_{i+1}^2 = ||Kv_i - Kv_{i+1}||^2 \to 0 \ (i \to \infty)$$

□

Lemma 5.3.8. *Let*

(S1) V is a Hilbert space.

(A1) T is a compact self adjoint operator from V to V.

(S2) V_+ is the minimum closed subspace of V such that V_+ contains all eigenspaces whose eigenvalue is positive. V_- is the minimum closed subspace of V such that V_+ contains all eigenspaces whose eigenvalue is negative.

Then

$$V = V_+ \oplus Ker(T) \oplus V_-$$

Proof. We set $V_* := (V_+ \oplus Ker(T) \oplus V_-)^{\perp}$. Because $(V_+ \oplus Ker(T) \oplus V_-)$ is T-invariant and T is self-adjoint, V_* is T-invariant. By Proposition5.3.5, $(Tv, v) = 0$ $(\forall v \in V_*)$. By Proposition5.3.4, $T|V_* = 0$. So, $V_* = \{0\}$. □

5.4 Matrix coefficient and Character of representation

Definition 5.4.1 (Character)**.** *Let G be a topological group and (π, V) be a finite dimensional continuous representation of G. Then*

$$\chi_\pi(g) := Trace\pi(g) \ (g \in G)$$

We call χ_π a character of π.

Definition 5.4.2 (Matrix Coefficient)**.** *Let G be a topological group and (π, V) be a finite dimensional irreducible continuous representation of G and let $v \in V$ and $f \in V^*$.*

$$\Phi_\pi(v, f)(g) := f(\pi(g)^{-1}v)$$

Because π is a continuous representation, $\Phi_\pi(v, f)$ is a continuous function on G.

The following clearly holds.

Proposition 5.4.3. *We succeed notations in Definition5.4.2. Then Φ_π is a bilinear form on $\mathbb{C}$.*

Proposition 5.4.4. *Let*

(S1) G is a topological group.

(S2) (π, V) is a finite dimensional unitary representation of G.

(S3) $\{v_1, v_2, ..., v_m\}$ is an orthonomal basis of V.

(S4) $\pi_{i,j}(g) := (\pi(g)v_j, v_i)$ $(g \in G, i, j \in \{1, 2, ..., m\})$

then

(i) $\chi_\pi = \sum_{i=1}^m \pi_{i,i}$.

(ii) $\pi_{i,j}(gh) = \sum_{k=1}^m \pi_{i,k}(g)\pi_{k,j}(h)$ $(\forall g, h \in G, \forall i, j \in \{1, 2, ..., m\})$.

(iii) $\pi_{i,j}(g^{-1}) = \overline{\pi_{j,i}(g)}$ $(\forall g \in G, \forall i, j \in \{1, 2, ..., m\})$.

Proof of (i). It is clear. □

Proof of (ii).

$$\pi_{i,j}(gh) = (\pi(gh)v_j, v_i) = (\pi(g)\pi(h)v_j, v_i) = (\pi(g)(\sum_{k=1}^m(\pi(h)v_j, v_k)v_k), v_i) = \sum_{k=1}^m(\pi(g)v_k, v_i)(\pi(h)v_j, v_k)$$

$$= \sum_{k=1}^m \pi_{i,k}(g)\pi_{k,j}(h)$$

□

Proof of (iii).

$$\pi_{i,j}(g^{-1}) = (\pi(g^{-1})v_j, v_i) = (v_j, \pi(g)v_i) = \overline{(\pi(g)v_i, v_j)} = \overline{\pi_{j,i}(g)}$$

□

5.5 Schur orthogonality relations

Proposition 5.5.1. *Let*

(S1) G is a compact Lie group.

(S2) (π_i, V_i) is a continous unitary representation of G on $\mathbb{C}$ $(i = 1, 2)$.

(S3) $f \in Hom_{\mathbb{C}}(V_1, V_2)$.

(S4) We set $\tilde{f}$ by

$$\tilde{f}(v) := \int_G \pi_2(g) \circ f \circ \pi_1(g)^{-1}(v)dg \ (v \in V_1)$$

Then $\tilde{f} \in Hom_G(V_1, V_2)$.

Proof. By Proposition5.2.3, $\tilde{f}(v)$ exists and

$$||\tilde{f}(v)|| \leq \int_G ||\pi_2(g)f\pi_1(g^{-1})v||dg$$

Because π_1 and π_2 are unitary representation,

$$\int_G ||\pi_2(g)f\pi_1(g^{-1})v||dg \leq \int_G ||\pi_2(g)f\pi_1(g^{-1})||||v||dg \leq \int_G ||f||||v||dg \leq ||f||||v||$$

So $\tilde{f}$ is continuous linear map. Becuase dg is a Haar measure on G, clearly, $\tilde{f}$ is G-invariant. □

Proposition 5.5.2 (Shur orthogonality relations)**.** *Let*

(S1) G is a compact Lie group.

(S2) (π_i, V_i) is a continous irreducible representation of G on $\mathbb{C}$ $(i = 1, 2)$.

(A1) Either V_1 or V_2 is finite dimensional.

(S3) $(u_i, v_i) \in V_i$ $(i = 1, 2)$.

Then

$$(\Phi(u_1, v_1), \Phi(u_2, v_2))_{L^2(G)} = \begin{cases} 0 & (\pi_1 \not\simeq \pi_2) \\ dimV(Tu_1, u_2)\overline{(Tv_1, v_2)} & (\pi_1 \not\simeq \pi_2) \end{cases}$$

Here T is a unitary G-isomorphism from V_1 to V_2.

STEP1. Case when $\pi_1 \not\simeq \pi_2$. We set $f \in Hom_{\mathbb{C}}(V_1, V_2)$ by

$$f(v) := (v, v_1)v_2 \ (v \in V_1)$$

Proposition5.5.1, $\tilde{f} \in Hom_G(V_1, V_2)$ exists. In this case, by Shur Lemma, $\tilde{f} = 0$.

$$0 = (\tilde{f}(u_1), u_2) = \int_G (\pi_2(g)f\pi_1(g)^{-1}u_1, u_2)dg = \int_G (f\pi_1(g)^{-1}u_1, \pi_2(g)^{-1}u_2)dg = \int_G (v_2, \pi_2(g)^{-1}u_2)dg$$
$$= \int_G (\pi_1(g)^{-1}u_1, v_1)(v_2, \pi_2(g)^{-1}u_2)dg = \int_G (\pi_1(g)^{-1}u_1, v_1)\overline{(\pi_2(g)^{-1}u_2, v_2)}dg$$

□

STEP2. Case when $\pi_1 \simeq \pi_2$. In this case, by Shur Lemma, there is $\lambda \in \mathbb{C}$ such that $T^{-1} \circ \tilde{f} = \lambda id_{V_1}$. By the argument in STEP1,

$$(\Phi(u_1, v_1), \Phi(u_2, v_2))_{L^2(G)} = \lambda(Tu_1, u_2)$$

And

$$Trace(T^{-1} \circ \tilde{f}) = \lambda dimV_1$$

By Proposition5.2.2 and T^{-1} is G-invariant,

$$T^{-1} \circ \tilde{f} = \widetilde{T^{-1}f}$$

So,

$$Trace(T^{-1} \circ \tilde{f}) = Trace(T^{-1}f) = T^{-1}f(\frac{v_1}{||v_1||}) = (T^{-1}(\frac{v_1}{||v_1||}, v_1)v_2, \frac{v_1}{||v_1||}) = ||v_1||(T^{-1}v_2, \frac{v_1}{||v_1||})$$
$$= (T^{-1}v_2, v_1) = (v_2, Tv_1) = \overline{(Tv_1, v_2)}$$

So,

$$(\Phi(u_1, v_1), \Phi(u_2, v_2))_{L^2(G)} = (Tu_1, u_2)\overline{(Tv_1, v_2)}$$

□

By Shur orthogonality Relations, the following three holds.

Proposition 5.5.3. *Let*

(S1) G *is a compact Lie group.*

(S2) $R(G) := \left\langle \{\Phi_\pi(u,v)|\ (\pi,V)\in \hat{G}_f, u,v\in V\}\right\rangle$. *Here,* $\hat{G}_f$ *is the set of all finite dimensional irreducible continous unitary representations of* G.

Then

(i) *Let* $\{u_i\}_{i=1}^{dimV}$ *is a orthonormality base of* V. *For any* $(\pi,V)\in\hat{G}_f$,

$$\{\frac{1}{\sqrt{dimV}}\Phi_\pi(u_i,u_j)|\ i,j=1,2,...,dimV\}$$

is a basis of $\Phi(V,V^*)$.

(ii) *The following is well-defined.*

$$\Phi_\pi(u\otimes v) := \Phi_\pi(u,v)$$

(iiii) *The following holds.*

$$R(G) = \oplus_{(\pi,V)\in\hat{G}_f}\Phi_\pi(V\otimes V^*)$$

Proposition 5.5.4. *Let*

(S1) G *is a compact Lie group.*

(S2) (π,V) *is a finite irreducible continuos representation of* G *and* χ_π *is the character of* π.

Then

$$(\chi_\pi,\chi_\pi) = 1$$

Proposition 5.5.5. *Let*

(S1) G *is a compact Lie group.*

(S2) (π_i,V_i) *are two finite irreducible continuos representation of* G *and* χ_{π_i} *is the character of* π_i $(i=1,2)$.

(A1) $\chi_{\pi_1} = \chi_{\pi_2}$.

Then

$$\pi_1 \simeq \pi_2$$

5.6 Orthogonal projection by character

Proposition 5.6.1. *We succeed notations in Definition5.4.1.*

(i) χ_π *is continuous.*

(ii) *If* $\pi_1\simeq\pi_2$ *then* $\chi_{\pi_1} = \chi_{\pi_2}$.

(iii) $\chi_\pi(gxg^{-1}) = \chi_\pi(x)$ $(\forall g,x\in G)$.

(iv) $\chi_\pi(g^{-1}) = \chi_{\pi^*}(g)$ $(\forall g\in G)$.

Proof of (i). (i) is from Proposition5.6.1. □

Proof of (ii). Let us take $T:(\pi_1,V_1)\to(\pi_2,V_2)$ be a G-isomorphism. Then $T\circ\pi_1 = \pi_2\circ T$. This means $T\circ\pi_1\circ T^{-1} = \pi_2$. So, $\chi_{\pi_1} = \chi_{\pi_2}$. □

Proof of (iii). For any $g,x\in G$,

$$\chi_\pi(gxg^{-1}) = Trace(\pi(gxg^{-1}) = Trace(\pi(g)\pi(x)\pi(g)^{-1}) = Trace(\pi(x)) = \chi_\pi(x)$$

□

Proof of (iv). For any $g\in G$,

$$\chi_\pi(g^{-1}) = Trace(\pi(g^{-1})) = Trace({}^t\pi(g^{-1})) = \chi_{\pi^*}(g)$$

□

Definition 5.6.2 (τ-component)**.** *Let*

(S1) G is a topological group.

(S2) (π, V) is a continous representation of G.

(S3) (τ, W) is a continous irreducible representation of G.

We set

$$V_\tau := \sum_{A \in Hom_G(W,V)} ImA$$

We call this τ-component of V.

Proposition 5.6.3. *We succeed settings in Definition5.6.2. And if $dimW < \infty$, for any $A \in Hom_G(W,V)$, $ImA = \{0\}$ or $A : (\tau, W) \to (\pi|ImA, ImA)$ is G-isomorphism.*

Proof. Let us assume $ImA \neq \{0\}$. Because W is irreducible, $Ker(A) = \{0\}$. And, because A is G-linear, $Im(A)$ is G-invariant. So, A is bijective and A is G-linear and A^{-1} is G-linear. Because $Im(A)$ is finite dimensional, A^{-1} is continuous. So, $A : (\tau, W) \to (\pi|ImA, ImA)$ is G-isomorphism. □

Definition 5.6.4 (Projection by character)**.** *Let*

(S1) G is a compact Lie group.

(S2) (τ, V) is a continuous finite dimensional unitary representation of G.

We set

$$P_{\pi,\tau}(v) := P_\tau(v) := dim\tau \int_G \overline{\chi_\tau(g)}\tau(g)vdg$$

We call P_τ the projection by τ.

Lemma 5.6.5. *Let*

(S1) G is a compact Lie group.

(S2) (τ, W) is a continuous finite dimensional irreducible unitary representation of G.

(S2) (π, V) is a continuous finite dimensional unitary representation of G.

then $ImP_\tau \subset V_\tau$.

Proof. By Proposition2.6.21, there is $\pi_1, ..., \pi_n \in \hat{G}_f$ such that

$$\pi = \oplus_{i=1}^n \pi_i$$

This implies that

$$P_{\pi,\tau} = \sum_i P_{\pi_i,\tau}$$

Let us fix any $i \in \{1, 2, ..., n\}$. By Shur orthogonality relation, if $\tau \not\simeq \pi_i$, $P_{\pi_i,\tau} = 0$. If there is $T : (\tau, W) \to (\pi_i, V_i)$ which is an unitary map and G-isomorphism. Let us take $w_1, ..., w_m$ which is a orthonomality basis of W. By Shur orthogonality relation, for any j,

$$P_{\pi_i,\tau}(Tw_j) = \dim\tau \int_G \overline{\chi_\tau(g)}\pi_i(g)Tw_jdg = \dim\tau \sum_{k,l} \int_G \overline{(\tau(g)w_k, w_k)}(\pi_i(g)Tw_j, Tw_l)Tw_ldg$$

$$= \dim\tau \sum_{k,l} \int_G \overline{(\pi_i(g)Tw_k, Tw_k)}(\pi_i(g)Tw_j, Tw_l)Tw_ldg = Tw_j$$

So, $P_{\pi_i,\tau} = id_{V_i}$. By this, $P_{\pi_i,\tau}(V_i) = ImT \subset V_\tau$. □

Lemma 5.6.6. *Let*

(S1) G is a compact Lie group.

(S2) (τ, W) is a continuous finite dimensional irreducible unitary representation of G.

(S3) (τ', W) is a continuous finite dimensional irreducible unitary representation of G.

(S4) (π, V) is a continuous finite dimensional unitary representation of G.

(A1) $(\tau, W) \not\simeq (\tau', W)$.

then $P_\tau|V'_\tau = 0$.

5.7 Peter-Weyl theorem

5.7.1 Irreducible decomposition

Theorem 5.7.1. *Let*

- *(S1)* G *is a compact Lie group.*
- *(S2)* (π, V) *is a continous finite dimensional representation of* G*.*
- *(S3)* $(\cdot, \cdot)$ *is an inner product of* V*.*

Then

- *(i)* (π, V) *is a unitary representation with respect to the following inner product*

$$(u, v)_\pi := \int_G (\pi(g)u, \pi(g)v)dg$$

  Here, dg *is a Haar measure on* G*. By Proposition3.6.13, there is a Haar measure on* G*.*
- *(ii)* (π, V) *is irreducible* $\iff$ $(\pi, V, (\cdot, \cdot)_\pi)$*.*
- *(ii) If* π' *is a continuous representation of* G *such that* π *and* π' *are equivalent as continuous representations,* $(\pi, V, (\cdot, \cdot)_\pi)$ *and* $(\pi', V', (\cdot, \cdot)_{\pi'})$ *are equivalent as unitary representations.*

Proof. Because G is unimodular and $C(G) \subset L^\infty(G)$, (i) holds. Because $(\cdot, \cdot)$ and $(\cdot, \cdot)_\pi)$ are equivalent, (ii) holds. □

The following Proposition clealy holds.

Proposition 5.7.2. *Let*

- *(S1)* G *is a topological group.*
- *(S2)* (π, V) *is a continous finite dimensional representation of* G*.*
- *(S3)* $P \in Hom_G(V, V)$*.*

Then

- *(i) Any eigenvalue space of* P *is* G*-invariant.*
- *(ii)* ImP *is* G*-invariant.*

Proposition 5.7.3. *Let*

- *(S1)* G *is a compact Lie group.*
- *(S2)* $(\pi, V, (\cdot, \cdot))$ *is a unitary representation of* G*.*
- *(S3)* $v_0 \in V$ *and* $||v_0|| = 1$
- *(S4)* $P : V \ni v \to (v, v_0)v_0 \in V$
- *(S5)* $\Phi : G \ni g \to \pi(g)P\pi(g)^* \in B_0(V)$*.*

Then

- *(i)* Φ *is a continuous. And for any* $g \in G$*,* $\Phi(g)$ *is self adjoint.*
- *(ii)* Φ *is Bochner integrable with respact to a Haar measre on* G*.*
- *(iii)* $\tilde{P} := \int_G \Phi(g)dg$ *is* G*-invariant.*
- *(iv)* $\tilde{P}$ *is a self-adjoint compact operator.*
- *(v)* $\tilde{P}$ *is a nonzero map.*
- *(vi) There is* $\lambda \neq 0$ *such that eigenspace of* $\tilde{P}$ *with respect to* λ *is not zero.*

Proof of (i). For any $v \in V$ and $g, h \in G$

$$\begin{aligned}
&||\pi(g)P\pi(g)^*v - \pi(h)P\pi(h)^*v|| = ||\pi(g)(\pi(g)^*v, v_0)v_0 - \pi(h)(\pi(h)^*v, v_0)v_0|| \\
= \ & ||(v, \pi(g^{-1})v_0)\pi(g)v_0 - (v, \pi(h^{-1})v_0)\pi(h)v_0|| \\
= \ & ||(v, \pi(g^{-1})v_0)\pi(g)v_0 - (v, \pi(h^{-1})v_0)\pi(g)v_0|| + ||(v, \pi(h^{-1})v_0)\pi(g)v_0 - (v, \pi(h^{-1})v_0)\pi(h)v_0|| \\
\leq \ & ||v||||\pi(g^{-1})v_0 - \pi(h^{-1})v_0||||\pi(g)v_0|| + ||v||||\pi(h^{-1})v_0||||\pi(g)v_0 - \pi(h)v_0|| \\
= \ & ||v||(||\pi(g^{-1})v_0 - \pi(h^{-1})v_0|| + ||\pi(g)v_0 - \pi(h)v_0||)
\end{aligned}$$

So Φ is continuous. By Proposition, for any $g \in G$, $\Phi(g)$ is compact. Because P is self-adjoint and $\pi(g)$ is unitary operator, $\Phi(g)$ is self adjoint. □

Proof of (ii). This is from Proposition5.2.3 and (i). ☐

Proof of (iii). Let us fix any $h \in G$ and $v, u \in V$. By Proposition5.2.2,

$$(\pi(h)\int_G \pi(g)P\pi(g)^* dgv, u) = \int_G (\pi(h)\pi(g)P\pi(g)^* v, u)dg = \int_G (\pi(hg)P\pi(hg)^{-1}\pi(h)v, u)dg$$
$$= \int_G (\pi(g)P\pi(g)^{-1}\pi(h)v, u)dg = (\int_G \pi(g)P\pi(g)^{-1}dg\pi(h)v, u)$$

So, $\pi(h)\tilde{P} = \tilde{P}\pi(h)$ ☐

Proof of (iv). By the simila argument to the proof of (iii), $\tilde{P}$ is self-adjoint. By the argument of proof of Proposition5.2.3, $\tilde{P} \in \overline{B_0(V)}$. By Proposition5.7.1, $\tilde{P} \in B_0(V)$. ☐

Proof of (v).

$$(\int_G \pi(g)P\pi(g)^* dgv_0, v_0) = \int_G (\pi(g)P\pi(g)^* v_0, v_0)dg = \int_G (P\pi(g)^* v_0, \pi(g)^* v_0)dg$$
$$= \int_G (P^* P\pi(g)^* v_0, \pi(g)^* v_0)dg = \int_G (P^* P\pi(g)^* v_0, \pi(g)^* v_0)dg = \int_G (P\pi(g)^* v_0, P\pi(g)^* v_0)dg = \int_G ||P\pi(g)^* v_0||^2 dg$$

Because $||P\pi(e)^* v_0||^2 = 1$, $\int_G ||P\pi(g)^* v_0||^2 dg > 0$. ☐

Proof of (vi). By (v) and Lemma5.3.8, (vi) holds. ☐

In the following proposition, we give a proof for the general case as well as for the finite group case. The proof of the finite group case shown here follows the same policy as the proof of the general case, but uses only knowledge of linear algebra. Therefore, this proof has the advantage that the essence of the proof of the general case can be easily understood. Note that the finite group case can be easily shown from the fact that $< \pi(G)v >$ is finite dimensional G-invariant subspace for any vector v, apart from the proof given below.

Proposition 5.7.4. *Let*

(S1) G is a compact Lie group.

(S2) $(\pi, V, (\cdot, \cdot))$ is a unitary representation of G.

Then there is a finie irreducible G-subspace of V.

Proof in general case. By (v) of Proposition5.7.3, this Proposition holds. ☐

We will show a proof that does not knowledge of bochner integrals and self-adjoint compact operators in the case when G is a finite group.

Proof in the case when G is a finite group. We will succeed notations the proof of Proposition5.7.3 . Then

$$\tilde{P} = \sum_{g \in G} \pi(g^{-1}) \circ P \circ \pi(g)$$

For any $h \in G$,

$$\tilde{P} \circ \pi(h) = \sum_{g \in G} \pi(g^{-1}) \circ P \circ \pi(gh) = \sum_{g \in G} \pi(h) \circ \pi(gh^{-1}) \circ P \circ \pi(gh) = \pi(h) \circ \tilde{P}$$

So, $\tilde{P}$ is G linear.

For each $g \in G$, $\pi(g^{-1}) \circ P \circ \pi(g)$ is finite rank operator. So, $\tilde{P}$ is G finite rank operator. Then $\{v_1, .., v_m\}$ such that $\sum_{i=1}^m \mathbb{C}\tilde{P}(v_i) = Im(\tilde{P})$. Let us fix $\{w_1, ..., w_n\}$ which is an orthonormal basis of $Im(\tilde{P}) + \sum_{i=1}^n w_i$. Because $\tilde{P}| \sum_{i=1}^m w_i$ is not zero, $\tilde{P}| \sum_{i=1}^n w_i$ has nonzero eiggenvalue $\lambda \neq 0$.

For any $u \in Ker(\tilde{P} - \lambda I)$,

$$u = \frac{1}{\lambda}(\tilde{P}u) = \frac{1}{\lambda}\sum_{i=1}^m (\tilde{P}u, u_i)u_i$$

So,

$$Ker(\tilde{P} - \lambda I) \subset \sum_{i=1}^n \mathbb{C}u_i$$

These imply that $Ker(\tilde{P} - \lambda I)$ is finite dimensional G-invariant subspace. ☐

By Proposition5.7.4 and the same argument as the proof of Proposition4.3.8, the following holds.

Theorem 5.7.5 (Peter-weyl theorem I). *Let (π, V) be a continuous unitary representation of a compact Lie group G. Then there is $\mathcal{W}$ which is a subset of G-invariant finite dimensional irreducible subspaces such that*

$$V = \overline{\bigoplus_{W \in \mathcal{W}} W}$$

In specail, if π is irreducible, $dim(\pi) < \infty$.

5.7.2 Orthonormal basis of $L^2(G)$

Proposition 5.7.6. *Let*

(S1) G is a compact Lie group.

(S2) $(\pi, V, (\cdot, \cdot))$ is a finite dimensional unitary representation of G.

Then

$$\{\Phi_\pi(u, v) | u, v \in V\}$$

is $G \times G$-invariant subspace of $L^2(G)$.

Proof. For any $x, y, g \in G$,

$$L_x \times R_y \Phi_\pi(u, v)(g) = (\pi(xgy^{-1})^{-1}u, v) = (\pi(g)^{-1}\pi(x)^{-1}u, \pi(y)^{-1}v) = \Phi_\pi(\pi(x)^{-1}u, \pi(y)^{-1}v)(g)$$

So,

$$\{\Phi_\pi(u, v) | u, v \in V\}$$

is $G \times G$-invariant subspace of $L^2(G)$. □

By Proposition5.5.3, the following two holds.

Proposition 5.7.7. *Let*

(S1) G is a compact Lie group.

(S2) (π, V) is a finite dimensional G-invariant space of $L^2(G)$.

Then $V \subset \Phi_\pi(V \otimes V^*)$.

Proof. Let us fix $\{f_1, ..., f_m\}$ which is an orthonormal basis of V. Then there are $\{h_{i,n}\}_{n=1}^\infty \subset C(G)$ such that

$$\lim_{n\to\infty} ||h_{i,n} - f_i||_\infty = 0 \ (\forall i)$$

Let us fix any $\epsilon > 0$.
For any $g \in G$

$$L(g^{-1})f_i = \sum_{j=1}^m (L(g^{-1})f_i, f_j)f_j = \sum_{j=1}^m \Phi(f_i, f_j)(g)f_j$$

Then there exists n such that

$$||L(\cdot^{-1})h_{i,n}(e) - \sum_{j=1}^m \Phi(f_i, f_j)(\cdot)h_{j,n}(e)||_\infty < \frac{1}{2}\epsilon$$

and

$$||L(\cdot^{-1})h_{i,n}(e) - f_i||_{L^2(G)} < \frac{1}{2}\epsilon$$

So,

$$||f_i - \sum_{j=1}^m \Phi(f_i, f_j)(\cdot)h_{j,n}(e)||_2 < \epsilon$$

This means $f_i \in \overline{\Phi_\pi(V \otimes V^*)}$. Since $\Phi_\pi(V \otimes V^*)$ is a closed subspace of $L^2(G)$, $f_i \in \Phi_\pi(V \otimes V^*)$. □

In the proof of Proposition5.7.7, we need only the fact that $C(G)$ is dense in $L^2(G)$. Therefore, the following clearly holds.

Proposition 5.7.8. *Let*

(S1) *G is a compact Lie group.*

(S2) *(π, V) is a finite dimensional continuous unitary representation of G such that $V \subset L^2(G)$.*

(A1) *$V \cap C(G)$ is dense in V.*

Then $V \subset \Phi_\pi(V \otimes V^*)$.

Proposition 5.7.9. *Let*

(S1) *G is a compact Lie group.*

(S2) *$R(G) := \oplus_{(\pi,V)\in\hat{G}} \Phi_\pi(V \otimes V^*)$. Here $\hat{G}$ is the set of all equivalent classes of irreducible representation of G.*

Then $R(G)$ is dense in $L^2(G)$.

Proof. Be Proposition5.7.6, $R(G)^\perp$ is G-invariant. Let us assume $R(G)^\perp \neq \{0\}$. By Proposition5.7.4 and Proposition5.7.7, there are $\{f_1, ..., f_m\} \subset L^2(G)$ such that $\{f_1, ..., f_m\}$ is an orthonormality and $\langle f_1, ..., f_m\rangle$ is a irreducible G-invariant subspace and $\langle f_1, ..., f_m\rangle \subset R(G)$. So,

$$1 = (f_i, f_i) = 0$$

This is contradiction. □

Theorem 5.7.10 (Peter-Weyl Theorem II)**.** *Let*

(S1) *G is a compact Lie group.*

Then

$$\Phi : (L, \oplus_{\tau\in\hat{G}} V \otimes V^*) \to (L, L^2(G))$$

is an isomorphism as continuous unitary representaions. And $(L, V \otimes V^*)$ is isomorphic to a direct sum of $dim\tau$ of V.

Proof. The first part is directly followed from Proposition5.7.9. Let us take an orthonormal basis $\{v_1, ..., v_m\}$ of V. Then $V \otimes V^* = \oplus_{i=1}^m V \otimes (v_i)^*$ since $V \otimes (v_i)^* \perp V \otimes (v_j)^*$ for any $i \neq j$. Clearly $V \otimes (v_i)^*$ is isomorphic to V as continuous unitary representations for any i. The latter half part holds. □

Notation 5.7.11. *Let*

(S1) *G is a compact Lie group.*

(S2) *(τ, W) is an irreducible unitary representation of G.*

then we define Φ_τ, Φ'_τ, $\tilde{\Phi}_\tau$

(i) *$\Phi_\tau : W \otimes W^* \ni v \otimes w \mapsto (G \ni g \mapsto (\tau(g)v, w) \in \mathbb{C}) \in C(G)$.*

(ii) *$\Phi'_\tau := dimW\Phi_\tau$.*

(iii) *$\tilde{\Phi}_\tau := \sqrt{dimW}\Phi_\tau$.*

Proposition 5.7.12. *Let*

(S1) *G is a compact Lie group.*

(S2) *$(\tau, W) \in \hat{G}_f$.*

Then

$$(\tau_{i,j}, \tau_{k,l}) = \frac{1}{dim\tau}\delta_{i,j}\delta_{k,l}$$

Proof. Because for any $i, j \in \{1, ..., dim\tau\}$ and $g \in G$

$$\tau_{i,j}(g) = \Phi_\tau(v_i, v_j)(g^{-1})$$

by Proposition3.6.14 and Shur orthognality relation,

$$(\tau_{i,j}, \tau_{k,l}) = (\Phi_\tau(v_i, v_j), \Phi_\tau(v_k, v_l)) = \frac{1}{dim\tau}\delta_{i,j}\delta_{k,l}$$

□

By Proposition5.7.9 and Shur orthogonality relations and Proposition5.7.12, the following holds.

Theorem 5.7.13 (Peter Weyl Theorem II, matrix coefficient version). *Let*

(S1) G is a compact Lie group.

(S2) $(\tau, W) \in \hat{G}_f$.

Then

(i) The following is a completely orthonomal system of $L^2(G)$.

$$\{\sqrt{\dim\tau}\tau_{i,j}|i,j=1,2,..,\dim\tau,(\tau,W)\in\hat{G}_f\}$$

(ii) $\hat{G}$ is at most countable.

(iii) For any $f \in L^2(G)$,

$$f = dim\tau \sum_{\tau\in\hat{G}_f,i,j=1,...,dim\tau}(f,\tau_{i,j})\tau_{i,j}\ (L^2\text{-convergence})$$

Proof of (i). This is followed by Proposition5.7.9 and Shur orthogonality relations and Proposition5.7.12. □

Proof of (ii). Because $L^2(G)$ is separable, $L^2(G)$ has a countable complete orhonormal basis. So, this is followed by (i) and Peter-Weyl I and Proposition2.5.12(iii). □

Proof of (iii). This is followed by (i) and (ii) and Proposition2.5.12(ii). □

5.7.3 Uniform approximate of continuous function

Theorem 5.7.14 (Peter-Weyl Theorem III). *Let G be a compact Lie group. Then the $\mathbb{C}$-vector space generated by the following set is dense subset of $C(G)$ in uniformly convergence topology.*

$$\{(\tau(\cdot)u,v)|(\tau,V) \text{ is a continuous finite dimensional irreducible unitary representation of } G,\ u,v\in V \text{ such that } ||u||=||v||=1\}$$

Proof. By Peter-Weyl I and Proposition4.5.24,

$$ex(\mathbb{P}_1)=\{(\tau(\cdot)v,v)|(\tau,V) \text{ is a continuous finite dimensional irreducible unitary representation of } G,\ v\in V \text{ such that } ||v||=1\}$$

Because the trivial representation of G is finite dimensional irreducible, $ex(\mathbb{P}_1)$ contains 1 which is (A2) in Theorem5.1.1. Because $\varphi \in ex(\mathbb{P}_1) \implies \bar{\varphi} \in ex(\mathbb{P}_1)$, $ex(\mathbb{P}_1)$ satisfies (A3) in Theorem5.1.1. By Proposition4.5.6, $ex(\mathbb{P}_1)$ satisfies (A4) in Theorem5.1.1. By Gelfand-Raikov Theorem, $ex(\mathbb{P}_1)$ satisfies (A5) in Theorem5.1.1. So, by Theorem5.1.1, the $\mathbb{C}$-vector space generated by $ex(\mathbb{P}_1)$ is dense subset of $C(G)$ in uniformly convergence topology. □

Definition 5.7.15 (Class function). *Let G be a group and f be a function on G. We say f is a class function if*

$$f(x^{-1}gx)=f(g)\ (\forall x,g\in G)$$

We denote the set of all squared integrable class functions by $L^2(G)^{Ad}$. We denote the set of all continuous class functions by $C(G)^{Ad}$.

Clearly the following holds.

Proposition 5.7.16. *Any character of compact Lie group is a class function.*

Proposition 5.7.17. *Let G be a compact group. Then $L^2(G)^{Ad}$ is closed subset of $L^2(G)$ and $C(G)^{Ad}$ is closed subset of $C(G)$.*

Proof. Because $f(x^{-1}gx) = L_x \circ R_x f\ (\forall x,g\in G,\forall f\in C(G))$ and $L_x \circ R_x$ is continuous operator of $L^2(G)$ and $C(G)$. So, this Proposition holds. □

Proposition 5.7.18. *Let G be a compact Lie group. We set*

$$P(f)(g) := \int_G f(x^{-1}gx)dg(x)\ (g\in G)$$

then

(i) P *is the orthogonal projection of* $L^2(G)^{Ad}$.

(ii) $P(C(G)) = C(G)^{Ad}$.

(iii) $P : C(G) \to C(G)^{Ad}$ *is surjective continuous in uniform convergence topology.*

Proof of (i). Clearly $P(L^2(G)) \subset L^2(G)^{Ad}$, and $P \circ P = P$ and P is linear. For any $g, f \in L^2(G)$,

$$|(g, P(f))| = |\int_G g(x) \int_G \overline{f(y^{-1}xy)} dg(y) dg(x)| = |\int_G \int_G g(x) \overline{f(y^{-1}xy)} dg(x) dg(y)|$$
$$\leq \int_G ||g||_{L^2} ||L_y \circ R_y f||_{L^2} dg(y) = \int_G ||g||_{L^2} ||f||_{L^2} dg(y) = ||g||_{L^2} ||f||_{L^2}$$

and

$$(g, P(f)) = \int_G g(x) \int_G \overline{f(y^{-1}xy)} dg(y) dg(x) = \int_G \int_G g(x) \overline{f(y^{-1}xy)} dg(y) dg(x)$$
$$= \int_G \int_G g(yxy^{-1}) \overline{f(x)} dg(x) dg(y) = \int_G \int_G g(yxy^{-1}) dg(y) \overline{f(x)} dg(x) = \int_G \int_G g(y^{-1}xy) dg(y) \overline{f(x)} dg(x)$$
$$= (P(g), f)$$

So, P is continuous and self adjoint. Because of these result, (i) holds. ☐

Proof of (ii). Clearly $P(C(G)) \subset C(G)^{Ad}$ and and $P|C(G)^{Ad} = id|C(G)^{Ad}$. ☐

Proof of (iii). For any $f \in C(G)$, f is uniformly continuous. So, $P|C(G)$ is continuous in uniformy convergence topology. By (ii), $P|C(G)$ is surjective. So (iii) holds. ☐

Proposition 5.7.19. *We will succeed notations in Proposition5.7.18. And let* $(\tau, V) \in \hat{G}_f$. *then for any* $i, j \in \{1, 2, ..., dim\tau\}$

$$P(\tau_{i,j}) = \frac{\delta_{i,j}}{dim\tau} \chi_\tau$$

Proof. For any $g \in G$,

$$P(\tau_{i,j})(g) = \int_G \tau_{i,j}(x^{-1}gx) dg(x)$$

by Proposition5.4.4

$$= \sum_{a,b} \int_G \tau_{i,a}(x^{-1}) \tau_{a,b}(g) \tau_{b,j}(x) dg(x)$$

by Proposition5.4.4

$$= \sum_{a,b} \int_G \overline{\tau_{a,i}(x)} \tau_{a,b}(g) \tau_{b,j}(x) dg(x) = \sum_{a,b} \tau_{a,b}(g) \int_G \overline{\tau_{a,i}(x)} \tau_{b,j}(x) dg(x)$$

by Shur orthogonality relations

$$= \delta_{i,j} \frac{1}{dim\tau} \sum_{i=1}^{dim\tau} \tau_{i,i}(g) = \delta_{i,j} \frac{1}{dim\tau} \chi_\tau$$

☐

Theorem 5.7.20. *Let*

(S1) G *is a compact Lie group.*

(S2) $(\tau, W) \in \hat{G}_f$.

Then

(i) $\sum_{\tau \in \hat{G}_f} \mathbb{C}\chi_\tau$ *is dense in* $C(G)^{Ad}$.

(ii) $\{\chi_\tau | \tau \in \hat{G}_f\}$ *is an orthonomal basis of* $L^2(G)^{Ad}$.

Proof of (i). Let us fix any $f \in C(G)^{Ad}$, $\epsilon > 0$. Because P is continuous, there is $\delta > 0$ such that

$$g \in C(G) \text{ and } ||g - f||_\infty < \delta \implies ||P(g) - P(f)||_\infty < \epsilon.$$

Because $f \in C(G)^{Ad}$, $P(f) = f$. By Theorem5.7.14, there is $g \in \sum_{\tau \in \hat{G}_f} \sum_{i,j \in \{1,2,...,dim\tau\}} \mathbb{C}\tau_{i,j}$ such that $||g - f||_\infty < \delta$. By Proposition5.7.19, $P(g) \in \sum_{\tau \in \hat{G}_f} \mathbb{C}\chi_\tau$. □

Proof of (ii). Let us fix any $f \in L^2(G)^{Ad} \setminus \{0\}$. By Theorem5.7.13, there is $\tau \in \hat{G}_f$ and $i, j \in \{1, 2, ..., dim\tau\}$ such that $(f, \tau_{i,j}) \neq 0$. Because P is the orthogonal projection of $L^2(G)^{Ad}$, there is $g \in (L^2(G)^{Ad})^\perp$ such that $\tau_{i,j} = P(\tau_{i,j}) + g$. So,

$$0 \neq (f, \tau_{i,j}) = (f, P(\tau_{i,j})) = \frac{\delta_{i,j}}{dim\tau}(f, \chi_\tau)$$

This implies $(f, \chi_\tau) \neq 0$. □

5.7.4 Component of irreducible decomposition

Proposition 5.7.21. *Let*

(S1) G is a compact Lie group.

(S2) (π, V) is an continuous unitary representation of G.

(S3) $(\tau, W) \in \hat{G}_f$.

(S4) $A \in Hom_G(W, V)$.

(S5) $P_\tau(v) := \int_G \overline{\chi(g)}\tau(g)vd\mu(g)$ $(v \in W)$.

Then

$$P_\tau|ImA = id|ImA$$

Proof. By Proposition5.7.12

$$\begin{aligned} P_\tau(Aw_i) &= dim\tau \int_G \overline{\chi_\tau(g)}\pi(g)Aw_i dg = dim\tau \int_G \overline{\chi_\tau(g)}A\tau(g)w_i dg = dim\tau \sum_{j=1}^m \int_G \overline{\chi_\tau(g)}A(\tau(g)w_i, w_j)w_j dg \\ &= dim\tau \sum_{j=1}^m \int_G \overline{\chi_\tau(g)}\tau_{i,j}(g)dgAw_j = dim\tau \sum_{k=1}^m \sum_{j=1}^m \int_G \overline{\tau_{k,k}(g)}\tau_{i,j}(g)dgAw_j = Aw_i \end{aligned}$$

□

Proposition 5.7.22. *Let*

(S1) G is a compact Lie group.

(S2) $(\tau, W_1), (\pi, W_2) \in \hat{G}_f$.

then

$$\chi_\tau * \chi_\pi = \begin{cases} \dfrac{1}{dim\tau}\chi_\tau & (\tau \simeq \pi) \\ 0 & (\tau \not\simeq \pi) \end{cases}$$

Proof. For any $h \in G$,

$$\int_G \chi_\tau(g)\chi_\pi(g^{-1}h)dg = \sum_{i,j} \int_G \tau_{i,i}(g)\pi_{j,j}(g^{-1}h)dg$$

For any j,

$$\pi_{j,j}(g^{-1}h) = (\pi(g^{-1}h)v_j, v_j) = (\pi(h)v_j, \tau(g)v_j) = \sum_k \pi_{j,k}(h)(v_k, \pi(g)v_j) = \sum_k \pi_{j,k}(h)\overline{\pi_{j,k}(g)}$$

So, by Shur orthogonality relations,

$$\sum_{i,j} \int_G \tau_{i,i}(g)\pi_{j,j}(g^{-1}h)dg = \sum_{i,j,k} \tau_{j,k}(h) \int_G \tau_{i,i}(g)\overline{\pi_{j,k}(g)}dg = \delta\tau, \pi\frac{1}{dim\tau}\sum_{i=1}^{dim\tau} \tau_{i,i}(h) = \delta\tau, \pi\frac{1}{dim\tau}\chi_\tau(h)$$

□

Proposition 5.7.23. *Let*

(S1) G is a compact Lie group.

(S2) $(\tau, W), (\pi, V) \in \hat{G}$.

Then

$$P_\tau \circ P_\pi = \begin{cases} P_\tau & (\tau = \pi) \\ 0 & (\tau \not\cong \pi) \end{cases}$$

Proof. Let us fix an orthonormal basis of V. For any $v_i \in V$, by Shur orthogonality relations,

$$\begin{aligned} P_\pi(P_\tau(v_i)) &= \sum_{j=1}^{dim\pi} (dim\tau)(dim\pi) \int_G \overline{\chi_\tau(g)}\tau(g) \int_G \overline{\chi_\pi(h)}(\pi(h)v_i, v_j)v_j dhdg \\ &= \sum_{j,k} (dim\tau)(dim\pi) \int_G \overline{\chi_\tau(g)} \int_G \overline{\chi_\pi(h)}(\pi(h)v_i, v_j)(\tau(g)v_j, v_k)v_k dhdg \\ &= \sum_{j,k} (dim\tau)(dim\pi) \int_G \overline{\chi_\tau(g)} \int_G \overline{\chi_\pi(h)}\pi_{j,i}(h)\tau_{k,j}(g)v_k dhdg \\ &= \sum_{j,k,a,b} (dim\tau)(dim\pi) \int_G \overline{\tau_{a,a}(g)} \int_G \overline{\pi_{b,b}(h)}\pi_{j,i}(h)\tau_{k,j}(g) dhdg v_k \\ &= (dim\tau)(dim\pi) \sum_{j,k,a,b} (\tau_{k,j}, \tau_{a,a})(\pi_{k,j}, \pi_{a,a}) = \delta_{\tau,\pi} v_i \end{aligned}$$

□

Theorem 5.7.24. *Let*

(S1) G is a compact Lie group.

(S2) (π, V) *is an continuous unitary representation of G.*

(S3) $(\tau, W) \in \hat{G}$.

then P_τ *is the orthogonal projection of* V_τ.

Proof. By Proposition5.7.21,

$$P_\tau | V_\tau = id_{V_\tau}$$

Let us fix any $v \in V$. We will show there is V' which is a finite dimensional G-invariant subspace of V such that $P_\tau(v) \in V'$. Let us fix $\{v_1, ..., v_m\}$ which is a orthogonality basis of (τ, W). For any $x \in G$,

$$\begin{aligned} \pi(x)P_\tau(v) &= \int_G \overline{\chi_\tau(g)}\pi(xg)vdg = \int_G \overline{\chi_\tau(x^{-1}g)}\pi(g)vdg = \sum_i \int_G \overline{\tau_{i,i}(x^{-1}g)}\pi(g)vdg = \sum_i \int_G \overline{(\tau(x^{-1}g)v_i, v_i)}\pi(g)vdg \\ &= \sum_i \int_G (\tau(x)v_i, \tau(g)v_i)\pi(g)vdg = \sum_{i,j} \tau_{i,j}(x) \int_G (v_j, \tau(g)v_i)\pi(g)vdg \in \sum_{i,j} \mathbb{C} \int_G (v_j, \tau(g)v_i)\pi(g)vdg =: V' \end{aligned}$$

By Proposition5.7.23 and Proposition5.6.5, $P_\tau(v) = P_\tau(P_\tau(v)) \in P_\tau(V') \subset V'_\tau \subset V_\tau$.

Lastly, we will show $P_\tau^* = P_\tau$. Let us fix any $u, v \in V$. By Proposition3.6.14 and Proposition5.6.1,

$$\begin{aligned} (P_\tau(u), v) &= (\int_G \overline{\chi_\tau(g)}\pi(g)udg, v) = \int_G \overline{\chi_\tau(g)}(\pi(g)u, v)dg = \int_G (u, \chi_\tau(g)\pi(g^{-1})v)dg \\ &= (u, \int_G \overline{\chi_\tau(g^{-1})}\pi(g^{-1})vdg) = (u, P_\tau(v)) \end{aligned}$$

So, $P_\tau^* = P_\tau$. □

Proposition 5.7.25. *Let*

(S1) G is a compact Lie group.

(S2) (π, V) *is an continuous unitary representation of G.*

(S3) (τ, V) *is an continuous finite dimensional unitary representation of G.*

then $P_{\pi,\tau}$ *is G-linear.*

Proof. For any $x \in G$ and $v \in V$,

$$\begin{aligned} \pi(x)P_{\pi,\tau}(v) &= \int_G \overline{\chi_\tau(y)}\pi(x)\pi(y)vdg(y) = \int_G \overline{\chi_\tau(xx^{-1}yxx^{-1})}\pi(xyx^{-1})\pi(x)vdg(y) \\ &= \int_G \overline{\chi_\tau(xyx^{-1})}\pi(y)\pi(x)vdg(y) = \int_G \overline{\chi_\tau(y)}\pi(y)\pi(x)vdg(y) = P_{\pi,\tau}(\pi(x)v) \end{aligned}$$

□

Theorem 5.7.26. *Let*

(S1) G is a compact Lie group.

(S2) (π, V) is a continuous unitary representation of G.

then

$$V = \oplus_{\tau \in \hat{G}_f} V_\tau$$

Proof. By Proposition5.7.23, $V_\tau \perp V_\pi$ $(\tau \not\simeq \pi)$. So, it is enough tho show $\cap_{\tau \in \hat{G}_f} V_\tau^\perp = \{0\}$. Let us fix any $v \in \cap_{\tau \in \hat{G}_f} V_\tau^\perp$. Then for any $x \in G$ and $\tau \in \hat{G}_f$, by Proposition5.7.25,

$$0 = \int_G (P_\tau(\pi(x^{-1})\pi(x)w), w)dg(x) = \int_G (\pi(x^{-1})P_\tau(\pi(x)w), w)dg(x) == \int_G (P_\tau(\pi(x)w), \pi(x)w)dg(x)$$
$$= \int_G \int_G \overline{\chi_\tau(g)}(\pi(g)\pi(x)w, \pi(x)w)dg(g)dg(x) = (f, \chi_\tau)$$

Here,

$$f(x) := \int_G (\pi(x)\pi(g)v, \pi(g)v)dg \ (x \in G)$$

For any $x, y \in G$,

$$f(y^{-1}xy) = \int_G (\pi(y^{-1}xy)\pi(g)v, \pi(g)v)dg = \int_G (\pi(x)\pi(yg)v, \pi(yg)v)dg = f(x)$$

So, $f \in C(G)^{Ad}$. By Theorem5.7.20, $f = 0$. So, $||w||^2 = f(e) = 0$. □

5.7.5 Expansion formula of L^2 functions

Proposition 5.7.27. *Let*

(S1) G is a compact Lie group.

(S2) (τ, W) is an irreducible unitary representation of G.

Then

$$\Phi_\tau(W \otimes W^*) = L^2(G)_\tau$$

Proof. Firstly, we will show that

$$\Phi_\tau(W \otimes W^*) \subset L^2(G)_\tau$$

For each $f \in W^*$, we set $\Phi_{\tau,f} : W \to L^2(G)$ by

$$\Phi_{\tau,f}(w) := \Phi_(w, f) \ (w \in W)$$

Let us fix any $f \in W^*$. Clearly $\Phi_{\tau,f}$ is linear. By shur orthogonality relations, $\Phi_{\tau,f}$ is continous. And for any $h \in G$

$$\Phi_{\tau,f}(\tau(h)w) = f(\tau(\cdot)^{-1}\tau(h)w) = f(\tau(h^{-1}\cdot)^{-1}w) = L_h\Phi_{\tau,f}(\tau(h)w)$$

This means that $\Phi_{\tau,f}$ is G-linear. So, $\Phi_\tau(W \otimes W^*) \subset L^2(G)_\tau$.

Lastly, we will show that

$$L^2(G)_\tau \subset \Phi_\tau(W \otimes W^*)$$

Let us fix $w_1, ..., w_m \in W$ which is a basis of W and $A \in Hom_G(W, V)$. For any i and $x \in G$,

$$(Aw_i)(x) = (L_{x^{-1}}Aw_i)(e) = (A\tau(x^{-1})w_i)(e) = (A(\sum_{j=1}^m \tau(x^{-1})w_i, w_j)w_j)(e) = (A(\sum_{j=1}^m \Phi_{i,j}(x)w_j)(e)$$
$$= \sum_{j=1}^m (Aw_j)(e)\Phi_{i,j}(x)$$

So, $L^2(G)_\tau \subset \Phi_\tau(W \otimes W^*)$. □

Proposition 5.7.28. *Let*

(S1) G is a compact Lie group.

(S2) $\tau \in \hat{G}$.

for any $f \in L^2(G)$

$$P_{L,\tau}(f)(x) = dim\tau\overline{\chi_\tau} * f(x) \text{ (a.e. } x \in G)$$

Proof. For any $f \in L^2(G)$ and $a.e$ $x \in G$,

$$P_{L,\tau}(f)(x) = \int_G \overline{\chi_\tau(g)} f(g^{-1}x)dg = \int_G \overline{\chi_\tau(g^{-1})} f(gx)dg = \int_G \overline{\chi_\tau(xg^{-1})} f(g)dg = \overline{\chi_\tau} * f(x)$$

□

Proposition 5.7.29 (Operator Valued Fourier Transform)**.** *Let*

(S1) G *is a compact Lie group.*

(S2) (τ, W) *is a continuous unitary representation of* G*.*

(S3) $f \in L^2(G)$.

Then

(i) For each $w \in W$*, there is the unique element* $I(\tau, f)w$ *such that*

$$(u, I(\tau, f)w) = \int_G (u, f(g)\tau(g)w)dg_l(g) \ (\forall u \in W)$$

(ii) $I(\tau, f)$ *is bounded and* $||I(\tau, f)|| \leq ||f||_{L^2(G)}$.

Without fear of misinterpretation, we denote $I(\tau, f)$ *by* $\tau(f)$*. We call* $\hat{G} \ni \pi \mapsto I(\pi, f)$ *the operator valued fourier transform of* f*.*

Proof of (i).

$$|\int_G (u, f(g)\tau(g)w)dg_l(g)| \leq ||f||_{L^2(G)}||u|| \cdot ||w|| \ (\forall u \in W)$$

So, by Riez representation theorem, (i) holds. □

Proof of (ii). (ii) is followed by the above equation. □

Proposition 5.7.30. *Let*

(S1) G *is a compact Lie group.*

(S2) (π, V) *is a continuous unitary representation of* G*.*

(S3) $f \in L^2(G)$.

Then

(i) $\pi(f * g) = \pi(f)\pi(g)$ *is a compact Lie group.*

(ii) $\pi(R_x f) = \pi(f)\pi^*(x)$ $(\forall x \in G)$.

(iii) $\pi(L_x f) = \pi(x)\pi(f)$ $(\forall x \in G)$.

Proof of (i).

$$\pi(f * g) = \int_G f * g(x)\pi(x)dg(x) = \int_G \int_G f(xy^{-1})g(y)dg(y)\pi(x)dg(x) = \int_G \int_G f(y^{-1})g(yx)dg(y)\pi(y^{-1})\pi(yx)dg(x)$$
$$= \int_G f(y^{-1})\pi(y^{-1}) \int_G g(yx)\pi(yx)dg(x)dg(y) = \int_G f(y^{-1})\pi(y^{-1}) \int_G g(x)\pi(x)dg(x)dg(y) = \int_G f(y^{-1})\pi(y^{-1})\pi(g)dg(x)$$
$$= \int_G f(y)\pi(y)\pi(g)dg(x) = \pi(f)\pi(g)$$

□

Proof of (ii).

$$\pi(R_x f) = \int_G f(gx)\pi(g)dg(g) = \int_G f(gx)\pi(gx)\pi(x^{-1})dg(g) = \int_G f(gx)\pi(gx)dg(g)\pi^*(x) = \pi(f)\pi^*(x)$$

□

Proof of (iii).

$$\pi(L_x f) = \int_G f(x^{-1}g)\pi(g)dg(g) = \int_G f(x^{-1}g)\pi(xx^{-1}g)dg(g) = \pi(x)\int_G f(g)\pi(g)dg(g) = \pi(x)\pi(f)$$

□

(i)(ii) in Proposition5.7.30 characterize the operator valued fourier transformation. See Theorem3.1 in [28].

Proposition 5.7.31. *Let*

(S1) G is a compact Lie group.

(S2) (τ, W) is a continuous finite dimensional unitary representation of G.

Then

$$P_{L,\tau}(f) = dimW\Phi_{L,\tau}(\tau(f))\ (\forall f \in L^2(G))$$

Proof. For any $y \in G$,

$$\begin{aligned}
\Phi_\tau(\tau(f))(y) &= \sum_{i,j}\Phi_\tau((\tau(f)v_j, v_i)v_i \otimes v_j)(y) = \sum_{i,j}\int_G \Phi_\tau((f(x)\tau(x)v_j, v_i)v_i \otimes v_j)dg(x)(y) \\
&= \int_G \sum_{i,j} f(x)(\tau(x)v_j, v_i)\Phi_\tau(v_i \otimes v_j)dg(x)(y) = \int_G \sum_{i,j} f(x)\tau_{i,j}(x)(\tau(y^{-1})v_j, v_i)dg(x) \\
&= \sum_{i,j}\int_G f(x)\tau_{i,j}(x)dg(x)(\tau(y^{-1})v_j, v_i) = \sum_{i,j}\int_G f(x)\tau_{i,j}(x)dg(x)\tau_{j,i}(y^{-1}) \\
&= \sum_i \int_G f(x)\tau_{i,i}(xy^{-1}) = \sum_i \int_G f(xy)\tau_{i,i}(x)dg(x) = \int_G f(xy)\overline{\chi_\tau(x^{-1})}dg(x) \\
&= \int_G f(x^{-1}y)\overline{\chi_\tau(x)}dg(x) = \int_G L_x f\overline{\chi_\tau(x)}dg(x)(y) = \frac{1}{dim\tau}P_{L,\tau}(f)(y)
\end{aligned}$$

□

Theorem 5.7.32 (Plancherel formula for compact Lie group)**.** *Let*

(S1) G is a compact Lie group.

(S2) $f \in L^2(G)$.

then

$$f = \sum_{\tau \in \hat{G}_f} \Phi'_\tau(\tau(f))\ (L^2\ convergence)$$

We set μ by the counting measure of $\hat{G}_f$. Then

$$f = \int_{\hat{G}_f} \Phi'_\tau(\tau(f))d\mu(\tau)$$

The right side is a bochner integral on the $L^2(G)$ valued function. We call μ the Plancherel measure on $\hat{G}$.

Proof by Peter-Weyl Theorem III.. This is followed by Theorem5.7.24 and Proposition5.7.26 and Proposition. □

Proof by Peter-Weyl Theorem II.. By Proposition5.7.27 and Theorem5.7.24, $P_\tau(L^2(G)) = \Phi_\tau(V \otimes V^*)$ for any $(\tau, V) \in \hat{G}$. By Proposition5.7.5, $P_\tau(f) = \Phi'_\tau(f)$ $(\forall f \in L^2(G))$. By Peter Weyl Theorem II and Proposition2.5.17,

$$f = \sum_{\tau \in \hat{G}_f} \Phi'_\tau(\tau(f))\ (\forall f \in L^2(G))$$

□

Proposition 5.7.33. *Let*

(S1) G is a compact Lie group.

(S2) (π, V) and (τ, W) are continuous unitary representations of G.

(S3) $T : V \to W$ *is an isomorphism as continuous unitary representations of* G.

(S4) $f \in L^2(G)$.

Then

$$\pi(f) = T^{-1}\tau(f)T$$

Proof. For any $u, v \in V$,

$$(u, \pi(f)v) = (Tu, T\pi(f)v) = \int_G (Tu, Tf(g)\pi(g)v)dg = \int_G (Tu, f(g)\tau(g)Tv)dg = \int_G (u, T^{-1}f(g)\tau(g)Tv)dg$$
$$= (u, T^{-1}\tau(f)Tv)$$

□

5.7.6 Example:Fourier series expansion

By Lemma3.6.7, the following holds.

Proposition 5.7.34. *The following* μ *is a Haar measure on* S^1.

$$\mu(f) := \frac{1}{2\pi}\int_0^{2\pi} f(exp(i\theta))d\theta \ (f \in C(S^1))$$

Proposition 5.7.35. *Let*

(S1) (τ, W) *is a unitary representation of* $\mathbb{T}^1$.

Then (τ, W) *is irreducible* $\iff$ $dim\tau = 1$ *and there is* $n \in \mathbb{Z}$ *such that*

$$\tau(exp(i\theta 2\pi))v = exp(in\theta 2\pi)v \ (\forall\theta \in \mathbb{R}, \forall v \in W)$$

We denote this irreducible representation by τ_n

Proof1 of $\Longrightarrow$. By Shur Lemma, $dim\tau = 1$. Since τ is unitary, $\tau(S^1)$ can been seen as elements of S^1. By Theorem3.4.14, τ is C^ω-class. We set $f(\theta) := \tau(i\theta 2\pi)$ $(\theta \in \mathbb{R})$. Because $f(\theta + h) = f(\theta)f(h)$ $(\forall\theta, h \in \mathbb{R})$,

$$f'(\theta) = f'(0)f(\theta) \ (\forall\theta \in \mathbb{R})$$

So, taylor series of f converges on $\mathbb{R}$. This implies that there is $\alpha \in C$ such that

$$f(\theta) = exp(i\alpha\theta 2\pi) \ (\forall\theta \in \mathbb{R})$$

Because $Im(f) \subset S^1$, $\alpha \in \mathbb{R}$. Because $f(1) = 1$, $\alpha \in \mathbb{Z}$. □

Proof2 of $\Longrightarrow$ *without Theorem3.4.14.* By Shur Lemma, $dim\tau = 1$. Since τ is unitary, $\tau(S^1)$ can been seen as elements of S^1. We set

$$f(\theta) := \tau(i\theta 2\pi) \ (\theta \in \mathbb{R})$$

and

$$\psi(\theta) := exp(i\theta) \ (\theta \in (-\pi, \pi))$$

There is $\delta > 0$ such that $f((-\delta, \delta)) \subset \psi((-\pi, \pi))$ We can assume $f|(-\delta, \delta) \neq 1$. So, there is $t_0 \in (-\delta, \delta) \setminus 0$ such that $f(t_0) \neq 1$. There is $\alpha \in (-\pi, \pi)$ such that $f(t_0) = exp(i\alpha)$. Because ψ is injective,

$$f(\frac{k}{2^m}t_0) = exp(i\frac{k}{2^m}\alpha) \ (\forall m \in \mathbb{Z}_+, \forall k \in \mathbb{Z} \text{ such that } |\frac{k}{2^m}| \leq 1)$$

Because the both sides are continuous,

$$f(\theta) = exp(i\frac{\alpha}{t_0 2\pi}\theta 2\pi) \ (\forall\theta \in (-|t_0|, |t_0|))$$

We set $\beta := \dfrac{\alpha}{t_0 2\pi}$. Becuase f is homomorphism,

$$f(\theta) = exp(i\beta\theta 2\pi) \ (\forall\theta \in \mathbb{R})$$

Because $f(1) = 1$, $\beta \in \mathbb{Z}$. □

Proof of $\Longleftarrow$. It is clear. □

By Proposition5.7.35, the following holds.

Proposition 5.7.36. *Let*

(S1) τ_n *is an irreducible unitary representation of* $\mathbb{T}^1$ *for* $n \in \mathbb{Z}$.

(S2) χ_n *is the character of* τ_n.

(S3) $\tau^n_{1,1}$ *is the matrix coefficient of* τ_n.

Then

(i)

$$\tau^n_{1,1}(z) = \chi_n(z) = z^n = exp(i \cdot n \cdot arg(z)) \ (\forall z \in S^1)$$

(ii)

$$(f, \tau^n_{1,1}) = \frac{1}{2\pi}\int_0^{2\pi} f(expi\theta)exp(-in\theta d\theta = \hat{f}(n) \ (\forall f \in L^2(S^1), \forall n\mathbb{N})$$

By Peter-Weyl II and Proposition5.7.36 and Proposition2.5.12, the following holds.

Theorem 5.7.37 (Fourier expansion formula)**.** *For any* $f \in L^2([0, 2\pi])$

$$f = \lim_{N\to\infty}\sum_{n=-N}^{N}\hat{f}(n)\chi_n \ (L^2\text{-}convergence)$$

By Peter-Weyl III and Proposition5.7.36 and Proposition2.5.12, the following holds.

Theorem 5.7.38 (Wierstrass Theorem)**.** *For any* $f \in C(S^1)$ *and* $\epsilon > 0$*, there is a finite subset* $N \subset \mathbb{N}$ *and* $a_{-N}, a_{-N+1}, ..., a_N$ *such that*

$$||f - \sum_{n=-N}^{N} a_n\chi_n||_\infty < \epsilon$$

5.7.7 Characterization of compact Lie group

Theorem 5.7.39. *Let us* G *be a compact topological group. Then* G *is a Lie group* $\Longleftrightarrow$ G *has a continuous finite dimensional faithful unitary representation. In special, if* G *is a compact Lie group, then there is a* C^ω*-class diffeomorphism from* G *to some closed subgroup of* $U(n)$ *for some* $n \in \mathbb{N}$.

Proof of $\Longrightarrow$. By Proposition3.4.11, there is an open neighborhood U which does not contain subgroups without $\{e\}$. By Peter-Weyl Theorem I, for any $\tau \in \hat{G}$, $Ker(\tau)$ is closed subset of G. By Gelfand-Raikov theorem, $G = \cup_{\tau\in\hat{G}}Ker(\tau)^c \cup U$. Because G is compact, there are finite $\tau_1, ..., \tau_m \in \hat{G}_f$ such that $G = \cup_{i=1}^m Ker(\tau_i)^c \cup U$. Because U does not contain subgroups without $\{e\}$, $\cap_{i=1}^m Ker(\tau_i) = \{e\}$. Then $\oplus_{i=1}^m \tau_i$ is a continuous finite dimensional faithful unitary representation of G. □

Proof of $\Longleftarrow$. Then G is isomorphic to closed subgroup of $U(n) \subset GL(n, \mathbb{C})$ as toplogical groups for some $n \in \mathbb{N}$. So, G is Lie group. □

5.8 Review

The main theorems of this chapter are Peter-Weyl's Theorem I-III, embedding any compact Lie group into $U(n)$, Plancherel formula for compact Lie groups. In this section, we review these theorems, noting their relationship to the Mautner-Teleman theorem. We also explain how this is a generalization of the theory of Fourier series expansions. The key facts in this chapter are various capabilities of 'averaging' by Haar measure in compact Lie groups, Shur Lemma, Gelfand-Raikov Theorem.

The Mautner-Teleman theorem guarantees that any unitary representation of a Lie group can be decomposed into a direct integral of irreducible unitary representations. The following Peter-Weyl Theorem I guarantees that this direct integral is a discrete direct sum of finite-dimensional irreducible unitary representations if the Lie group G is compact. In particular, the irreducible unitary representation of a compact Lie group is always finite-dimensional. This means $\hat{G} = \hat{G}_f$. Here $\hat{G}$ is the set of all equivalent classes of continuous irreducible unitary representation of G, and $\hat{G}_f$ is the set of all equivalent classes of continuous finite dimensional irreducible unitary representation of G.

Theorem 5.8.1 (Peter-weyl theorem I). *Let (π, V) be a continuous unitary representation of a compact Lie group G. Then there is D which is a subset of G-invariant finite dimensional irreducible subspaces such that*

$$V = \overline{\bigoplus_{W \in D} W}$$

The proof of Peter-Weyl's Theorem I, by using Zorn's Lemma, boils down to the proof of the claim that any unitary representation of a compact Lie group has a finite dimensional G-invariant subspace. Such an invariant subspace can be realized as the eigenspace of a G-linear map composed by acting on all group elements in their projection onto a suitable 1-dimensional space and averaging them. If the group is a finite group, this operator is a finite-dimensional matrix, its eigenspace will be one-dimensional. In the general case, this sum is the Bochner integral, and the operator formed by the sum is compact operator, so its eigenspace is finite-dimensional.

The irreducible unitary representation of S^1 is, by Shur's lemma and the real analyticity of finite dimensional representations of Lie groups(Theorem3.4.14), we find that it is exhausted by homomorphisms of the following form(Proposition5.7.35).

$$\tau_n : S^1 \ni z \mapsto z^n = exp(i \cdot n \cdot arg(z)) \in S^1 \ (n \in \mathbb{Z})$$

Thus, any unitary representation of S^1 can be decomposed into a direct sum of these representations.

Peter-Weyl's Theorem II gives the irreducible decomposition of $L^2(G)$ using Peter-Weyl's Theorem I.

Theorem 5.8.2 (Peter-weyl theorem II).

$$\Phi : (L, \oplus_{\tau \in \hat{G}_f} V \otimes V^*) \to (L, L^2(G))$$

Here, for each $(\tau, V) \in \hat{G}_f$ and $v \otimes f \in V \otimes V^*$,

$$\Phi(v \otimes f)(g) := f(\tau(g^{-1})v) \ (g \in G)$$

$$L_x(v \otimes f) = \tau(x)v \otimes f \ (x \in G)$$

$$L_x h(g) = h(x^{-1}g) \ (h \in L^2(G), g, x \in G)$$

We set

$A := \{\sqrt{dim\tau}\tau_{i,j} | (\tau, V)$ is an representative of $\hat{G}_f$ and $\{v_1, ..., v_{dim\tau}\}$ is an orthonormal basis of V and $1 \leq i, j \leq dim\tau\}$

Here, $\tau_{i,j}$ is defined as bellow for each i, j.

$$\tau_{i,j}(g) := (\tau(g)v_j, v_i) \ (g \in G)$$

The Peter-Weyl Theorem III guarantees that any continuous function f on G can be uniformly approximated by elements of a vector space B generated from the above set A.

Theorem 5.8.3 (Peter-Weyl Theorem III). *For any $\epsilon > 0$, there is a $a_1, ..., a_n \in \mathbb{C}$ and $\tau_{j_1,j_1}, ..., \tau_{j_n,j_n} \in A$*

$$|f(g) - \sum_{i,k=1,...,n} a_i \tau_{j_i,j_k}(g)| < \epsilon \ (\forall g \in G)$$

The proof of this theorem uses Stone Wierestrass's theorem(Theorem5.1.1) on uniform approximation of continuous functions on compact metric spaces. By Gelfand Raikov's theorem and the theory of positive definite functions, B contains constants and is closed by products and complex conjugates. Stone wierestrass theorem, such a space is , guarantees a uniform approximation of continuous functions on G. By applying Peter-Weyl's Theorem III to the case $G = S^1$, we obtain the following approximate theorem.

Theorem 5.8.4 (Wierstrass Theorem). *For any $f \in C(S^1)$ and $\epsilon > 0$, there is a finite subset $N \subset \mathbb{N}$ and $a_{-N}, a_{-N+1}, ..., a_N$ such that*

$$|f(z) - \sum_{n=-N}^{N} a_n z^n| < \epsilon \ (\forall z \in S^1)$$

By Peter-Weyl Theorem I and Gelfand-Raikov Theorem, the following is shown(Theorem5.7.39).

Theorem 5.8.5. *Any compact Lie group is isomophic to a closed subgroup of $U(n)$ for some $n \in \mathbb{N}$*

By Peter-Weyl Theorem II and Shur's Lemma, the above set A of matrix coefficients corresponding to all irreducible unitary representations is guaranteed to be an orthonormal basis of $L^2(G)$. Since $L^2(G)$ is separable, by Peter-Weyl's Theorem II, $\hat{G}_f$ is at most countable set. Due to the real analyticity of finite-dimensional representations of Lie groups, each $\tau_{i,j}$ is real analytic. From the above, we can say that this family of functions is an easy-to-handle family of functions. By the theory on orthonormal bases of Hilbert spaces, The square integrable function on G can be expanded by such a tractable function as by such an easy-to-handle function.

$$f = \sum_{\tau \in \hat{G}_f, 1 \leq i,j \leq dim\tau} dim\tau (f, \tau_{i,j}) \tau_{i,j} \ (L^2\text{-convergence})$$

This equation has two other expression. The one is the expression by characters(Proposition5.7.28 and Theorem5.7.20).

$$f = \sum_{\tau \in \hat{G}_f} \dim \tau \overline{\chi_\tau} * f \ (L^2\text{-convergence})$$

The another one is the expression by operator valued fourier transform.

Theorem 5.8.6 (Plancherel formula for compact Lie group)**.** *Let*

(S1) G is a compact Lie group.

(S2) $f \in L^2(G)$.

then

$$f = \sum_{\tau \in \hat{G}_f} \Phi'_\tau(\tau(f)) \ (L^2 \ convergence)$$

Here,

$$\tau(f) := \int_G \overline{\chi_\tau}(g) \tau(g) f dg \ (f \in L^1(G))$$

$$\Phi'_\tau(v \otimes f)(g) := dim\tau f(\tau(g^{-1})v)$$

We set μ by the counting measure of $\hat{G}_f$. Then

$$f = \int_{\hat{G}_f} \Phi'_\tau(\tau(f)) d\mu(\tau)$$

The left side is a bochner integral on the $L^(G)$ valued function. We call μ the Plancherel measure on $\hat{G}$.

The mapping $\hat{G} \ni \tau \mapsto \tau(f)$ is called the operator valued fourier transform of f. Operator valued fourier transform have the following properties.

(i) $\pi(f * g) = \pi(f)\pi(g) \ (\forall f, g \in L^2(G))$.

(ii) $\pi(R_x f) = \pi(f)\pi^*(x) \ (\forall x \in G)$.

It is known operator valued fourier transform is characterized by these properties[28]. In the case when $G = S^1$, $\tau_n(f) = \hat{f}(n) = (f, \tau_n)$ and $P_{\tau_n}(f)(\theta) = \hat{f}(n) exp(in\theta)$.

By applying Peter-Weyl's Theorem II to the case $G = S^1$, we obtain the following Fourier series expansion formula.

Theorem 5.8.7 (Fourier series expansion formula)**.** *For any $f \in L^2([0, 2\pi])$*

$$f = \lim_{N \to \infty} \sum_{n=-N}^{N} \hat{f}(n) \chi_n \ (L^2\text{-convergence})$$

Chapter 6

Homogeneous space

6.1 C^ω-class structure

Theorem 6.1.1. *Let*

(S1) G_1 *is a Le group which is locally isomorphic to a Lie subgroup of* $GL(n,\mathbb{C})$ G_2.

(A1) H *is a closed subgroup of* G_1 *such that* $dimLie(H) > 0$.

(S2) $\mathfrak{h} := Lie(H)$.

(S3) $\mathfrak{g}_1$ *is a complementary space of* $\mathfrak{h}$ *in* $\mathfrak{g} := Lie(G_1)$.

(S4) $k := dim\mathfrak{g}_1$ *and* $l := dim\mathfrak{h}$.

Then there is a C^ω*-class manifold structure of* G/H *such that*

(i) $p : G_1 \ni g \mapsto gH \in G_1/H$ *is a continuous map and an open map.*

(ii) $G_1 \times G_1/H \ni (g_1, g_2H) \mapsto g_1g_2H$ *is* C^ω*-class.*

(iii) *For any* $g \in G$ *and* $h \in H$*, there is* $\epsilon > 0$ *such that*

$$B_k(O,\epsilon) \times B_l(O,\epsilon) \ni (X,Y) \mapsto gExp(X)hExp(Y) \in G$$

and

$$B_k(O,\epsilon) \ni X \mapsto \pi(gExp(X)) \in G/H$$

are C^ω*-class diffeomorphism.*

We call G/H *homogeneous space or homogeneous manifold.*

STEP1. Definition of the topology of G/H*.* We set

$$p : G \ni g \to gH \in G/H$$

and

$$\mathcal{O}(G/H) := \{A \subset G/H | p^{-1}(A) \in \mathcal{O}(G)\}$$

Clearly, p is continuous. Also, for each $O \in \mathcal{O}(G)$,

$$p^{-1}(p(O)) = \cup_{h\in H} Oh$$

So, p is an open map. Because p is surjective, for any $O_1 \in \mathcal{O}(G/H)$, there is $O_2 \in \mathcal{O}(G)$ such that

$$p(O_2) = O_1$$

And clearly, for any $O \in \mathcal{O}(G)$ and $g \in G$,

$$L_g \circ p(O) = p \circ L_g(O)$$

So, L_g is a homeomorphism of G/H.

We will show G/H is a Hausdorff space. Let us fix $g_1, g_2 \in G$ such that $g_1H \neq g_2H$. So, $g_2^{-1}g_1 \notin H$. Because H is a closed set, there is U which is an open neighborhood of e such that

$$U^{-1}g_2^{-1}g_1U \cap H = \phi$$

This implies that

$$g_1UH \cap g_2UH = \phi$$

So, G/H is a Hausdorff space. □

STEP2. Construction of a local coordinate system of G/H. There is $\epsilon_0 > 0$ and $\epsilon > 0$ such that $Exp|B(O,\epsilon)$ is a C^ω-class homeomorphism to an open set of G and

$$Exp(B(O,\epsilon))Exp(B(O,\epsilon)) \subset Exp(B(O,\epsilon_0))$$

and

$$\rho : (\mathfrak{g}_1 \cap B(O,\epsilon_0)) \oplus (\mathfrak{h} \cap B(O,\epsilon_0)) \ni X + Y \to Exp(X)Exp(Y)$$

is a C^ω-class homeomorphism. We set for each $g \in G$

$$\rho_g : (\mathfrak{g}_1 \cap B(O,\epsilon)) \ni X \to gExp(X)H \in gExp(B(O,\epsilon_0))H$$

Clearly, $gExp(B(O,\epsilon_0))H \in \mathcal{O}(G/H)$ and ρ_g is surjective. We will show ρ_g is injective. Let us fix any $X_1, X_2 \in \mathfrak{g}_1$ such that $\rho_g(X_1) = \rho_g(X_2)$. Then, becaluse $Exp(B(O,\epsilon))Exp(B(O,\epsilon)) \subset Exp(B(O,\epsilon_0))$,

$$Exp(-X_2)Exp(X_1) \in H \cap Exp(B(O,\epsilon_0))$$

By von-Neumann-Cartan's theorem, we can assume

$$H \cap Exp(B(O,\epsilon_0)) = Exp(B(O,\epsilon_0) \cap \mathfrak{h})$$

So,

$$Exp(X_1) = Exp(X_2)Exp(B(O,\epsilon_0) \cap \mathfrak{h})$$

Because ρ is injective, $X_1 = X_2$.

We can assume for any $X \in B(O,\epsilon)\mathfrak{g}_1$, there is C^ω-class π_1 and π_2 such that for any $Z \in B(O,\epsilon)\mathfrak{g}_1$

$$Exp(X_2 + Z) = Exp(X_2 + \pi_1(Z))Exp(\pi_2(Z)), \pi_1(Z) \in \mathfrak{g}_1, \pi_2(Z) \in \mathfrak{h}$$

Let us fix any $g_1, g_2 \in G$ such that

$$g_1Exp(\mathfrak{g}_1 \cap B(O,\epsilon))H \cap g_2Exp(\mathfrak{g}_1 \cap B(O,\epsilon))H \neq \phi$$

Let us fix any $X_1 \in \rho_{g_1}^{-1}(g_1Exp(\mathfrak{g}_1 \cap B(O,\epsilon))H \cap g_2Exp(\mathfrak{g}_1 \cap B(O,\epsilon))H)$. There is $X_2 \in \mathfrak{g}_1 \cap B(O,\epsilon)$ and $h \in H$ such that

$$g_2^{-1}g_1Exp(X_1)h = Exp(X_2)$$

So, there is $\delta > 0$ such that

$$g_2^{-1}g_1Exp(X_1 + B(O,\delta))h \subset Exp(B(O,\epsilon_0))$$

We set

$$\psi(Y) := log(\tau(g_2^{-1}g_1Exp(X_1 + Y)h)) - X_2 \ (Y \in B(O,\delta) \cap \mathfrak{g}_1)$$

Then ψ is C^ω-class and

$$g_1Exp(X_1 + Y)h = g_2Exp(X_2 + \psi(Y))$$

So,

$$g_2Exp(X_2 + \psi(Y)) = g_2Exp(X_2 + \pi_1(\psi(Y)))Exp(\pi_2(\psi(Y)))$$

This implies that

$$\rho_{g_2}^{-1} \circ \rho_{g_1}(Y) = \pi_1(\psi(Y))$$

Consequently, $\{\rho_g\}_{g \in G}$ defines the C^ω-class structure of G/H. □

STEP3. Showing $G \times G/H \ni (g_1, g_2H) \to g_1g_2H$ is C^ω-class. For any $Y \in Lie(G) \cap B(O,\epsilon)$ and $X_1 \in \mathfrak{g}_1 \cap B(O,\epsilon)$

$$\rho_{g_1g_2}(g_1ExpYg_2Exp(X_1)H) = \rho_{g_1g_2}(g_1g_2Exp(Ad(g^{-1}Y)Exp(X_1)H) = \rho_{g_1g_2}(g_1g_2Exp(\xi(Ad(g^{-1}Y, X_1))) = \xi(Ad(g^{-1}Y, X_1)$$

Here, ξ is C^ω-class mapping such that $Exp(Y')Exp(X'_1) = \xi(Y', X'_1)$ $(\forall Y' \in Lie(G) \cap B(O,\epsilon), \forall X'_1 \in \mathfrak{g}_1 \cap B(O,\epsilon))$. □

STEP4. Proof of (iii). By STEP2., there is $\delta > 0$ such that

$$\sigma : \mathfrak{g}_1 \cap B_k(O,\delta) \times \mathfrak{h} \cap B_l(O,\delta) \ni (X, Y) \mapsto Exp(X)Exp(Y) \in G$$

is C^ω-class diffeomorphism and

$$\mathfrak{g}_1 \cap B_k(O,\delta) \ni X \mapsto \pi(Exp(X)) \in G/H$$

is C^ω-class diffeomorphism. So,

$$B_k(O,\delta) \ni X \mapsto \pi(gExp(X)) \in G/H$$

is C^ω-class diffeomorphism. There is $\epsilon > 0$ such that

$$Ad(h)B_l(O,\epsilon) \subset B_l(O,\delta)$$

Let us fix any $g \in G$ and $h \in H$. We set

$$\rho : B_k(O,\epsilon) \times B_l(O,\epsilon) \ni (X,Y) \mapsto gExp(X)hExp(Y) \in G$$

Then ρ is clearly C^ω-class and $Im\rho$ is an open set. Because $gExp(X)hExp(Y) = gExp(X)Exp(Ad(h)Y)h$,

$$Im\rho \ni x \mapsto (p_1(\sigma^{-1}(g^{-1}xh^{-1})), Ad(h^{-1})p_2(\sigma^{-1}(g^{-1}xh^{-1})) \in \mathfrak{g}_1 \cap B_k(O,\delta) \times \mathfrak{h} \cap B_l(O,\delta)$$

is the inverse of σ and C^ω-class diffeomorphism. □

Proposition 6.1.2. *Let G be a Lie group and N is a closed normal subgroup of G. Then G/N is a topological group such that*

$$G/N \times G/N \ni (g_1N, g_2N) \mapsto g_1{g_2}^{-1}N \in G/N$$

is C^ω-class.

Proof. From Theorem6.1.1, C^ω-class and the action of G on G/N is C^ω-class. Since

$$G \times G \ni (g_1, g_2) \mapsto g_1{g_2}^{-1} \in G$$

,

$$G/N \times G/N \ni (g_1N, g_2N) \mapsto g_1{g_2}^{-1}N \in G/N$$

is C^ω-class. □

Theorem 6.1.3. *Here are the settings and assumptions.*

(S1) G is a Lie group.
(S2) M is a locally compact Hausdorff space.
(S3) G continuously and transitively acts on M.
(S4) $x_0 \in M$.
(S4) $H := \{h \in G | h \cdot x_0 = x_0\}$. We call H the isotropy subgroup regarding x_0.

Then

(i) H is a closed subgroup of G.
(ii) $\pi : G/H \ni gH \mapsto gx \in M$ is a homeomorphism.
(iii) In addition, let us assume M is a C^∞ class manifold and the action of G on M is C^∞-class. Then π is a C^∞-class diffeomorphism.

Proof of (i). We set

$$\bar{\pi} : G \ni g \mapsto gx_0 \in M$$

Since M is a Hausdorff space, $\{x_0\}$ is a closed subset of M. In addition, $\bar{\pi}$ is continuous. Therefore, $H = \bar{\pi}^{-1}(\{x_0\})$ is closed. □

Proof of (ii). Clearly π is well-defined and bijective. Let $p : G \to G/H$ denote the natural projection. For any open subset U in M,

$$p^{-1}\bar{\pi}^{-1}(U) = \pi^{-1}(U)$$

and $\pi^{-1}(U)$ is open set since π is continuous. From the definition of the topology of G/H, $\bar{\pi}^{-1}(U)$ is open set. Therefore, π is continuous.

So, it is enough to show π is open map. We set $\mathfrak{g} := Lie(G)$ and $\mathfrak{h} := Lie(H)$ and pick a complement of $\mathfrak{h}$ $\mathfrak{q}$. By the proof of Theorem6.1.1, there is $\epsilon > 0$ such that for any $g \in G$

$$\phi_g : \mathfrak{q} \cap B(O, 2\epsilon) \ni X \mapsto g\exp(X)H \in G/H$$

is a local diffeomorphism to an open neighborhood of gH. Since $M = \cup_{g\in G}\pi(\phi_g(\mathfrak{q} \cap \overline{B(O,\epsilon)}))$, by Baire Category Theorem, there is $g \in G$ such that $\pi(\phi_g(\mathfrak{q} \cap B(O, 2\epsilon)))^\circ \neq \phi$. Since

$$g^{-1}\pi(\phi_g(\mathfrak{q} \cap B(O, 2\epsilon)))^\circ = \pi(\phi_e(\mathfrak{q} \cap B(O, 2\epsilon)))^\circ$$

, $\pi(\phi_e(\mathfrak{q} \cap B(O, 2\epsilon)))^\circ \neq \phi$. So, π is an open map. □

Proof of (iii). Since

$$\pi(\phi_g(X)) = \bar{\pi}(g\exp(X)) \ (\forall X \in \mathfrak{q} \cap B(O, 2\epsilon)))$$

π is C^∞-class. We set

$$\tau : M \ni g \cdot x_0 \mapsto gH \in G/H$$

Then τ is clearly well-defined and C^∞ class and the inverse map of π. □

6.2 Topological Properties

Proposition 6.2.1. *Let G be a Lie group and H is a closed subgroup of G such that H and G/H are connected. Then G is connected.*

Proof. We will show the contraposition. Let us assume that G is not connected. Let G_0 denote the connected component of G that contains e and $p : G \to G/H$ be the projection. Since G_0 is an open subset of G and p is an open map, $p(G_0)$ is an open subset. Next, we will show

$$p(G_0)^c = p(G_0^c)$$

Clearly $p(G_0)^c \subset p(G_0^c)$ holds. For aiming contradiction, let us assume $p(G_0^c) \cap p(G_0) \neq \phi$. Then there is $g' \in G_0^c$ and $g \in G_0$ such that $g'H \cap gH \neq \phi$. That implies $g' \in gH$. Since H is connected and contains e, $H \in G_0$. Therefore, $g' \in G_0$. That is contradiction. So $p(G_0)^c = p(G_0^c)$.

Since G_0^c is an open subset of G and p is an open map, $p(G_0)^c = p(G_0^c)$ is an open subset. Since $p(G_0)$ contains H, G/H is not connected. □

Proposition 6.2.2. *Here are the settings and assumptions.*

(S1) G is a Lie group.

(S2) H is a closed subgroup of G.

(S3) $p : G \ni g \mapsto gH \in G/H$.

(S4) $\tilde{c} \in C([0, 1], G/H)$.

Then there is a $c \in C([0, 1], G/H)$ such that $p \circ c = \tilde{c}$.

Proof. We set $\mathfrak{g} := Lie(G), \mathfrak{h} := Lie(H)$. Let us pick a complement of $\mathfrak{h}$ in $\mathfrak{g}$, denoted by $\mathfrak{q}$. Since $Im\tilde{c}$ is compact, from Theorem6.1.1, there are $\epsilon > 0$, $g_1, ..., g_m \in G$, $0 = t_0 < t_2 < ... < t_m = 1$ such that

$$\mathfrak{q} \cap B(O, \epsilon) \times \mathfrak{h} \cap B(O, \epsilon) \ni (X, Y) \mapsto exp(X)exp(Y) \in G$$

is a diffeomorphism to an open subset of G and

$$\tilde{c}(t_{i-1}, t_i) \subset g_i exp(\mathfrak{q} \cap B(O, \epsilon))H \ (i = 1, 2, ..., m)$$

Then, for each $i \in \{1, 2, ..., m\}$, there is $a_i \in C([t_{i-1}, t_i], G)$ such that

$$\tilde{c}(t) = g_i exp(a_i(t))H \ (\forall t \in [t_{i-1}, t])$$

And for each $i > 1$, there is $h_i \in H$ such that

$$g_{i-1} exp(a_{i-1}(t_{i-1}))h_i = g_i exp(a_i(t_{i-1}))$$

We set

$$c(t) := \begin{cases} g_i exp(a_i(t)h_i & t \in [t_{i-1}, t_i], i < m \\ g_m exp(a_i(t)) & t \in [t_{m-1}, t_m] \end{cases}$$

Then $c \in C([0, 1], G)$ and $p \circ c = \tilde{c}$. □

Lemma 6.2.3. *Here are the settings and assumptions.*

(S1) G is a Lie group.

(S2) H is a closed subgroup of G.

(S3) $p : G \ni g \mapsto gH \in G/H$.

(S4) Let $C_0([0, 1], G/H) := \{c \in C([0, 1], G/H) | c(0) = c(1) = H\}$.

(S5) *For each* $c \in C_0([0,1], G/H)$,

$$\lambda(c) := \begin{cases} 0 & d(0), d(1) \text{ are in the same connected components. Where, } d \in C_0([0,1], G) \text{ such that } p \circ d = c. \\ 1 & \text{Otherwise.} \end{cases}$$

Then

(i) λ *is well-defined.*

(ii) *For any* $c_1, c_2 \in C_0([0,1], G)$ *such that there is a homotopy from* c_1 *to* c_2 *preserving start point and end point,* $\lambda(c_1) = \lambda(c_2)$.

Proof of (i). For aiming contradiction, let us assume that there are $d_1, d_2 \in C([0,1], G)$ such that $p \circ d_i = c$ $(i = 1, 2)$ and $d_1(0), d_1(1)$ are in the same connected components of H and $d_2(0), d_2(1)$ are in different connected components of H. Since $H \ni h \mapsto h^{-1} \in H$ is connected, $d_1(0)^{-1}, d_1(1)^{-1}$ are in the same connected components of H. From Proposition3.4.6, any connected component of H is path-connected. Therefore there is $\alpha \in C([0,1], H)$ such that $\alpha(0) = d_1(0)^{-1}$ and $\alpha(1) = d_1(1)^{-1}$. By setting $c \cdot \alpha$, we can assume that $d_1(0) = d_1(1) = e$.

We set $e := d_1^{-1} \cdot d_2$. Then $e \in C([0,1], H)$. And, $e(0) = d_2(0)$ and $e(1) = d_2(1)$ are in different connected components of H. That is a contradiction. □

Proof of (ii). Let us fix a homotopy Φ from c_1 to c_2. It is enough to show that for each t, there is $\epsilon > 0$ such that $\lambda(\Phi(s, \cdot))$ is constant for any $s \in [t - \epsilon, t + \epsilon]$. We set $\mathfrak{g} := Lie(G), \mathfrak{h} := Lie(H)$. Let us pick a complement of $\mathfrak{h}$ in $\mathfrak{g}$, denoted by $\mathfrak{q}$. Let us fix any $s_0 \in [0,1]$.

Since $Im\Phi$ is compact, from Theorem6.1.1, there are $\epsilon > 0, \delta > 0$, $g_1, ..., g_m \in G$, $0 = t_0 < t_2 < ... < t_m = 1$ such that

$$\mathfrak{q} \cap B(O, \epsilon) \times \mathfrak{h} \cap B(O, \epsilon) \ni (X, Y) \mapsto exp(X)exp(Y) \in G$$

is a diffeomorphism to an open subset of G and

$$\Phi([s_0 - \delta, s_0 + \delta] \times [t_{i-1}, t_i]) \subset g_i exp(\mathfrak{q} \cap B(O, \epsilon))H \ (i = 1, 2, ..., m)$$

Then, for each $i \in \{1, 2, ..., m\}$, there is $a_i \in C([s_0 - \delta, s_0 + \delta] \times [t_{i-1}, t_i], G)$ such that

$$\Phi(s, t) = g_i exp(a_i(s, t))H \ (\forall s \in [s_0 - \delta, s_0 + \delta], \forall t \in [t_{i-1}, t])$$

And for each $i > 1$, there is $h_i \in C([s_0 - \delta, s_0 + \delta], H)$ such that

$$g_{i-1} exp(a_{i-1}(t_{i-1}))h_i(s) = g_i exp(a_i(t_{i-1})) \ (\forall s \in [s_0 - \delta, s_0 + \delta])$$

For each $s \in [s_0 - \delta, s_0 + \delta]$, we set

$$c_s(t) := \begin{cases} g_i exp(a_i(t))h_i(s) & t \in [t_{i-1}, t_i], i < m \\ g_m exp(a_i(t)) & t \in [t_{m-1}, t_m] \end{cases}$$

Then $c_s \in C([0,1], G)$ and $p \circ c_s = \Phi(s, \cdot)$. Since h_1 is continuous, $c_.(0)$ and $c_.(1)$ is continuous. Then $\{c_s(0) | s \in [s_0 - \delta, s_0 + \delta]\}$ are in the sample connected components of H. And $\{c_s(1) | s \in [s_0 - \delta, s_0 + \delta]\}$ are in the sample connected components of H. Therefore, $\lambda(\Phi(s, \cdot))$ is constant for any $s \in [t - \epsilon, t + \epsilon]$. □

Proposition 6.2.4. *Here are the settings and assumptions.*

(S1) G *is a connected Lie group.*

(S2) H *is a closed subgroup of* G.

(S3) $p : G \ni g \mapsto gH \in G/H$.

(A1) G/H *is simply connected.*

Then H *is connected.*

Proof. For aiming contradiction, let us assume that H is not connected. Let H_0 denote the connected component of H which contains e. Pick another connected component of H, H_1. Since G is connected, there exists $c \in C([0,1], G)$ such that $c(0) = e$ and $c(1) =: h_1 \in H_1$. We set $\tilde{c} := p \circ c$. From the definition of c, $\lambda(\tilde{c}) = 1$. On the other hand, since G/H is simply connected, $\tilde{c} \sim \{H\}$. From Lemma6.2.3, $\lambda(\tilde{c}) = 0$. That is a contradiction. □

Example 6.2.5. *If* $G := \mathbb{R}^+$ *and* $H := \mathbb{Z}$, *then it is known that* $\pi_1(G/H) = \mathbb{Z}$ *(See [24]). Therefore,* G/H *is not simply connected.*

Proposition 6.2.6. *Here are the settings and assumptions.*

- *(S1)* G *is a connected Lie group.*
- *(S2)* H *is a closed subgroup of* G*.*
- *(S3)* $p : G \ni g \mapsto gH \in G/H$*.*
- *(A1)* G/H *is simply connected.*
- *(S4) Let* $\tilde{G}$ *denote the universal covering of* G*.*
- *(S5) Let* $\tilde{H}$ *denote the analytical subgroup of* $\tilde{G}$*.*
- *(S6) Let* π *denote the projection from* $\tilde{G}$ *to* G*.*

Then $\tilde{Z} := \ker(\pi) \subset \tilde{H}$*.*

Proof. First, we will show

$$\tilde{Z}\tilde{H} = \pi^{-1}(H)$$

Clearly, $\tilde{Z}\tilde{H} \subset \pi^{-1}(H)$. Let us fix any $\tilde{g} \in \pi^{-1}(H)$. Then there is $c_1 \in C([0,1], G)$ such that $c_1(1) \in H$ and $\tilde{g} = [c_1]$. From Proposition6.2.4 , there is $c_2 \in C([0,1], H)$ such that $c_2(1) = c_1(1)$. From the definition of analytic subgroup, $[c_2] \in \tilde{H}$.

Then

$$\tilde{g} = [c_1 c_2^{-1}][c_2]$$

Since $[c_1 c_2^{-1}] \in \pi^{-1}(e)$, $\tilde{g} \in \tilde{Z}\tilde{H}$. Therefore, $\tilde{Z}\tilde{H} = \pi^{-1}(H)$.

So, $\tilde{Z}\tilde{H}$ is a closed subgroup of $\tilde{G}$. Since the definition of π, $Lie(\tilde{Z}\tilde{H}) = \mathfrak{h}$. Next, we will show $\tilde{Z}\tilde{H} = \tilde{H}$. From the uniqueness of analytic subgroup, it is enough to show that $\tilde{Z}\tilde{H}$ is connected. Since G/H is simply connected and $\tilde{G}$ is connected, from Proposition6.2.4 , it is enouth to show G/H is homeomorphic to $\tilde{G}/\tilde{Z}\tilde{H}$. We set for each $g \in G$, $[c] \in \tilde{G}$,

$$[c] \cdot gH := c(1)gH$$

Clearly the action is well-defined and continuous and the isotoropy group is $\pi^{-1}(H) = \tilde{Z}\tilde{H}$. From Theorem6.1.3, we get

$$G/H \simeq \tilde{G}/\tilde{Z}\tilde{H}$$

Consequently, $\tilde{Z}\tilde{H} = \tilde{H}$. So, $\tilde{Z} \subset \tilde{H}$. □

6.3 Various Types of Homogeneous Space

Definition 6.3.1 (Involutive automorphism)**.** *Let* G *be a Lie group. We call* $\sigma \in Auto(G)$ *a involutive or involution if* $\sigma \circ \sigma = id_G$*. We set* $G^\sigma := \{g \in G | \sigma(g) = g\}$*. And we denote the connected component of* G^σ *which contains the unit element by* G_0^σ*.*

Clealy the following hold.

Proposition 6.3.2. G^σ *and* G_0^σ *a closed subgroup of* G*.*

The following definition is from [9].

Definition 6.3.3 (Symmetric Pair, Symmetric Space, Riemann Symmetric Pair)**.** *Let* G *be a Lie group and* σ *be a involution of* G*. If* H *is a closed subgroup of* G *such that* $G_0^\sigma \subset H \subset G^\sigma$*. Then we call* (G, H) *be a symmetric pair and* G/H *be a symmetric space. In addition, if* $Ad(H)$ *is compact, we call* (G, H) *be a Riemannian symmetric pair.*

Example 6.3.4. *Let* G *be a Lie group and* $H := \{e\}$ *and* $\sigma := id$*. Then* (G, H) *is a symmetric pair.*

The following two definition is from [2].

Definition 6.3.5 (Reductive Homogeneous Space)**.** *The followings are settings and assumptions.*

- *(S1)* $G, H \subset GL(n, \mathbb{R})$ *be connected linear reductive Lie groups.*

Then we call G/H *a reductive homogeneous space.*

Definition 6.3.6 (Reductive Symmetric Space)**.** *The followings are settings and assumptions.*

- *(S1)* $G, H \subset GL(n, \mathbb{R})$ *be connected linear reductive Lie groups.*
- *(S2)* σ *is a involution of* G*.*

Then we call G/H *a reductive symmetric space if* H *is an open set and is a subgroup of* G^σ*.*

6.4 Invariant measure

6.4.1 Existence of Invariant measure

Definition 6.4.1 (Invariant measure)**.** *Here are the settings and assumptions.*

- *(S1) G is a Lie group and $m := Lie(G)$.*
- *(S2) H is a closed subgroup of G.*
- *(S3) μ is a Baire measure on G/H.*

We say μ is a invariant measure on G/H if for any $f \in C_c(G/H)$ and any $g_0 \in G$

$$\int_G f(g_0 \cdot x) d\mu(x) = \int_G f(x) d\mu(x)$$

We say μ is a right invariant measure on G

Notation 6.4.2. *Let G be a Lie group and $g_0 \in G$. For each $x \in G/H$, $\tau_{g_0}(x) := g_0 \cdot x$.*

Lemma 6.4.3. *Here are the settings and assumptions.*

- *(S1) G is a Lie group and $\mathfrak{g} := Lie(G)$ and $m := dim\mathfrak{g}$.*
- *(S2) H is a closed subgroup of G and $\mathfrak{h} := Lie(H)$ and $k := dim\mathfrak{h}$.*
- *(S3) $\pi : G \ni g \mapsto gH \in G/H$.*
- *(S4) $\tau_g : G/H \ni xH \mapsto gxH \in G/H$ $(g \in G)$.*
- *(S5) $\mathfrak{q}$ is a complement space of $\mathfrak{h}$ in $\mathfrak{g}$ and $l := dim\mathfrak{q}$.*
- *(S6) $x \in G$.*
- *(S7) $\delta > 0$ such that $\Phi_x : B_l(O, \delta) \cap \mathfrak{q} \ni X \mapsto xexp(X)H \in G/H$ is a local coordinate around $\pi(x)$ in G/H. We set $U := B_l(O, \delta) \cap \mathfrak{q}$.*
- *(S8) $\omega_{\pi(e)}$ is a m-th antisymmetric tensor field on $T_{\pi(e)}(G/H)$.*
- *(S9) For each $X \in U$,*

$$\omega^x_{\Phi_x(X)}(v_1, ..., v_l) := \omega_e(((d\tau_{xExp(X)})_{\pi(e)})^{-1}v_1, ..., ((d\tau_{xExp(X)})_{\pi(e)})^{-1}v_l) \ (v_1, ..., v_l \in T_{\Phi_x(X)}(G/H))$$

Then ω^x is C^ω-class l-form on $\Phi_x(U)$.

Proof. It is enough to show a representation matrix $(d\tau_{xExp(X)})_{\pi(e)}$ is C^ω-class. For each $y \in G/H$, we denote the local coordinate around y defined in the proof of 6.1.1 by ψ_y. So, it is enough to show

$$U \times U \ni (X, Y) \mapsto \psi^{-1}_{\pi(x)}(\tau_{xExp(X)}\psi_{\pi(e)}(Y)) \in \mathfrak{q}$$

is C^ω-class. By the proof of 6.1.1, there is $\epsilon \in (0, \delta)$ such that

$$\Theta : \mathfrak{q} \cap B_k(O, \epsilon) \times \mathfrak{h} \cap B_l(O, \epsilon) \ni (X, Y) \mapsto exp(X)exp(Y) \in G$$

is a C^ω-class homeomorphism to an open neighborhood of e. We can assume $Exp(U)Exp(U) \in Im\Theta$. For each $(X, Y) \in U \times U$, there is the unique $(\alpha(X), \beta(Y)) \in \mathfrak{q} \cap B_k(O, \epsilon) \times \mathfrak{h} \cap B_l(O, \epsilon)$ such that

$$\tau_{xExp(X)}\psi_{\pi(e)}(Y) = Exp(\alpha(X, Y))Exp(\beta(X, Y))$$

and α and β are C^ω-class. And for any $X, Y \in U$,

$$\psi^{-1}_{\pi(x)}(\tau_{xExp(X)}\psi_{\pi(e)}(Y)) = \alpha(X, Y)$$

So,

$$U \times U \ni (X, Y) \mapsto \psi^{-1}_{\pi(x)}(\tau_{xExp(X)}\psi_{\pi(e)}(Y)) \in \mathfrak{q}$$

is C^ω-class. □

Lemma 6.4.4. *We will succeed notations in 6.4.4. And here are the settings and assumptions.*

(A1) For any $x, y \in G$, there is $\sigma \in \{-1, 1\}$ such that

$$\omega^x = \sigma\omega^y \text{ in } \Phi_x(U) \cap \Phi_y(U)$$

(S1) For any $x \in G$, there is $\phi_x \in C^\omega(\Phi_x(U))$ such that for any $q \in \Phi_x(U)$,

$$\omega^x_q = \phi_x(q)d(\Psi^1_x)_q \wedge ... \wedge d(\Psi^k_x)_q$$

Here $\Psi_x := \Phi_x^{-1}$.

(S2) We set

$$\tilde{\omega}_q = |\phi_x(q)|d(\Psi^1_x)_q \wedge ... \wedge d(\Psi^k_x)_q \ (x \in G, q \in \Phi_x(U))$$

and define $\rho : G/H \to \{-1, 1\}$ by

$$\tilde{\omega}_q = \rho(q)\omega_q(x \in G, q \in \Phi_x(U))$$

Then $\tilde{\omega}$ is C^∞-class form on G/H and for any $q \in G/H$ and $g \in G$ there is $\sigma_{g,q} \in \{-1, 1\}$

$$(d\tau_g)\tilde{\omega}_q = \sigma_{g,q}\tilde{\omega}_q$$

and G/H is orientable.

Proof. Let us fix any $g, x \in G$. We set $q := \pi(x)$ and $p := \pi(e)$. Then for any $v_1, ..., v_k \in T_q(G/H)$,

$$((d\tau_g)\tilde{\omega})_q(v_1, ..., v_k) = \tilde{\omega}_{gq}((d\tau_g)_q v_1, ..., (d\tau_g)_q v_k) = \rho(gq)\omega_e((d\tau_{gx})_e^{-1}(d\tau_g)_q v_1, ..., (d\tau_{gx})_e^{-1}(d\tau_g)_q v_k)$$
$$= \omega_e((d\tau_x)_e^{-1}v_1, ..., (d\tau_x)_e^{-1}v_k) = \rho(gq)\rho(q)\tilde{\omega}_q(v_1, ..., v_k)$$

□

Lemma 6.4.5. *We will succeed notations in 6.4.4. Then*

$$\omega^x_{xExp(X)H} = det(d\tau_{xExp(X)})^{-1}(d\Psi^1_x)_{xExp(X)H} \wedge ... \wedge (d\Psi^k_x)_{xExp(X)H} \ (\forall X \in U)$$

Proof. Let us fix any $X \in U$. We set $g := xExp(X)$ and $q := \pi(g)$.

$$\omega^x_q = det(\{\omega^x_q((\frac{\partial}{\partial\Psi^j_x})_q e_i)\}^k_{i,j=1})(d\Psi^1_x)_q \wedge ... \wedge (d\Psi^k_x)_q$$

We denote the inverse of jacobi matrix of $(d\tau_g)_p$ with respect to $\{(\frac{\partial}{\partial\Psi^j_x})_q\}_j$ and $\{(\frac{\partial}{\partial\Psi^j_e})_p\}_j$ by $\{a_{j,r}\}^k_{j,r=1}$. Then

$$(d\tau_g)_p^{-1}(\frac{\partial}{\partial\Psi^j_x})_q = \sum_{r=1}^k a_{j,r}(\frac{\partial}{\partial\Psi^r_e})_p$$

So,

$$\omega^x_q((\frac{\partial}{\partial\Psi^j_x})_q e_i) = a_{j,i}$$

Consequently,

$$\omega^x_{xExp(X)H} = det(d\tau_{xExp(X)})^{-1}(d\Psi^1_x)_{xExp(X)H} \wedge ... \wedge (d\Psi^k_x)_{xExp(X)H}$$

□

Lemma 6.4.6. *We will succeed notations in 6.4.4. And here are the settings and assumptions.*

(A1) For any $h \in H$,

$$|det((d\tau_h)_p)| = 1$$

Then for any $x, y \in G$, there is $\sigma \in \{-1, 1\}$ such that

$$\omega^x = \sigma\omega^y \text{ in } \Phi_x(U) \cap \Phi_y(U) \tag{6.4.1}$$

Proof. Let us fix any $q \in \Phi_x(U) \cap \Phi_y(U)$. Then there are $X, Y \in U$ such that

$$\pi(xExp(X)) = q = \pi(yExp(Y))$$

We set $x_0 := xExp(X)$ and $y_0 := yExp(Y)$ and $h := y_0^{-1}x_0$. Then by Lemma6.4.5,

$$\begin{aligned} &(6.4.1) \\ \iff &|det((d\tau_{x_0})_p)| = |det((d\tau_{y_0})_p)| \\ \iff &|det((d\tau_h)_p)| = |det((d\tau_{y_0})_p^{-1})det((d\tau_{x_0})_p)| = 1 \end{aligned}$$

□

Lemma 6.4.7. *We will succeed notations in 6.4.4. Then*

$$(d\tau_h)_p = Ad_{\mathfrak{g}/\mathfrak{h}}(h) \ (\forall h \in H)$$

and

$$det((d\tau_h)_p) = \frac{det(Ad_G(h))}{det(Ad_H(h))} \ (\forall h \in H)$$

Proof. Let us fix any $h \in H$. For any $t \in \mathbb{R}$ and $X \in \mathfrak{g}$,

$$\tau_h\pi(exp(tX)) = hExp(tX)H = hExp(tX)h^{-1}H = Exp(tAd(h)X)$$

So,

$$(d\tau_h)_p = Ad_{\mathfrak{g}/\mathfrak{h}}(h)$$

Let A, B, C be the representation matrices corresponding to $Ad_G(h), Ad_{\mathfrak{g}/\mathfrak{h}}$ and $Ad_H(h)$ with respect to $\mathfrak{g}$, respectively. Let us fix any $X \in \mathfrak{g}$. There are $Y \in \mathfrak{q}$ and $Z \in \mathfrak{h}$ such that $X = Y + Z$. $Ad_G(h)X - Ad_{\mathfrak{g}/\mathfrak{h}}(h)X \in \mathfrak{h}$ and $Ad_H(h)Z \in \mathfrak{h}$. So,

$$A = \begin{pmatrix} B & O \\ * & C \end{pmatrix}$$

This implies $det(A) = det(B)det(C)$. □

Lemma 6.4.8. *We will succeed notations in 6.4.4. And here are the settings and assumptions.*

(A1) For any $x, y \in G$, there is $\sigma \in \{-1, 1\}$ such that

$$\omega^x = \sigma\omega^y \text{ in } \Phi_x(U) \cap \Phi_y(U)$$

(S1) $g \in G$.

(S2) (U_α, ψ_α) and (U_β, ψ_β) are local coordinates on G/H and $gU_\beta \cap U_\alpha \neq \phi$.

(S5) For any $x \in U_\alpha$ and $y \in U_\beta$

$$\omega_x = \Phi_\alpha(x)d\phi_{\alpha,1} \wedge ... \wedge d\phi_{\alpha,m}, \ \omega_y = \Phi_\beta(y)d\phi_{\beta,1} \wedge ... \wedge d\phi_{\beta,m}$$

Then, for any $x \in U_\beta \cap L_g^{-1}U_\alpha$,

$$\Phi_\beta(x) = |det(J(\psi_\alpha \circ \tau_g \circ \phi_\beta)(\psi_\beta(x)))|\Phi_\alpha(gx)$$

Proof. Let us fix any $x \in U_\beta \cap \tau_g^{-1}U_\alpha$. Then

$$\omega_x = \Phi_\beta(x)(d\phi_{\beta,1} \wedge ... \wedge d\phi_{\beta,m})_x$$

and

$$\omega_{gx} = \Phi_\alpha(gx)(d\phi_{\alpha,1} \wedge ... \wedge d\phi_{\alpha,m})_{gx}$$

So,

$$\omega_x((\frac{\partial}{\partial\psi_{\beta,1}})_x, ..., (\frac{\partial}{\partial\psi_{\beta,m}})_x) = \omega_{gx}(dL_g((\frac{\partial}{\partial\psi_{\beta,1}})_x), ..., dL_g((\frac{\partial}{\partial\psi_{\beta,m}})_x))$$

and

$$\omega_{gx}(dL_g((\frac{\partial}{\partial\psi_{\beta,1}})_x), ..., dL_g((\frac{\partial}{\partial\psi_{\beta,m}})_x)) = |detJ(\psi_\alpha \circ \tau_g \circ \phi_\beta)(\psi_\beta(x))|(d\phi_{\beta,1} \wedge ... \wedge d\phi_{\beta,m})_x$$

These implies that

$$\Phi_\beta(x) = \Phi_\alpha(gx)|detJ(\psi_\alpha \circ \tau_g \circ \phi_\beta)(\psi_\beta(x))|$$

□

Theorem 6.4.9. *Here are the settings and assumptions.*

(S1) G be a Lie group.

(S2) H be a closed subgroup of G.

(A1) For any $h \in H$,

$$|detAd_G(h)| = |det(Ad_H(h))|$$

Then

(i) There is C^∞-class form $\tilde{\omega}$ on G such that for any $g \in G$ there is $\sigma_g \in C(G/H, \{-1, 1\})$

$$d\tau_g \tilde{\omega} = \sigma_g \tilde{\omega}$$

(ii) G/H is orientable by $\tilde{\omega}$.

(iii) The measure induced from $\tilde{\omega}$ is G invariant. Specially, G/H has a invariant measure.

Proof. (i) is from Lemma6.4.4. (ii) is from Lemma6.4.6. We will show (iii). We set $k := dim(G/H)$. Let us fix $f \in C_c^\infty(G/H)$ and $g_0 \in G$. For $x \in G/H$,

$$(\tau_{g_0} f)(x) := f(g_0 x)$$

By (ii) and the second contable axiom, there is $\{U_i, \psi_i, V_i, \Phi_i, \rho_i\}_{i=1}^\infty$ such that $\{U_i, \psi_i\}_{i=1}^\infty$ is a local coordinate system of G/H and $\{U_i, \psi_i\}_{i=1}^\infty$ is local finite and for each i $V_i \in \mathcal{O}(\mathbb{R}^k)$

$$\psi_i : U_i \to V_i$$

is an homeomorphism and $\{U_i, \psi_i\}_{i=1}^\infty$ preserves a orientation of G and for each i and $x \in U_i$

$$\omega_x = \Phi_i(x)(d\psi_{i,1} \wedge ... \wedge d\psi_{i,k})_x$$

and $\Phi_i > 0$ and $\{\rho_i\}_{i=1}^\infty$ is a partition of unity subordinating $\{U_i\}_{i=1}^\infty$. We set for each i, $f_i := f\rho_i$. By Lebesgue's convergence theorem,

$$\int_{G/H} f\omega = \sum_{i=1}^\infty \int_{G/H} f_i\omega, \ \int_{G/H} \tau_{g_0} f\omega = \sum_{i=1}^\infty \int_{G/H} \tau_{g_0} f_i\omega$$

So, it is enough to show for each i

$$\int_{G/H} f_i\omega = \int_{G/H} \tau_{g_0} f_i\omega$$

By Lemma 3.6.10, we can assume that for each i, there is j such that $supp(\tau_{g_0} f_i) \subset U_j$. Because $supp(f_i)$ is compact, there is an open set U_i' such that

$$supp(f_i) \subset U_i' \subset U_i$$

and

$$supp(\tau_{g_0} f_i) = \tau_{g_0}^{-1} supp(f_i) \subset \tau_{g_0}^{-1} U_i' \subset U_j$$

We set $\phi_i := \psi_i^{-1}$ and $V_i := \psi_i(U_i)$ and $\phi_j := \psi_j^{-1}$ and $V_j := \psi_j(U_j)$. By change-of-variables formula for integral and Lemma6.4.8,

$$\begin{aligned}
\int_G \tau_{g_0} f_i\omega &= \int_{\psi_j(\tau_{g_0}^{-1} U_i')} f_i(g_0\phi_j(x))\Phi_j(x)dx \\
&= \int_{\psi_j(\tau_{g_0}^{-1} U_i')} f_i(\phi_i(\psi_i(g_0\phi_j(x))))\Phi_j(x)dx \\
&= \int_{\psi_j(\tau_{g_0}^{-1} U_i')} f_i(\phi_i(\psi_i \circ \tau_{g_0} \circ \phi_j(x))) \\
&\quad \times|det(J(\psi_i \circ \tau_{g_0} \circ \phi_j))(\psi_j \circ \tau_{g_0}^{-1}\phi_i \circ \psi_i \circ \tau_{g_0} \circ \phi_j(x))))|^{-1} \\
&\quad \times\Phi_j(\psi_j \circ \tau_{g_0}^{-1}\phi_i \circ \psi_i \circ \tau_{g_0} \circ \phi_j(x)))) \\
&= \int_{V_i'} f_i(\phi_i(y))det(J(\psi_i \circ \tau_{g_0} \circ \phi_j))(\psi_j \circ \tau_{g_0}^{-1} \circ \phi_i(y))^{-1} \\
&\quad \times\Phi_j(\psi_j \circ \tau_{g_0}^{-1}\phi_i(y))dy \\
&= \int_{V_i'} f_i(\phi_i(y))\Phi_i(y)dy \\
&= \int_G f_i\omega
\end{aligned}$$

□

Proposition 6.4.10. *Here are the settings and assumptions.*

(S1) G be a Lie group.

(S2) H be a closed subgroup of G such that $dimLie(H) > 0$.

(S3) $\epsilon > 0$.

(S4) $\mathfrak{g} := Lie(G)$, $\mathfrak{h} := Lie(H)$.

(S5) $\mathfrak{q}$ is a complement subspace of $\mathfrak{h}$ in $\mathfrak{g}$.

Then there are $\{g_i\}_{i=1}^{\infty} \subset G$ and $\{U_i\}_{i=1}^{\infty}$ such that U_i is a open neighborhood of 0_k $(\forall i)$ and $U_i \subset B_k(O, \epsilon) \cap \mathfrak{q}$ $(\forall i)$ and $\{\pi(g_i Exp(U_i))\}_{i \in \mathbb{N}}$ is an open covering of G/H and for any $i \in \mathbb{N}$ $\#\{j \in \mathbb{N} | \pi(g_i Exp(U_i)) \cap \pi(g_j Exp(U_j)) \neq \phi\} < \infty$.

Proof. There is V which an open neighborhood of e in G such that $V^4 \subset Exp(B(O, \epsilon))$ and $\bar{V}$ is compact. There are $\{g_{0,i}\}_{i=1}^{N_0}$ and $\{\epsilon_{0,i}\}_{i=1}^{N_0} \subset (0, \infty)$ such that $\pi(\bar{V}^4) \subset \cup_{i=1}^{N_0} \pi(g_{0,i} Exp(B_k(O, \epsilon_{0,i})))$ and $g_{0,i} Exp(B_k(O, \epsilon_{0,i}) \subset Exp(B_k(O, \epsilon) g_{0,i}$ $(\forall i)$.

And for each $s \in \mathbb{N}$ there are $\{g_{s,i}\}_{i=1}^{N_s}$ and $\{\epsilon_{s,i}\}_{i=1}^{N_s} \subset (0, \infty)$ such that $\pi(\bar{V}^{4+s}) \backslash \pi(V^{3+s}) \subset \cup_{i=1}^{N_s} \pi(g_{s,i} Exp(B_k(O, \epsilon_{s,i})))$ and $g_{s,i} Exp(B_k(O, \epsilon_{s,i}) \subset Exp(B_k(O, \epsilon) g_{s,i}$ $(\forall i)$.

We set $\{g_i\}_{i=1}^{\infty} := \{g_{s,i} | s, i \in \mathbb{N}, 1 \leq i \leq N_s\}$ and $\{U_i\}_{i=1}^{\infty} := \{U_{s,i} | s, i \in \mathbb{N}, 1 \leq i \leq N_s\}$. We will show for any $i \in \mathbb{N}$ and $s \in \mathbb{N}$,

$$\pi(g_{s,i}) \notin \pi(V^{s+2})$$

For aiming contradiction, let us assume $s \in \mathbb{N}$ and $i \in \mathbb{N}$ such that $\pi(g_{s,i}) \in \pi(V^{s+2})$. So,

$$\pi(g_{s,i} Exp(B_k(O, \epsilon_{s,i}))) \subset \pi(Exp(B_k(O, \epsilon)) g_{s,i}) \subset \pi(V^{s+3})$$

This contradicts with

$$\pi(g_{s,i} Exp(B_k(O, \epsilon_{s,i}))) \cap \pi(V^{s+3})^c \neq \phi$$

Nextly, we will show for any $i \in \mathbb{N}$ and $s \in \mathbb{N}$,

$$\pi(g_{s,i}) \cap \pi(V^{s+1}) = \phi$$

For aiming contradiction, let us assume $s \in \mathbb{N}$ and $i \in \mathbb{N}$ such that $\pi(g_{s,i} Exp(B_k(O, \epsilon_{0,i})) \cap \pi(V^{s+1}) \neq \phi$. Then there is $X \in B_k(O, \epsilon)$ and $u \in V^{s+2}$ such that $\pi(Exp(X) g_{s,i}) = \pi(u)$. So, $\pi(g_{s,i}) = \pi(Exp(X) u) \in \pi(V^{s+2})$. This is a contradiction. So,

$$(g_{s,i} Exp(B_k(O, \epsilon_{s,i}))) \cap \pi(V^s) = \phi$$

□

By the same argument as the proof of Proposition6.4.10, the following holds.

Proposition 6.4.11. *Here are the settings and assumptions.*

(S1) G be a Lie group such that $dimLie(G) > 0$.

(S2) $\epsilon > 0$.

(S3) $\mathfrak{g} := Lie(G)$ and $m := dim\mathfrak{g}$.

Then there are $\{g_i\}_{i=1}^{\infty} \subset G$ and $\{U_i\}_{i=1}^{\infty}$ such that U_i is a open neighborhood of 0_m $(\forall i)$ and $U_i \subset B_m(O, \epsilon) \cap \mathfrak{g}$ $(\forall i)$ and $\{g_i Exp(U_i)\}_{i \in \mathbb{N}}$ is an open covering of G and for any $i \in \mathbb{N}$ $\#\{j \in \mathbb{N} | g_i Exp(U_i) \cap g_j Exp(U_j) \neq \phi\} < \infty$.

Proposition 6.4.12. *Here are the settings and assumptions.*

(S1) G be a Lie group.

(S2) H be a closed subgroup of G such that $dimLie(H) > 0$.

(S3) $\epsilon > 0$.

(S4) $\mathfrak{g} := Lie(G)$, $\mathfrak{h} := Lie(H)$.

(S5) $\mathfrak{q}$ is a complement subspace of $\mathfrak{h}$ in $\mathfrak{g}$.

Then there are $\{g_i\}_{i=1}^{\infty} \subset G$ and $\{U_i\}_{i=1}^{\infty}$ and $\{h_j\}_{j=1}^{\infty} \subset H$ and $\{V_j\}_{j=1}^{\infty}$ such that U_i is a open neighborhood of 0_k $(\forall i)$ and $U_i \subset B_k(O, \epsilon) \cap \mathfrak{q}$ $(\forall i)$ and V_j is a open neighborhood of 0_l $(\forall j)$ and $V_j \subset B_l(O, \epsilon) \cap \mathfrak{h}$ $(\forall j)$ and V_j is a open neighborhood of 0_l $(\forall j)$ and $g_i Exp(U_i) h_j Exp(V_j) \in \mathcal{O}(G)$ $(\forall i, j)$ and for any $i, j \in \mathbb{N}$

$$U_i \times V_j \ni (X, Y) \mapsto g_i Exp(X) h_j Exp(Y) \in g_i Exp(U_i) h_j Exp(V_j)$$

is a C^{ω}-class diffeomorphism and $\{g_i Exp(U_i) h_j Exp(V_j)\}_{i,j \in \mathbb{N}}$ is a local finite open covering of G and $\{\pi(g_i Exp(U_i))\}_{i \in \mathbb{N}}$ is a local finite open covering of G/H and $\{h_j Exp(V_j))\}_{j \in \mathbb{N}}$ is a local finite open covering of H.

Proof. Let $\{g_i\}_{i=1}^{\infty}$ and $\{U_i\}_{i=1}^{\infty}$ be the one in Proposition6.4.10. Let $\{h_j\}_{j=1}^{\infty}$ and $\{V_j\}_{j=1}^{\infty}$ be the one in Proposition6.4.11. By Theomrem6.1.1, we can assume for each $i, j \in \mathbb{N}$

$$U_i \times V_j \ni (X, Y) \mapsto g_i Exp(X) h_j Exp(Y) \in G$$

is a C^{ω}-class diffeomorphism to an open neighborhood of $g_i h_j$. So, it is enough to show $\{g_i U_i h_j V_j\}_{i,j \in \mathbb{N}}$ is local finite. Let us fix any $i, j \in \mathbb{N}$. For each $i', j' \in \mathbb{N}$,

$$g_i U_i h_j V_j \cap g_{i'} U_{i'} h_{j'} V_{j'} \neq \phi \implies \pi(g_i U_i) \cap \pi(g_{i'} U_{i'}) \neq \phi$$

So,

$$\#\{i' \in \mathbb{N} | \exists j' \text{ s.t } g_i U_i h_j V_j \cap g_{i'} U_{i'} h_{j'} V_{j'} \neq \phi\} < \infty$$

We denote this set by I. Let us fix any $i_0 \in I$. Because $(g_{i_0} \bar{U_{i_0}})^{-1} g_i \bar{U}_i h_j \bar{V}_j \cap H$ is compact, there are $j_1, ..., j_M$ such that

$$(g_{i_0} \bar{U}_{i_0})^{-1} g_i \bar{U}_i h_j \bar{V}_j \cap H \subset \cup_{a=1}^{M} h_{j_a} V_{j_a}$$

This implies

$$\{j' | g_i U_i h_j V_j \cap g_{i_0} U_{i_0} h_j V_{j'} \neq \phi\} \subset \cup_{a=1}^{M} \{j' | h_{j_a} V_{j_a} \cap h_{j'} V_{j'} \neq \phi\}$$

So,

$$\#\{j' | g_i U_i h_j V_j \cap g_{i_0} U_{i_0} h_j V_{j'} \neq \phi\} < \infty$$

□

Theorem 6.4.13. *Here are the settings and assumptions.*

(S1) G be a Lie group.

(S2) H be a closed subgroup of G such that $dimLie(H) > 0$.

(A1) For any $h \in H$,

$$|detAd_G(h)| = |det(Ad_H(h))|$$

(S3) μ_H is a left invariant meausre induced by a left invariant form on H.

(S4) $\mu_{G/H}$ is a invariant measure induced by Theorem6.4.9.

(S5) μ_G is a left invariant meausre induced by a left invariant form ω_0 on G.

Then there is $c \in \mathbb{R}$ such that for any $f \in C_c(G)$

$$\int_G f(g) d\mu_G(g) = c \int_{G/H} \bar{f}(x) d\mu_{G/H}(x)$$

Here

$$\bar{f}(gH) = \int_H f(gh) d\mu_H(h) \ (gH \in G/H)$$

$\bar{f}$ is well-defined and $\bar{f}$ is continuous.

STEP1. $\bar{f}$ is well-defined and $\bar{f}$ is continuous. If $g_1 H = g_2 H$, because $g_2^{-1} g_1 \in H$,

$$\int_H f(g_1 h) d\mu_H(h) = \int_H f(g_2 g_2^{-1} g_1 h) d\mu_H(h) = \int_H f(g_2 h) d\mu_H(h)$$

So, $\bar{f}$ is well-defined. Because f is uniformly continuous and $gExp(U)H$ is an open neighborhood of gH for any open neighborhood of e U, $\bar{f}$ is continuous. □

STEP2. Construction of a left invariant measure μ from invariant measures on G/H and H. We set

$$I : C_c^+(G) \ni f \mapsto \int_G \bar{f}(x) d\mu_{G/H}(x) \in \mathbb{R}_+$$

By Riez-Markov-Kakutani Theorem, I induces the baire measure μ on G. □

STEP3. Construction of a local coordinates system. We set $\mathfrak{g} := Lie(G)$ and $\mathfrak{h} := Lie(H)$. We fix $\mathfrak{q}$ which is the complement of $\mathfrak{h}$. $k := dim\mathfrak{q}$ and $m := \mathfrak{g}$ and $l := \mathfrak{q}$. There is $\delta_1 > 0$ such that

$$B_k(O,\delta_1) \cap \mathfrak{q} \times B_l(O,\delta_1) \cap \mathfrak{h} \ni (Y,Z) \mapsto exp(Y)exp(Z) \in G$$

is a C^ω-class diffeomorphism to an open neighborhood of e. For each $g \in G$ and $h \in H$,

$$g(Exp(B_k(O,\delta_1) \cap \mathfrak{q})h(B_l(O,\delta_1) \cap \mathfrak{h}) = gh(Exp(Ad_G(h^{-1})B_k(O,\delta_1) \cap \mathfrak{q})(B_l(O,\delta_1) \cap \mathfrak{h})$$

So, there is $\delta_2 > 0$ such that

$$B_k(O,\delta_2) \cap \mathfrak{q} \times B_l(O,\delta_2) \cap \mathfrak{h} \ni (Y,Z) \mapsto gexp(Y)hexp(Z) \in G$$

is a C^ω-class diffeomorphism to an open neighborhood of gh. There are $\{g_i\}_{i=1}^\infty \subset G \setminus H \cup \{e\}$ and $\{h_i\}_{i=1}^\infty \subset H$ and $\{U_i\}_{i=1}^\infty$ and $\{V_i\}_{i=1}^\infty$ such that s U_i is an open neighborhood of 0_k $(\forall i)$ and V_i is an open neighborhood of 0_k $(\forall i)$ and $\{\pi(g_iU_i)\}_{i=1}^\infty$ is a local finite covering of G/H and $\{h_iV_i\}_{i=1}^\infty$ is a local finite covering of H and $\{g_iU_ih_jV_j\}_{i,j=1}^\infty$ is a local finite covering of G. We denote a partition of unity correspoinding to $\{\pi(g_iU_i)\}_{i=1}^\infty$ by $\{\alpha_i\}_{i=1}^\infty$ and denote a partition of unity correspoinding to $\{h_jV_j)\}_{j=1}^\infty$ by $\{\beta_j\}_{j=1}^\infty$. Then clearly $\{\alpha_i\beta_j\}_{i,j=1}^\infty$ is a partition of unity correspoinding to $\{g_iU_ih_jV_j\}_{i,j=1}^\infty$. □

STEP4. Construction of a C^∞-form ω. We set for each $i,j \in \mathbb{N}$,

$$\omega_{g_iExp(X)h_jExp(Y)} := \Phi_{1,i}(g_iExp(X))\Phi_{2,j}(h_jExp(Y))d\phi_{1,i}^1 \wedge d\phi_{1,i}^2 \wedge ... \wedge d\phi_{1,i}^k \wedge d\phi_{2,j}^1 \wedge d\phi_{2,j}^2 \wedge ... \wedge d\phi_{1,j}^l \ (X \in U_i, Y \in V_j, i,j \in \mathbb{N})$$

We will show ω is well-defined. Let us fix any $i_1, j_1, i_2, j_2 \in \mathbb{N}$, $X_1 \in U_{i_1}$, $Y_1 \in V_{j_1}$, $X_2 \in U_{i_2}$, $Y_2 \in V_{j_2}$ such $g_{i_1}Exp(X_{i_1})h_{j_1}Exp(Y_{j_1}) = g_{i_2}Exp(X_{i_2})h_{j_2}Exp(Y_{j_2})$. We set

$$g_1 := g_{i_1}Exp(X_{i_1}), g_2 := g_{i_2}Exp(X_{i_2}), h_1 := h_{j_1}Exp(Y_{j_1}), h_2 := h_{j_2}Exp(Y_{j_2})$$

Because $h_0 := g_2^{-1}g_1 \in H$, $\pi(g_1) = \pi(g_2)$. So, by Lemma6.4.4,

$$\Phi_{1,i_1}(g_1)d\phi_{1,i_1}^1 \wedge d\phi_{1,i_1}^2 \wedge ... \wedge d\phi_{1,i_1}^k = \Phi_{1,i}(g_2)d\phi_{1,i_2}^1 \wedge d\phi_{1,i_2}^2 \wedge ... \wedge d\phi_{1,i_2}^k$$

So, $h_0h_1 = h_2$. Because μ_H is left invariant, by Lemma3.6.6,

$$\begin{aligned}
&\Phi_{2,j_2}(h_2)d\phi_{2,j_2}^1 \wedge d\phi_{2,j_2}^2 \wedge ... \wedge d\phi_{2,j_2}^l = \Phi_{2,j_2}(h_0h_1)d\phi_{2,j_2}^1 \wedge d\phi_{2,j_2}^2 \wedge ... \wedge d\phi_{2,j_2}^l \\
&= det(J(\phi_1 \circ \text{Ł}_{h_0^{-1}} \circ \psi_2)(\phi_2(h_1)))\Phi_{1,j_1}(h_1)d\phi_{2,j_2}^1 \wedge d\phi_{2,j_2}^2 \wedge ... \wedge d\phi_{2,j_2}^l \\
&= \Phi_{1,j_1}(h_1)d\phi_{1,j_1}^1 \wedge d\phi_{1,j_1}^2 \wedge ... \wedge d\phi_{1,j_1}^l
\end{aligned}$$

So, ω is well-defined. □

STEP5. The measure induced by ω is equal to μ. Let us fix any $f \in C_c(G)$.

$$\begin{aligned}
&\int_G f\omega = \sum_{i,j=1}^\infty \int_{g_iU_ih_iV_i} f\alpha_1\alpha_2\omega \\
&= \sum_{i,j=1}^\infty \int_{\psi_{1,i}(U_i)\times\psi_{2,j}(V_j)} f(g_iExp(X)h_jExp(Y))\alpha_1(g_iExp(X))\alpha_2(h_jExp(Y))\Phi_{1,i}(g_iExp(X))\Phi_{2,i}(h_jExp(Y))dXdY \\
&= \sum_{i=1}^\infty \int_{\psi_{1,i}(U_i)} \Phi_{1,i}(g_iExp(X))\alpha_1(g_iExp(X)) \sum_{j=1}^\infty \int_{\psi_{2,j}(V_j)} f(g_iExp(X)h_jExp(Y))\alpha_2(h_jExp(Y))\Phi_{2,i}(h_jExp(Y))dYdX \\
&= \sum_{i=1}^\infty \alpha_1(g_iExp(X)) \int_{\psi_{1,i}(U_i)} \Phi_{1,i}(g_iExp(X)) \int_H f(g_iExp(X)h)d\mu_H(h)dX \\
&= \sum_{i=1}^\infty \int_{\psi_{1,i}(U_i)} \alpha_1(g_iExp(X))\Phi_{1,i}(g_iExp(X))\bar{f}(g_iExp(X))dX = \int_{G/H} \bar{f}(x)d\mu_{G/H}(x) = I(f)
\end{aligned}$$

So, ω introduces μ. By Proposition3.6.8, ω is left invariant form. Consequently, there is $c \in \mathbb{R}$ such that $\omega = c\omega_0$. This implies $\mu = c\mu_G$. □

In speciality, the following holds.

Proposition 6.4.14. *Here are the settings and assumptions.*

(S1) G be a compact Lie group.

(S2) H be a closed subgroup of G.

Then G/H has a invariant measure induced by a C^∞ form.

6.4.2 $L^p(G/H)$

By the same argument as the proof of Proposition3.6.16, the following holds.

Proposition 6.4.15. *Here are the settings and assumptions.*

(S1) G *be a Lie group.*

(S2) H *be a closed subgroup of* G.

(A1) For any $h \in H$,

$$|detAd_G(h)| = |det(Ad_H(h))|$$

Then $L^p(G/H)$ *is separable for any* $p \in \mathbb{N} \cap [1, \infty)$.

By the proof of Proposition6.4.15, the following holds.

Proposition 6.4.16. *Here are the settings and assumptions.*

(S1) G *be a Lie group.*

(S2) H *be a closed subgroup of* G.

(A1) For any $h \in H$,

$$|detAd_G(h)| = |det(Ad_H(h))|$$

Then there is at most countable subset of $C_c(G/H)$ *which is dense in* $L^p(G/H)$ *for any* $p \in \mathbb{N} \cap [1, \infty)$.

6.5 Basics of Fiber bundle

Definition 6.5.1 (Topological transformation group)**.** *Let* G *be a topological group. And let* Y *be a topological space. If* $\eta : G \times Y \to Y$ *satisfies the following conditions, we say* G *is a topological transformation group of* Y *respects to* η.

(i) $\eta(e, \cdot) = id_Y$.

(ii) $\eta(g_2, \eta(g_1, \cdot)) = \eta(g_2 g_1, \cdot)$ $(\forall g_1, g_2 \in G)$.

If is clear what η *is, we denote* $gy := \eta(g, y)$.

Definition 6.5.2 (Effective topological transformation group)**.** *Let* G *be a topological transformation group of a topological space* Y *respects to* η. *We say that* G *is effective if* $\eta(g, \cdot) = id_Y$ *only if* $g = e$.

Definition 6.5.3 (Coordinate bundle)**.** *We call*

$$\mathfrak{B} := (B, X, Y, p, \{V_j\}_{j \in J}, \{\phi_j\}_{j \in J}, G)$$

a coordinate bundle if

(i) B, X, Y *are toplogical spaces.* B *is called a bundle space or total space.* X *is called a base space.* Y *is called a fibre.*

(ii) $p : B \to X$ *is a surjective and continuous map.* p *is called a projection.*

(iii) G *is a topological transformation group of* Y *respects to* η *and* G *is effective.*

(iii) $\{V_j\}_{j \in J}$ *is an open covering of* X. *We call each* V_j *a coordinate neighborhood.*

(iv) $\phi_j : V_j \times Y \to p^{-1}(V_j)$ *is an isomorphism. We call* $\phi_j^{-1} : p^{-1}(V_j) \to V_j \times Y$ *a local trivialization or a coordinate function. For each* $x \in V_j$, *we call* $Y_x := p^{-1}(x)$ *a fiber on* x.

(v) $p \circ \phi_j(x, y) = x$ $(\forall j \in J, \forall x \in V_j, \forall y \in Y)$

(vi) If $V_i \cap V_j \neq \phi$, *for each* $x \in V_i \cap V_j$, *we define* $\phi_{i,x} : Y \to Y$ *by*

$$\phi_{i,x}(y) := \phi_i(x, y)$$

Then there is the unique $g_{j,i}(x) \in G$ *such that*

$$\phi_{j,x}^{-1} \circ \phi_{i,x}(\cdot) = \eta(g_{j,i}(x), \cdot)$$

is an isomorphism.

(vii) $g_{j,i} : V_i \cap V_j \to G$ *is continuous.*

Memo 6.5.4. *I think, roughly speaking, a coordinate bundle is a pair (B, X, Y, p) with local trivializations $(\{V_i\}_{i\in I}, \{\phi_i\}_{i\in I}$ which induce a system of coordinate transformations $\{g_{i,j}\}_{i,j\in I}$. Steenrod Theorem, which is showed later, states a system of coordinate transformations induces a local trivializations.*

Definition 6.5.5 (Equivalent in the strict sense between two coordinate bundles)**.** *Let*

$$\mathfrak{B}_1 := (B_1, X_1, Y, p_1, \{V_{1,j}\}_{j\in J_1}, \{\phi_j\}_{j\in J_1}, G)$$

and

$$\mathfrak{B}_2 := (B_2, X_2, Y, p_2, \{V_{2,j}\}_{j\in J_2}, \{\phi_j\}_{j\in J_2}, G)$$

are coordinate bundles. We say that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are equivalent in the strict sense if

(i) $B_1 = B_2, X_1 = X_2, Y_1 = Y_2, G_1 = G_2$.

(ii) Fix any $j_1 \in J_1$ and $j_2 \in J_2$ such that $V_{1,j_1} \cap V_{2,j_2} \neq \phi$. For any $x \in V_{1,j_1} \cap V_{2,j_2}$, there is unique $g_{j_2,j_1}(x) \in G$ such that

$$g_{j_2,j_1}(x) = \phi_{2,x}^{-1} \circ \phi_{1,x}$$

and

$$g_{j_2,j_1} : V_{1,j_1} \cap V_{2,j_2} \to G$$

is continuous.

Proposition 6.5.6. *The relation in Definition6.5.5 is equivalent relation.*

Definition 6.5.7 (Fibre bundle)**.** *We define that a fibre bundle is a equivalent class by strict sense equivalent of coordinate bundles.*

Clearly the following holds.

Proposition 6.5.8 (Smooth bundle, Holomorphic bundle.)**.** *Let*

(S1)

$$\mathfrak{B} := (B, X, Y, p, \{V_j\}_{j\in J}, \{\phi_j\}_{j\in J}, G) isacoordinatebundle.$$

(S2) X, Y, G are C^∞-class manifolds.

(A1) G is a Lie group.

(A2) The action of G on X is C^∞-class.

Then B is a C^∞-class manifold. We call B a smooth corrdinate bundle.

And we call B is a holomorphic coordinate bundle if X, Y are complex manifolds and G is complex Lie group.

Definition 6.5.9 (Bundle map)**.** *Let*

$$\mathfrak{B}_1 := (B_1, X_1, Y, p_1, \{V_{1,j}\}_{j\in J_1}, \{\phi_{1,j}\}_{j\in J_1}, G)$$

and

$$\mathfrak{B}_2 := (B_2, X_2, Y, p_2, \{V_{2,j}\}_{j\in J_2}, \{\phi_{2,j}\}_{j\in J_2}, G)$$

are coordinate bundles. We call $(h, \bar{h})$ a bundle map from $\mathfrak{B}_1$ to $\mathfrak{B}_2$ if

(i) $h : B_1 \to B_2$ is a continuous map.

(ii) $\bar{h} : X_1 \to X_2$ is a continuous map.

(iii) For each $x \in X$, $x' := h(x)$ and $Y_x := p^{-1}(x)$ and $Y_{x'} := p^{-1}(x')$ and $h_x := h|Y_x$. Then $h_x : Y_x \to Y_{x'}$ is an homeomorphism.

(iv) For any $x \in V_{1,j} \cap \bar{h}^{-1}(V_{2,k})$, there is unique $\bar{g}_{k,j}(x) \in G$ such that

$$\phi_{2,\bar{h}(x)}^{-1} \circ h_x \circ \phi_{1,x} = \bar{g}_{k,j}(x)\cdot$$

(iv) $\bar{g}_{k,j} : V_{1,j} \cap \bar{h}^{-1}(V_{2,k}) \to G$ is continuous. We call $\bar{g}_{k,j}$ a mapping transformation.

We also call h itself a bundle map and call $\bar{h}$ a map induced by h or call $\bar{h}$ the induced map from h.

Proposition 6.5.10. *The followings hold.*

(i) The identity map of any coordinate bundle is a bundle map.

(ii) The composition of any two bundle maps is a bundle map.

Proof of (i). This is clear because of the definition of coordinate bundle. □

Proof of (ii). Let

$$\mathfrak{B}_i := (B_i, X_i, Y, p_i, \{V_{i,j}\}_{j\in J_i}, \{\phi_j\}_{j\in J_i}, G)\ (i = 1, 2, 3)$$

be corrdinate bundles and $(h_1, \bar{h_1})$ be a bundle map from $\mathfrak{B}_1$ to $\mathfrak{B}_2$ and $(h_2, \bar{h_2})$ be a bundle map from $\mathfrak{B}_2$ to $\mathfrak{B}_3$. We set $h_3 := h_2 \circ h_1$ and $\bar{h_3} := \bar{h_2} \circ \bar{h_1}$. Clearly, h_3 and $\bar{h_3}$ are continuous. For any $x \in X$, clearly,

$$h_{3,x} = h_{2,\bar{h_1}(x)} \circ h_{1,x}$$

So, $h_{3,x}$ is a homeomorphism from Y_x to $Y_{h_3(x)}$.

Let us fix any $x \in V_{1,j} \cap \bar{h_3}^{-1}(V_{3,k})$. Clearly

$$\bar{h_3}^{-1}(V_{3,k}) = \bar{h_1}^{-1}(\bar{h_2}^{-1}(V_{3,k}))$$

This implies

$$\bar{h_1}(x) \in \bar{h_2}^{-1}(V_{3,k})$$

Because $\{V_{2,j}\}_{j\in J_2}$ is an open covering of X, there is $j \in J_2$ such that

$$\bar{h}_1(x) \in V_{2,j}$$

So,

$$\begin{aligned}
& \phi^{-1}_{3,\bar{h_3}(x)} \circ h_{3,x} \circ \phi_{1,x} \\
= \ & \phi^{-1}_{3,\bar{h_2}(\bar{h_1}(x))} \circ h_{2,x} \circ h_{2,x} \circ \phi_{1,x} \\
= \ & \phi^{-1}_{3,\bar{h_2}(\bar{h_1}(x))} \circ h_{2,x} \circ \phi_{2,\bar{h_1}(x)} \circ \phi^{-1}_{2,\bar{h_1}(x)} \circ h_{2,x} \circ \phi_{1,x} \\
= \ & \bar{g}_{2,k,j}(\bar{h}_1(x))\bar{g}_{1,j,i}(x)
\end{aligned}$$

Clearly $\bar{g}_{2,k,j}(\bar{h}_1(\cdot))\bar{g}_{1,j,i}(\cdot)$ is continuous on $V_{1,j} \cap \bar{h_3}^{-1}(V_{3,k}) \cap \bar{h}_1^{-1}(V_{2,j})$. □

Definition 6.5.11 (Equivalent between two coordinate bundles)**.** *Let*

$$\mathfrak{B}_1 := (B_1, X_1, Y, p_1, \{V_{1,j}\}_{j\in J_1}, \{\phi_j\}_{j\in J_1}, G)$$

and

$$\mathfrak{B}_2 := (B_2, X_2, Y, p_2, \{V_{2,j}\}_{j\in J_2}, \{\phi_j\}_{j\in J_2}, G)$$

are coordinate bundles. We say that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are equivalent if there is h such that (h, id_X) is a bundle map from $\mathfrak{B}_1$ to $\mathfrak{B}_2$.

The following is clear from the definition of bundle map.

Proposition 6.5.12. *Let*

$$\mathfrak{B}_1 := (B_1, X_1, Y, p_1, \{V_{1,j}\}_{j\in J_1}, \{\phi_{1,j}\}_{j\in J_1}, G)$$

and

$$\mathfrak{B}_2 := (B_2, X_2, Y, p_2, \{V_{2,j}\}_{j\in J_2}, \{\phi_{2,j}\}_{j\in J_2}, G)$$

are coordinate bundles. And $(h, \bar{h})$ is a bundle map from $\mathfrak{B}_1$ to $\mathfrak{B}_2$. Then the followings hold.

$$\bar{g}_{j,i}(x)g_{i,k}(x) = \bar{g}_{j,k}(x)\ (\forall x \in V_{1,i} \cap V_{1,k} \cap \bar{h}^{-1}(V_{2,j})) \tag{6.5.1}$$

$$g_{j,i}(\bar{h}(x))\bar{g}_{i,k}(x) = \bar{g}_{j,k}(x)\ (\forall x \in V_{1,k} \cap \bar{h}^{-1}(V_{1,i} \cap V_{2,j})) \tag{6.5.2}$$

Lemma 6.5.13. *Let*

$$\mathfrak{B}_1 := (B_1, X_1, Y, p_1, \{V_{1,j}\}_{j\in J_1}, \{\phi_{1,j}\}_{j\in J_1}, G)$$

and

$$\mathfrak{B}_2 := (B_2, X_2, Y, p_2, \{V_{2,j}\}_{j\in J_2}, \{\phi_{2,j}\}_{j\in J_2}, G)$$

are coordinate bundles. And let us assume $\bar{h}$ is a continous map from X_1 to X_2. and there is $\{\bar{g}_{i,j}\}_{i,j\in J}$ such that for each $i, j \in J$ $\bar{g}_{i,j} \in C(V_j \cap \bar{h}^{-1}(V_i), G)$ and the followings hold.

$$\bar{g}_{j,i}(x)g_{i,k}(x) = \bar{g}_{j,k}(x)\ (\forall x \in V_{1,i} \cap V_{1,k} \cap \bar{h}^{-1}(V_{2,j}))$$

$$g_{j,i}(\bar{h}(x))\bar{g}_{i,k}(x) = \bar{g}_{j,k}(x)\ (\forall x \in V_{1,k} \cap \bar{h}^{-1}(V_{1,i} \cap V_{2,j}))$$

Then there is a bundle map h from $\mathfrak{B}_1$ to $\mathfrak{B}_2$ such that $\bar{h}$ is the induced map from h and for each $i, j \in J$ $\bar{g}_{i,j}$ is a mapping transformations of h.

Proof. For each $i \in J_1$ and $j \in J_2$ such that $(V_{1,i} \cap \bar{h}^{-1}(V_{2,j})) \times Y \neq \phi$, we set

$$h(\phi_{1,i}(x,y)) = \phi_{2,j}(\bar{h}(x), \bar{g}_{j,i}(x)y) \ ((x,y) \in (V_{1,i} \cap \bar{h}^{-1}(V_{2,j})) \times Y)$$

We will show h is well-defined. Let us assume $(x,y) \in (V_{1,i} \cap \bar{h}^{-1}(V_{2,j})) \times Y$ and $(x',y') \in (V_{1,i'} \cap \bar{h}^{-1}(V_{2,j'})) \times Y$ and

$$\phi_{1,i}(x,y) = \phi_{1,i'}(x',y')$$

Then

$$x = p \circ \phi_{1,i}(x,y) = p \circ \phi_{1,i'}(x',y') = x'$$

So, $\phi_{1,i}(x,y) = \phi_{1,i'}(x,y')$. This implies

$$g_{i',i}(x)y = y'$$

So,

$$\bar{g}_{j,i}(x)y = \bar{g}_{j,i}(x)g_{i,i'}(x)y' = \bar{g}_{j,i'}(x)y'$$

So,

$$\begin{aligned} & \phi_{2,j}(\bar{h}(x), \bar{g}_{j,i}(x)y) = \phi_{2,j,\bar{h}(x)}(\bar{g}_{j,i}(x)y) = \phi_{2,j',\bar{h}(x)} \circ \phi^{-1}_{2,j',\bar{h}(x)} \circ \phi_{2,j,\bar{h}(x)}(\bar{g}_{j,i'}(x)y') = \phi_{2,j',\bar{h}(x)}(g_{j',j}(\bar{h}(x))\bar{g}_{j,i'}(x)y') \\ = \ & \phi_{2,j',\bar{h}(x)}(\bar{g}_{j',i'}(x)y') = \phi_{2,j'}(\bar{h}(x), \bar{g}_{j',i'}(x)y') \end{aligned}$$

Consequently, h is well-defined. Clearly, h is continuous. Also, clearly, for any $x \in V_{1,i} \cap \bar{h}^{-1}(V_{2,j})$, $h|Y_x$ is an homeomorphism from Y_x to $Y_{\bar{h}(x)}$ and

$$\phi^{-1}_{2,j,\bar{h}(x)} \circ h \circ \phi_{1,i,x} = \bar{g}_{j,i}(x)$$

□

Lemma 6.5.14. *The followings are the settings and assumptions.*

(S1)

$$\mathfrak{B}_1 := (B_1, X_1, Y, p_1, \{V_{1,j}\}_{j \in J_1}, \{\phi_{1,j}\}_{j \in J_1}, G)$$

and

$$\mathfrak{B}_2 := (B_2, X_2, Y, p_2, \{V_{2,j}\}_{j \in J_2}, \{\phi_{2,j}\}_{j \in J_2}, G)$$

are coordinate bundles.

(A1) $X_1 = X_2$.

(A2) There are $\bar{g}_{k,j} : V_j \cap V'_k \to G$*:continuous map(*$j \in J_1$, $k \in J_2$*) such that*

$$\bar{g}_{k,j}(x)g_{j,i}(x) = \bar{g}_{k,i}(x) \ (\forall x \in V_i \cap V_j \cap V'_k), g'_{l,k}(x)\bar{g}_{k,j}(x) = \bar{g}'_{l,j}(x) \ (\forall x \in V_i \cap V'_l \cap V'_k)$$

Then $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *are equivalent.*

Proof. It is from Proposition6.5.13. □

Lemma 6.5.15. *The followings are the settings and assumptions.*

(S1)

$$\mathfrak{B}_1 := (B_1, X_1, Y, p_1, \{V_j\}_{j \in J}, \{\phi_j\}_{j \in J_1}, G)$$

and

$$\mathfrak{B}_2 := (B_2, X_2, Y, p_2, \{V_j\}_{j \in J}, \{\phi_j\}_{j \in J_2}, G)$$

are coordinate bundles.

(A1) $X_1 = X_2$.

(A2) There are $\lambda_j : V_j \to G$*:continuous map(*$j \in J$*) such that*

$$g'_{i,j}(x) = \lambda_i(x)^{-1} g_{i,j}(x)\lambda_j(x) \ (\forall x \in V_i \cap V_j)$$

Then $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *are equivalent.*

Proof. We set

$$\bar{g}_{i,j}(x) := \lambda_i(x)^{-1} g_{i,j}(x) \ (x \in V_i \cap V_j)$$

Then

$$\bar{g}_{i,j}(x) g_{j,k}(x) = \lambda_i(x)^{-1} g_{i,j}(x) g_{j,k}(x) = \lambda_i(x)^{-1} g_{i,k}(x) = \bar{g}_{i,k}(x)$$

and

$$g'_{k,i}(x)\bar{g}_{i,j}(x) = \lambda_k(x)^{-1} g_{k,i}(x)\lambda_i(x)\lambda_i(x)^{-1} g_{i,j}(x) = \lambda_k(x)^{-1} g_{k,i}(x) g_{i,j}(x) = \lambda_k(x)^{-1} g_{k,j}(x) = \bar{g}_{k,j}(x)$$

So, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are equivalent from Lemma6.5.14. □

Definition 6.5.16 (System of coordinate transformations)**.** *Let*

(S1) *G is a topological group.*

(S2) *X is a topological space.*

We call $(\{V_j\}_{j\in J}, \{g_{i,j}\}_{i\in J})$ a system of coordinate transformations in X with values in G if

(i) *$\{V_j\}_{j\in J}$ is an open covering of X.*

(ii) *$g_{j,i} \in C(V_j \cap V_i, G)$ $(\forall i, j \in J)$.*

(iii) *$g_{k,j} \circ g_{j,i} = g_{k,i}$ in $V_k \cap V_j \cap V_i$ $(\forall i, j, k \in J)$.*

Clearly the following holds.

Proposition 6.5.17. *Let*

(S1) *G is a topological group.*

(S2) *X is a topological space.*

(S3) *$(\{V_j\}_{j\in J}, \{g_{i,j}\}_{i\in J})$ is a a system of coordinate transformations in X with values in G.*

Then the followings hold.

(i) *$g_{i,i} = e$ $(\forall i \in J)$.*

(ii) *$g_{i,j} = g_{j,i}^{-1}$ $(\forall i, j \in J)$.*

Theorem 6.5.18 (Steenrod's theorem)**.** *Let*

(S1) *G is a topological group.*

(S2) *X is a topological space.*

(S3) *$(\{V_j\}_{j\in J}, \{g_{i,j}\}_{i\in J})$ is a system of coordinate transformations in X with values in G.*

(S4) *Y is a topological space.*

(S5) *G is a topological transformation group of Y.*

(A1) *The action of G on Y is effective.*

Then

(i) *There is $B, p, \{\phi_j\}_{j\in J}$ such that $(B, X, p, \{V_j\}_{j\in J}, Y, \{\phi_j\}_{j\in J})$ is a coordinate bundle and for any $j, i \in J$ such that $V_i \cap V_j \neq \phi$, for any $x \in V_i \cap V_j$, in $V_i \cap V_j$,*

$$\phi_{j,x}^{-1} \circ \phi_{i,x} = g_{j,i}$$

(ii) *If B_1 and B_2 are topological spaces which individually satisfy (i), $(B_1, X, p, \{V_j\}_{j\in J}, Y, \{\phi_j^1\}_{j\in J})$ and $(B_2, X, p, \{V_j\}_{j\in J}, Y, \{\phi_j^2\}_{j\in J})$ are equivalent.*

STEP1. Construction of B and $\{\phi_j\}_{j\in J}$. Hereafter, let us assume the topology of J is the discrete topology. We set

$$T := X \times Y \times J$$

We define the relation of T by

$$(x, y, j) \sim (x', y', k) :\iff x = x' \text{ and } y' = g_{k,j}(x)y$$

We will show $\sim$ is a equivalent relation of T. Because $g_{i,i} = e$, the reflexivity of $\sim$ holds. Because $g_{i,i} = e$, by (S5), the reflexivity of $\sim$ holds. Because $g_{i,j} = g_{j,i}^{-1}$, by (S5), the symmetry of $\sim$ holds. Because $g_{k,j} \circ g_{j,i} = g_{k,i}$, by (S5), the transitivity of $\sim$ holds. So $\sim$ is a equivalent relation.

We set
$$B := T/\sim$$
and
$$q : T \ni (x, y, j) \mapsto [x, y, j] \in B$$
and
$$p : B \ni [x, y, j] \mapsto x \in X$$
By the definition of $\sim$, p is well-defined. And, clearly, p is surjective. Let us assume that the topology of B is the final topology of B induced by q. For any $O \in \mathcal{O}(X)$,
$$q^{-1}(p^{-1}(O)) = O \times Y \times \{j \in J | V_j \cap O \neq \phi\}$$
In this equation, the right side is an open set of T. So, p is continuous.

We define $\phi_j : V_j \times Y \to B$ by
$$\phi_j(x, y) = [x, y, j]$$
Clearly, ϕ_j is continuous and
$$\phi_j : V_j \times Y \subset B$$
and
$$p \circ \phi_j = id_{V_j}$$
□

STEP2. Proof of that ϕ_j is an isomorphism. By STEP1, it is enough to show that ϕ_j is bijective and an open map. We will show that $\phi_j : V_j \times Y \to p^{-1}(V_j)$ is surjective. Let us fix any $[x, y, k] \in p^{-1}(V_j)$. Clealy $x \in V_k$ and
$$(x, y, k) \sim (x, g_{j,k}(x)y, j)$$
So,
$$[x, y, k] = \phi_j(x, g_{j,k}(x)y)$$
So ϕ_j is surjective.

Nextly, we will show that ϕ_j is injective. Let us fix any $(x, y), (x', y') \in V_j \times Y$ such that $[x, y, j] = [x', y', j]$. Then $x = x'$ and
$$g_{j,j}(x)y = y'$$
Because $g_{j,j}(x) = id_{V_j}$, $y = y'$. So ϕ_j is injective.

Lastly, we will show that ϕ_j is an open map. Let us fix $W_1 \times W_2 \subset V_j \times Y$ which is an open set. For any $k \in J$ such that $V_k \cap V_j \neq \phi$, we set $r_{j,k} : (V_k \cap V_j) \times Y \to (V_k \cap V_j) \times Y$ by
$$r_{j,k}(x, y) := (x, g_{j,k}(x)y)$$
By (S5), $r_{j,k}$ is continuous.

We will show for any $W \in \mathcal{O}(V_j \times Y)$,
$$q^{-1}(\phi_j(W)) = \bigcup_{k \in J, V_k \cap V_j \neq \phi} r_{j,k}^{-1}(W) \times \{k\} \tag{6.5.3}$$
Let us fix any $(x, y) \in (V_j \cap V_k) \times Y$ such that $r_{j,k}(x, y) \in W$. Because
$$\phi_j(x, g_{j,k}(x)y) = [r_{j,k}(x), j] = q(x, y, k) \tag{6.5.4}$$
in (6.5.3), the right side is containd the left side. By (6.5.4), it is clear that in (6.5.3), the left side is containd the right side. So, (6.5.3) holds. Clearly, in (6.5.3), the right side is an open set. So, ϕ_j is an open map. □

STEP3. Proof of (i). By STEP1 and STEP2, it is enough to show that for any $i, j \in J$ such that $V_i \cap V_j \neq \phi$ and any $x \in V_i \cap V_j \neq \phi$
$$\phi_{j,x}^{-1} \circ \phi_{i,x} = g_{j,i} \tag{6.5.5}$$
For any $y \in Y$
$$\begin{aligned} & \phi_{j,x}^{-1} \circ \phi_{i,x}(y) \\ = \; & \phi_{j,x}^{-1}([x, y, i]) \\ = \; & \phi_{j,x}^{-1}([x, g_{j,i}(x)y, j]) \\ = \; & g_{j,i}(x)y \end{aligned}$$
So (6.5.5) holds. □

STEP3. Proof of (ii).

$$\phi_{1,j,x}^{-1} \circ \phi^{1,i,x} = g_{j,i}(x) = \phi_{2,j,x}^{-1} \circ \phi_{2,i,x} \ (\forall i, j, \forall x \in V_i \cap V_j)$$

When we set $\lambda_i(x) := e \ (\forall i, \forall x \in V_I)$, $\{\lambda_i\}_i$ satisfies the conditions of Lemma6.5.15. So, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are equivalent. □

Proposition 6.5.19 (Tangent bundle)**.** *The following are settings and assumptions.*

(S1) $\{M, \{(U_i, \psi_i)\}_{i \in I}\}$ *is a n-dimensional* C^∞*-class manifold.*

(S2) $B := \cup_{x \in M}\{x\} \times T_x(M))$.

(S3) $p : B \ni (x, X) \mapsto x \in M$.

(S4) $Y := \mathbb{R}^n$.

(S5) $\phi_i : U_i \times Y \ni (x, v) \mapsto (x, \sum_{j=1}^n v_j \left(\frac{\partial}{\partial \psi_i^j}\right)_x) \in B$.

Then $\{B, p, M, \mathbb{R}^n, \{(U_i, \phi_i)\}_{i \in I}, GL(n, \mathbb{R})\}$ *is a coordinate bundle. We call the fibre bundle of the coordinate bundle tangent bundle of* M.

Proof. Clearly,

$$p \circ \phi_i(x, v) = x \ (\forall i \in I, \forall x \in U_i, v \in Y)$$

and

$$\phi_i(U_i \times Y) = p^{-1}(U_i)$$

and ϕ_i is injective and ϕ_i is C^∞-class and ϕ_i^{-1} is C^∞-class. So, ϕ_i is a local trivialization. And

$$\phi_{j,x}^{-1} \circ \phi_{i,x}(v) = J(\phi_{j,x}^{-1} \circ \phi_{i,x})(v)$$

and

$$U_i \cap U_j \ni x \mapsto J(\phi_{j,x}^{-1} \circ \phi_{i,x}) \in GL(n, \mathbb{R})$$

is C^∞-class. So, $\{J(\phi_{j,x}^{-1} \circ \phi_{i,x})\}_{x \in U_i \cap U_j}$ is a system of coordinate transformations. Consequently,

$$\{B, p, M, \mathbb{R}^n, \{(U_i, \phi_i)\}_{i \in I}, GL(n, \mathbb{R})\}$$

is a coordinate bundle. □

Definition 6.5.20 (Cross section)**.** *Let*

$$\mathfrak{B} := (B, X, Y, p, \{V_j\}_{j \in J}, \{\phi_j\}_{j \in J}, G)$$

is a coordinate bundle. We say $s : X \to B$ *is a cross-section if* s *is continuous and* $p \circ s = id|X$.

If $\mathfrak{B}$ *is a smooth bundle ans* s *is* C^∞ *class the we call* s *a smooth section. If* $\mathfrak{B}$ *is a holomorphic bundle ans* s *is a holomorphic the we call* s *a holomorphic section.*

Definition 6.5.21 (Vector Bundle, Line Bundle)**.** *Let*

$$\mathfrak{B} := (B, X, Y, p, G)$$

be a fibre bundle. We say $\mathfrak{B}$ *is a vector bundle if* $Y = \mathbb{R}^n$ *and* $G = GL(n, \mathbb{R})$ *and* G *acts on* Y *with* $g \cdot v = gv \ (g \in G, v \in Y)$.

We say $\mathfrak{B}$ *is a complex vector bundle if* $Y = \mathbb{C}^n$ *and* $G = GL(n, \mathbb{C})$ *and* G *acts on* Y *with* $g \cdot v = gv \ (g \in G, v \in Y)$.

We say $\mathfrak{B}$ *is a holomorphic vector bundle if* $\mathfrak{B}$ *is a complex vector bundle and a holomorphic fibre bundle.*

We say $\mathfrak{B}$ *is a line bundle if* $\mathfrak{B}$ *is a vector bundle and* $\dim V = 1$.

Definition 6.5.22 (Principal Bundle)**.** *Let*

$$\mathfrak{B} := (B, X, Y, p, G)$$

be a fibre bundle. We say $\mathfrak{B}$ *is a principal bundle if* $Y = G$ *and* G *acts on* Y *with* $g \cdot h = gv \ (g, h \in G)$ *in* $\mathfrak{B}$.

Definition 6.5.23 (G-equivariant fibre bundle)**.** *The followings are settings and assumptions.*

(i) $\mathcal{B} := \{B, X, Y, \pi, \{V_i\}_{i \in I}, H, K\}$ *is a fibre bundle.*

(ii) G *is a topological group.*

(iii) G *acts on* B *and* X*, indivisually.*

Then we say $\mathcal{B}$ is G-equivariant fibre bundle if

$$\pi(g \cdot u) = g \cdot \pi(u) \ (\forall g \in G, \forall u \in B)$$

When $\mathcal{B}$ is G-equivariant we say π is G-equivariant.

Clearly, the following holds.

Proposition 6.5.24. *The followings are settings and assumptions.*

(i) *$\mathcal{B} := \{B, X, Y, \pi, \{V_i\}_{i\in I}, H, K\}$ is a fibre bundle.*
(ii) *G is a topological group.*
(iii) *$\mathcal{B}$ is G-equivariant.*

Then for any $x \in X$ and $g \in G$,

$$\pi^{-1}(x) \ni b \mapsto g \cdot b \in \pi^{-1}(g \cdot x)$$

is a homeomorphism.

Definition 6.5.25 (G-equivariant vector bundle)**.** *The followings are settings and assumptions.*

(i) *$\mathcal{B} := \{B, X, V, \pi, \{V_i\}_{i\in I}, H, K\}$ is a vector bundle.*
(ii) *G is a topological group.*
(iii) *$\mathcal{B}$ is G-equivariant.*

Then we say $\mathcal{B}$ is G-equivariant vector bundle if for any $x \in X$ and $g \in G$,

$$\pi^{-1}(x) \ni b \mapsto g \cdot b \in \pi^{-1}(g \cdot x)$$

is a linear isomorphism.

6.6 Homogeneous Vector Bundle

Definition 6.6.1 (local cross-section)**.** *Let G be a Lie group and H be a closed subgroup of G and $\pi : G \to G/H$ be the projection and U be an open neighborhood of $\pi(e)$. We say $s : U \to G$ is a local cross-section if s is C^∞-class and $p \circ s = id|U$.*

Theorem 6.6.2. *Let G be a Lie group and H be a closed subgroup of G and $\pi : G \to G/H$ be the projection. Then the followings hold.*

(i) *There is an open neighborhood of $\pi(e)$ U such that $\mathcal{B} := \{G, G/H, H, \{gU\}_{g\in G}, H\}$ is a principal bundle.*
(ii) *$\mathcal{B}$ has a local cross-section.*

Proof of (i). We set $\mathfrak{h} := Lie(H)$ and denote a complement of $\mathfrak{h}$ by $\mathfrak{q}$.

By the proof of Theorem6.1.1, there exists $r > \epsilon > 0$ such that

$$\psi : B(O, r) \cap \mathfrak{q} \times B(O, r) \cap \mathfrak{h} \ni (X, Y) \mapsto exp(X)exp(Y) \in G$$

is a C^ω-class diffeomorphism to an open neighborhood of $p := \pi(e)$ and $exp(B(O, \epsilon))exp(B(O, \epsilon)) \subset exp(B(O, r)$.

We set $U := \pi(exp(B(O, \epsilon) \cap \mathfrak{q}))$.

We set

$$\phi_p : U \times H \ni (\pi(exp(X)), h) \mapsto exp(X)h \in G$$

Since ψ is a diffeomorphism, X is identified uniquely. So, ϕ_p is well-defined and C^ω-class. And clearly $\pi \circ \phi_p = id|U$ and $Im\phi_p \subset \pi^{-1}(U)$. Let us fix any $g \in \pi^{-1}(U)$. Then $\exists X \in B(O, \epsilon) \cap \mathfrak{q}$ and $h \in H$ such that $g = exp(X)h = \phi_p(\pi(exp(X)), h)$. So, ϕ_p is surjective. Let uf fix any $X_1, X_2 \in \mathfrak{q} \cap B(O, \epsilon)$ and $h_1, h_2 \in H$ such that $exp(X_1)h_1 = exp(X_2)h_2$. Then $exp(X_1) = exp(X_2)h_2h_1^{-1}$ and $h_2h_1^{-1} = exp(-X_2)exp(X_1) \in exp(B(O, r)$. Since ψ is injective, $h_2h_1^{-1} = e$. That implies $X_1 = X_2$. For each $h \in H$, by von-Neuman Cartan Theorem, $\phi_p|U \times exp(\mathfrak{h} \cap B(O, \epsilon))h$ is a C^ω-class diffeomorphism to an open neighborhood of h. So, ϕ_p itself is C^ω-class diffeomorphism to $\pi^{-1}(U)$.

For each $g \in G$, we set

$$\phi_{\pi(g)} : gU \times H \ni (\pi(gexp(X)), h) \mapsto gexp(X)h \in G$$

As same as the above argument, $\phi_{\pi(g)}$ is a C^ω-class diffeomorphism from $gU \times H$ to an open subset $\pi^{-1}(gU)$.

Nextly, let us fix any $x \in g_1U \cap g_2U$. Then $w_1 \in g_1exp(B(O,\epsilon)\cap\mathfrak{q})$ and $w_2 \in g_2exp(B(O,\epsilon)\cap\mathfrak{q})$ such that $\pi(w_1) = \pi(w_2)$. Then $h_0 := w_2^{-1}w_1 \in H$. So,

$$\phi_{\pi(g_1)}(w_1, h) = w_1h = w_2h_0h = \phi_{\pi(g_2)}(w_2, h_0h)$$

This means that

$$\phi^{-1}_{\pi(g_2),x} \circ \phi_{\pi(g_1),x}(h) = (w_2^{-1}w_1)h$$

and $\phi^{-1}_{\pi(g_2),x} \circ \phi_{\pi(g_1),x} = L_{(w_2^{-1}w_1)}$. Since

$$\pi(g_1Exp(B(O,\epsilon) \cap \mathfrak{q})) \ni \pi(w_1) \mapsto w_1 \in g_1Exp(B(O,\epsilon) \cap \mathfrak{q})$$

and

$$\pi(g_2Exp(B(O,\epsilon) \cap \mathfrak{q})) \ni \pi(w_2) \mapsto w_2 \in g_2Exp(B(O,\epsilon) \cap \mathfrak{q})$$

are C^ω class,

$$g_1U \cap g_2U \ni x \mapsto \phi^{-1}_{\pi(g_2),x} \circ \phi_{\pi(g_1),x} = L_{w_2^{-1}w_1} \in H$$

is C^ω class. Consequently, $\pi : G \to G/H$ is a C^ω class principal bundle whose structure group is H. □

Proof of (ii). We succeed notations in the proof of (i). We set

$$s : \pi(exp(B(O,\epsilon) \cap \mathfrak{q})) \ni \pi(exp(X)) \mapsto exp(X) \in G$$

Then s is clearly a local cross-section. □

Theorem 6.6.3 (Homogeneous vector bundle)**.** *The followings are settings and assumptions.*

(i) G *is a Lie group.*

(ii) H *is a closed subgroup of* G*.*

(iii) (π, V) *is a continuous representation of* H*.*

(iv) $(g_1, v_1) \sim (g_2, v_2) :\iff \exists h \in H$ *s.t* $g_1 = g_2h$ *and* $v_1 = \pi(h)^{-1}v_2$*.*

(v) $p : G \times V \ni (g, v) \mapsto [g, v] \in G/\sim$*. Let us define* $\mathcal{O}(G/\sim)$ *by* p*. We set* $G \times_H V := G/\sim$*.*

(vi) $q : G \times_H V \ni [g, v] \mapsto gH \in G/H$*.*

Then

(i) $\sim$ *is an equivalent relation on* $G \times V$*.*

(ii) q *is a vector bundle whose fibre is* V *and whose structure group is* H*.*

(iii) G *acts on* $G \times_H V$ *by* $g \cdot [x, v] := [gx, v]$ $g, x \in G, v \in V$*.*

(iv) For each $g \in G, v \in V$, $\{p(gU \times (v + B))\}_{U:nei.\ of\ e,\ B:nei.\ of\ 0}$ *is a basis of neiborhoods of* $[g, v]$*.*

Proof of (i). It is clear from the def. of $\sim$. □

Proof of (ii):q is well-defined and continuous. We set $\mathfrak{h} := Lie(H)$. Let $\mathfrak{q}$ denote a complement of $\mathfrak{h}$. Firstly, from the def. of $\sim$, q is well-defined. By the proof of Theorem6.1.1, there is $\epsilon > 0$ such that for each $g \in G$ $\phi_g : \mathfrak{q} \cap B(O,\epsilon) \ni X \mapsto gexp(X)H \in G/H$ is a homeomorphism from $\mathfrak{q}_\epsilon := \mathfrak{q} \cap B(O,\epsilon)$ to an open neighborhood of gH.

For each $g \in G$, $q^{-1}(\phi_g(\mathfrak{q}_\epsilon)) = p(B(O,\epsilon) \times V)$. Because $p^{-1}(p(B(O,\epsilon) \times V)) = B(O,\epsilon)H \times V$ and $B(O,\epsilon)H \times V$ is an open set, $q^{-1}(\phi_g(\mathfrak{q}_\epsilon))$ is an open set. So, q is a continuous. □

Proof of (ii):Local trivializations. For each $g \in G$, we set $\psi_g : gexp(\mathfrak{q}_\epsilon)H \times V \ni (gexp(X)H, v) \mapsto [gexp(X), v] \in G \times_H V$. Clearly, ψ_g is well-defined and continuous and $Im\psi_g \subset q^{-1}(\phi_g(\mathfrak{q}_\epsilon))$ and $q \circ \psi_g(gexp(X)H, v) = gexp(X)H$ $(\forall X \in \mathfrak{q}_\epsilon)$. Let us fix any $[x, v] \in q^{-1}(\phi_g(\mathfrak{q}_\epsilon))$. Then $\exists h \in H$ and $X \in \mathfrak{q}_\epsilon$ such that $xh = gexp(X)$. So, $[x, v] = [gexp(X), \pi(h^{-1})v] = \psi_g(gexp(X), \pi(h^{-1})v)$. Consequently, ψ_g is a local trivialization. □

Proof of (ii):A system of coordinate transformation. Let us fix any

$$\psi_{g_1}(g_1exp(X_1)H, v_1) = \psi_{g_2}(g_2exp(\iota(X_1))H, v_2) \in q^{-1}(\phi_{g_1}(\mathfrak{q}_\epsilon)) \cap q^{-1}(\phi_{g_2}(\mathfrak{q}_\epsilon))$$

Then $v_2 = \pi((g_2exp(X_2))^{-1}g_1exp(X_1))v_1$. So, $\{\psi_g\}_{g\in G}$ defines a system of coordinate transformation with the Lie group H. □

Proof of (iii). It is clear from the def. of action. □

Proof of (iv). It is clear from the def. of topology of $G \times_H V$. □

Theorem 6.6.4. *The followings are settings and assumptions.*

(i) G *is a Lie group.*

(ii) H *is a closed subgroup of* G*.*

(iii) (π, V) *is a continuous representation of* H*.*

(iv) $\Gamma(G/H, G \times_H V)$ *is the set of all cross sections of* q*.*

(v) $\iota : H \ni h \mapsto (1, h, h) \in G \times H \times H$*.*

(vi) $(g, h_1, h_2) \cdot f(x) := \pi(h_2) f(g^{-1} x h_1)$ $(g, h_1, h_2) \in G \times H \times H, x \in G, f \in C(G, V)$*.*

(vii) $C(G, V)^{\iota(H)} := \{f \in C(G, V) | \iota(h) f = f \ (\forall h \in H)\}$*. In this note, we sometimes may denote* $C(G, V)^{\iota(H)}$ *by* $C(G, V)^H$*.*

Then

(i) $G \times H \times H$ *acts on* $C(G, V)$ *based on the def. of (vi).*

(ii) $C(G, V)^{\iota H} \simeq \Gamma(G/H, V)$ *as purely algebraic representation of* G*. Remark that here we don't care about any topology of them and* G *acts on* $\Gamma(G/H, V)$ *by* $g \cdot s(xH) := g s(g^{-1} x H)$ *for* $g, x \in G, s \in \Gamma(G/H, V)$*.*

Proof of (i). It is clear from the def. of action. □

Proof of (ii). Let us fix any $\phi \in C(G, V)^{\iota H}$. And let us $\Phi(\phi)(\bar{g}) := [g, \phi(g)]$. We will show $\Phi(\phi)$ is well-defined. Let us fix any $g_1, g_2 \in G$ such that $g_1 \sim g_2$. Then there is $h \in H$ such that $g_1 = g_2 h$. So,

$$\Phi(\phi)(g_1 H) = [g_1, \phi(g_1)] = [g_2 h, \phi(g_2 h)] = [g_2 h, \pi(h)^{-1} \phi(g_2)] = [g_2, \phi(g_2)] = \Phi(\phi)(g_2 H)$$

We set $\mathfrak{h} := Lie(H)$. Let $\mathfrak{q}$ denote a complement of $\mathfrak{h}$. Because $\Phi(\phi)(gexp(X)H) = [gexp(X), \phi(gexp(X))]$ $g \in G, X \in \mathfrak{q}$ such that $||X|| \ll 1$, $\Phi(\phi) \in C(G/H, G \times_H V)$. Clearly $q \circ \Phi(\phi) = id_{G/H}$, therefore $\Phi(\phi) \in \Gamma(G/H, G \times_H V)$.

Let us fix any $s \in \Gamma(G/H, G \times_H V)$. Let us fix any $g \in G$. Then there $\exists! v \in V$ such that $s(gH) = [g, v]$. We set $\Psi(s)(g) := v$. $\Psi(s)(g) := v$. Let us fix any $\epsilon > 0$. By (iv) of Theorem6.6.3, there is $\delta > 0$ such that for any $X \in \mathfrak{q}_\delta := \mathfrak{q} \cap B(O, \delta)$, $s(gexp(\mathfrak{q}_\delta)) \subset p(gexp(\mathfrak{q}_\epsilon \times (v + B(O, \epsilon)))$. So, there is $Y \in \mathfrak{q}_\epsilon$ and $u \in v + B(O, \epsilon)$ such that

$$s(gexp(X)) = [gexp(Y), u]$$

Because $s(gexp(X)) = [gexp(X), \Psi(s)(gexp(X))]$, there is $h \in H$ such that $gexp(X)h = gexp(Y)$ and $\pi(h)^{-1} u = \Psi(s)(gexp(X))$. Because of the proof of Theorem6.1.1, if we take δ to be sufficient small, then $h = e$. So, $\Psi(s)(gexp(X)) \in (v + B(O, \epsilon))$. Therefore, $\Psi(s)$ is continuous. And clearly $\Psi(s) \in C(G, V)^{\iota(H)}$.

Clearly, $\Phi \circ \Psi = id_{\Gamma(G/H, V)}$ and $\Psi \circ \Phi = id_{C(G, V)^{\iota(H)}}$. And

$$\begin{aligned}\Phi(g \cdot \phi)(x) &= [x, g \cdot \phi(x)] = [x, \phi(g^{-1} x)] = [g g^{-1} x, \phi(g^{-1} x)] = g \cdot [g^{-1} x, \phi(g^{-1} x)] \\ &= g \Phi(\phi)(g^{-1} x) = (g \cdot \Phi(\phi))(x) \ (\forall g, x \in G, \forall \phi \in C(G, V)^{\iota(H)})\end{aligned}$$

□

6.7 Invariant metric

6.7.1 Existence of Invariant metric

Notation 6.7.1 ($\mathcal{P}_+(V)$)**.** *Let* V *be a* $\mathbb{R}$ *vector space. Let* $\mathcal{P}_+(V)$ *denote the set of all positive definite symmetric bilinear function.*

Clearly the following holds.

Proposition 6.7.2 (Isotropy Representation)**.** *The followings are settings and assumptions.*

(i) G *is a Lie group and* $\mathfrak{g} := Lie(G)$*.*

(ii) H *is a closed subgroup of* G *and* $\mathfrak{h} := Lie(H)$*.*

Then for any $h \in H$

$$\mathfrak{g}/\mathfrak{h} \ni [X] \mapsto [Ad(h)X] \in \mathfrak{g}/\mathfrak{h}$$

is well-defined. Let denote $Ad_{\mathfrak{g}/\mathfrak{h}}(h)$ *the map. We call* $(Ad_{\mathfrak{g}/\mathfrak{h}}, \mathfrak{g}/\mathfrak{h})$ *the isotropy representation.*

Definition 6.7.3. *The followings are settings and assumptions.*

(i) G *is a Lie group and* $\mathfrak{g} := Lie(G)$.

(ii) H *is a closed subgroup of* G *and* $\mathfrak{h} := Lie(H)$.

Then for any $h \in H$

$$(h \cdot B)(X, Y) := B(Ad_{\mathfrak{g}/\mathfrak{h}}(h^{-1})X, Ad_{\mathfrak{g}/\mathfrak{h}}(h^{-1})Y)$$

The action is a representation of H. *Let us denote the* H*-invariant element of* $\mathcal{P}_+(V)$ *by* $\mathcal{P}_+(V)^H$.

Theorem 6.7.4. *The followings are settings and assumptions.*

(i) G *is a Lie group and* $\mathfrak{g} := Lie(G)$.

(ii) H *is a closed subgroup of* G *and* $\mathfrak{h} := Lie(H)$.

Then

(i) G/H *has a* G *invariant riemannian metric* $\iff$ $\mathfrak{g}/\mathfrak{h}$ *has an inner product with which* $Ad_{\mathfrak{g}/\mathfrak{h}}$ *is a unitary representation of* H*(i.e.* $\mathcal{P}_+(\mathfrak{g}/\mathfrak{h})^H \neq \phi$*).*

(ii) For each $P \in \mathcal{P}_+(\mathfrak{g}/\mathfrak{h})^H$, *we set*

$$\Phi(P)_p(\eta(g, X + \mathfrak{h}), \eta(g, Y + \mathfrak{h}) := P(X + \mathfrak{h}, Y + \mathfrak{h}) \ (p \in G/H, g \in G \ s.t \ gH = p, X, Y \in \mathfrak{g})$$

and

$$\eta(g, X + \mathfrak{h})(f) := \frac{d}{dt}|_{t=0} f(g \exp(tX)) \ (g \in G, f \in C_p(G/H))$$

Then $\Phi(P)$ *is well-defined and* G*-invariant* C^∞*-class riemannian metric and*

$$\Phi : \mathcal{P}_+(\mathfrak{g}/\mathfrak{h})^H \to M(G/H)^G$$

is an isomorphism as linear space. Here, $M(G/H)^G$ *denote the set of all* G*-invariant* C^∞*-class riemannian metric on* G/H.

Proof for sufficient condition in (i). Let us fix $P \in \mathcal{P}_+(V)^H$. Let us fix any $p \in G/H$ and $u, v \in T_p(G/H)$. Then there are $g \in G$ and $X, Y \in \mathfrak{g}$ such that

$$p = gH, u(f) = \frac{d}{dt}_{|t=0} f(g \exp(tX)H), v(f) = \frac{d}{dt}_{|t=0} f(g \exp(tY)H) \ (\forall f \in C_p(G/H))$$

We set

$$B_p(u, v) := P(X + \mathfrak{h}, Y + \mathfrak{h})$$

We will show B_p is well-defined ragarding X, Y, g. Clearly B_p is well-defined regarding X, Y. And any $h \in H$,

$$gh \exp(tX) = g \exp(tAd_{\mathfrak{g}/\mathfrak{h}}(h)X)h$$

and

$$P(X + \mathfrak{h}, Y + \mathfrak{h}) = P(Ad_{\mathfrak{g}/\mathfrak{h}}(h)X + \mathfrak{h}, Ad_{\mathfrak{g}/\mathfrak{h}}(h)Y + \mathfrak{h})$$

Therefore B_p is well-defined regarding g. And clearly B is C^∞-class and G-invariant. So B is a G-invariant rieamannian metric. □

Proof for necessary condition in (i). Let us fix B which is a G-invariant riemannian metric. We set

$$P(X + \mathfrak{h}, Y + \mathfrak{h}) := B_H(\iota(X + \mathfrak{h}), \iota(Y + \mathfrak{h}))$$

Here, ι is defined by

$$\iota(Z + \mathfrak{h})(f) := \frac{d}{dt}_{|t=0} f(\exp(tZ)H) \ (f \in C_H^\infty(G/H)$$

Since B is G-invariant,

$$B_H((d\tau_h)^* u, (d\tau_h)^* v) = B_H(u, v) \ (\forall h \in H, \forall u, v \in T_H(G/H))$$

Here, τ_h is defined by

$$\tau_h(gH) = hgH \ (gH \in G/H)$$

Clearly,

$$(d\tau_h)^* \iota(X) = \iota(Ad_{\mathfrak{g}/\mathfrak{h}}(h)(X + \mathfrak{h}) \ (\forall h \in H, \forall X \in \mathfrak{g}))$$

Therefore, $P \in \mathcal{P}_+(\mathfrak{g}/\mathfrak{h})$. □

Proof of (ii). (ii) is from the proof of (i). □

So, clearly the following holds from Theorem5.7.1.

Proposition 6.7.5. *The followings are settings and assumptions.*

(i) G is a Lie group and $\mathfrak{g} := Lie(G)$.

(ii) H is a compact subgroup of G.

Then G/H has a G invariant riemannian metric.

6.8 Induced representation

Theorem 6.8.1 (Induced Representation)**.** *The followings are settings and assumptions.*

(i) G is a compact Lie group.

(ii) H is a closed subgroup of G.

(iii) (π, V) is a continuous unitary representation of H.

(iv) For each $f_1, f_2 \in C(G, W)$, $(f_1, f_2) := \int_G (f_1(g), f_2(g))_W d\mu(g)$. Here, μ is the normalized Haar measre on G.

Then

(i) $C(G/H, W)^{\iota(H)}$ is a pre-Hilbert space and is an unitary representation space of G with the inner product. We call the completion of it the induced representation from π and denote the completion by $L^2(G, \mathcal{W})$ and denote the representation by L^2-$Ind(H \uparrow G)(\pi)$ or L^2-Ind_H^G.

(ii) For any $f_1, f_2 \in C(G/H, W)^{\iota(H)}$,

$$(f_1, f_2) = \int_{G/H} (f_1(g), f_2(g)) d\mu(gH)$$

Proof of (i). It is clear. □

Proof of (ii). It is clear from Theorem6.4.13. □

Induced Representation can be defined with homogeneous bundle as below.

Theorem 6.8.2. *The followings are settings and assumptions.*

(i) G is a compact Lie group.

(ii) H is a closed subgroup of G.

(iii) (π, V) is a continuous unitary representation of H.

(iv) For each $g \in G$ and $[g', v_1], [g', v_2] \in q^{-1}(gH)$, we set $([g', v_1], [g', v_2]) := (v_1, v_2)$.

(v) For $s_1, s_2 \in \Gamma(G/H, G \times_H V)$, $(s_1, s_2) := \int_{G/H} (s_1(gH), s_2(gH)) d\mu(gH)$. Here μ is the normalized invariant measure on G/H.

Then

(i) The inner product defined in (iv) is well-defined.

(ii) $\Gamma(G/H, G \times_H V)$ is a pre-Hilbert space and is an unitary representation space of G with the inner product defined in (v).

(iii) The completion is isomorphic to $L^2(G, \mathcal{W})$ as continuous unitary representations.

Proof of (i). For each $[g', v_1] = [g'', v_3], [g', v_2] = [g'', v_4] \in q^{-1}(gH)$,

$$([g', v_1], [g', v_2]) = (v_1, v_2) = (\pi(g'^{-1}g'')^{-1} v_3, \pi(g'^{-1}g'')^{-1} v_4) = (v_3, v_4) = ([g'', v_3], [g'', v_4])$$

Therefore, the inner product is well-defined. □

Proof of (ii). Clearly $\Gamma(G/H, G\times_H V)$ is a $\mathbb{C}$-linear space and G acts on $\Gamma(G/H, G\times_H V)$. Since G is compact, the inner product converges in any case. Since μ is G-invariant, G acts $\Gamma(G/H, G\times_H V)$ as unitary operator. Let denote the isomophism from $C(G,V)^{\iota(H)}$ to $\Gamma(G/H, G\times_H V)$ by Φ. Clearly, for each $s\in\Gamma(G/H, G\times_H V)$,

$$||s|| = 0 \iff \Phi^{-1}(s) = 0$$

Consequently, (ii) holds. □

Proof of (iii). It is clear from (i). □

Clearly the following holds.

Example 6.8.3. *The followings are settings and assumptions.*

(i) G is a compact Lie group.

(ii) H is a closed subgroup of G.

Then L^2-Ind$(H\uparrow G)(1)\simeq L^2(G/H)$. Here, 1 is the trivial representation of H.

6.8.1 Frobenius Reciprocity

Proposition 6.8.4. *The followings are settings and assumptions.*

(i) G is a compact Lie group.

(ii) H is a closed subgroup of G.

(iii) (π, V) is a finite dimensional continuous representation of H.

Then

(i) $C(G,V)\simeq C(G)\otimes V$ as representation of $G\times G\times H$.

(ii) $C(G,V)^{\iota(H)}\simeq (C(G)\otimes V)^{\iota(H)}$ as representation of G.

(iii) If π is an unitary representation, $L^2(G/H,\mathcal{W})^{\iota(H)}\simeq (L^2(G)\otimes V)^{\iota(H)}$ as representation of G.

Proof of (i). Let $\{v_i\}_{i=1}^m$ denote a basis of V. Let us fix $f\in C(G,V)$. Then for each $g\in G$, there are $\exists! f_1(g),...,f_m(g)$ such that $f(g)=\sum_{i=1}^m f_i(g)v_i$. We set $\Phi(f):=\sum_{i=1}^m f_i\otimes v_i$.

Let us fix $\phi\in C(G)\otimes V$. By Proposition2.1.3, there exists $\{f_i\}_{i=1}^m\subset C(G)$ such that $\phi=\sum_{i=1}^m \phi_i\otimes v_i$. We set $\Psi(\phi):=(f_1,...,f_m)$.

Clearly Φ,Ψ are $\mathbb{C}$-linear and $\Phi\circ\Psi = id_{C(G)\otimes V}$ and $\Psi\circ\Phi = id_{C(G,V)}$.

$$\pi(h)L_{g_1}R_{g_2}f(g)=\sum_{i=1}^m L_{g_1}R_{g_2}f_i(g)\pi(h)v_i=\sum_{i=1}^m\sum_{j=1}^m L_{g_1}R_{g_2}f_i(g)(\pi(h)v_i,v_j)v_j=\sum_{j=1}^m\sum_{i=1}^m L_{g_1}R_{g_2}f_i(g)(\pi(h)v_i,v_j)v_j$$

So,

$$\begin{aligned}\Phi((g_1,g_2,h)\cdot f)=\Phi(\pi(h)L_{g_1}R_{g_2}f)&=\sum_{j=1}^m\sum_{i=1}^m L_{g_1}R_{g_2}f_i(\pi(h)v_i,v_j)\otimes v_j=\sum_{i=1}^m L_{g_1}R_{g_2}f_i\otimes\sum_{j=1}^m(\pi(h)v_i,v_j)v_j\\ &=\sum_{i=1}^m L_{g_1}R_{g_2}f_i\otimes\pi(h)v_i=(g_1,g_2,h)\cdot\sum_{i=1}^m f_i\otimes v_i=(g_1,g_2,h)\Phi(f)\end{aligned}$$

Consequently, Φ is G-invariant. □

Proof of (ii). (ii) is clearly from (i). □

Proof of (iii). (iii) is clearly from (i). □

Proposition 6.8.5. *The followings are settings and assumptions.*

(i) G is a compact Lie group.

(ii) H is a closed subgroup of G.

(iii) (π, W) is a finite dimensional continuous representation of H.

(iv) (τ, V_τ) is an irreducible continuous representation of G.

Then, for each $\pi \in \hat{G}$,

$$Hom_G(V_\pi, V_\tau \otimes Hom_H(V_\pi|H, W)) \simeq \begin{cases} 0 & \tau \not\cong \pi \\ Hom_H(V_\pi|H, W) & \tau = \pi \end{cases}$$

as vector spaces.

STEP1: When $\tau \not\cong \pi$*:* By Peter-Weyl theorem, $Hom_H(V_\pi|H, W)$ is finite dimensional. Let us fix a basis of $Hom_H(V_\pi|H, W)$ $\{\psi_i\}_{i=1}^m$. Let us fix any $\phi \in Hom_G(V_\pi, V_\tau \otimes Hom_H(V_\pi|H, W))$. We define $\phi_1, ..., \phi_m$ by

$$\phi(v) = \sum_{i=1}^{m} \phi_i(g) \otimes \psi_i \ (v \in V_\pi).$$

Clearly, $\phi_1, ..., \phi_m \in Hom_G(V_\pi, V_\tau)$. By Shur Lemma, $\phi_1 = ... = \phi_m = 0$. □

STEP2: When $\tau = \pi$*:* I continue to use the notations from STEP1. In the case, by Shur Lemma, there exist $c_1, ..., c_m \in \mathbb{C}$ such that $\phi_i = c_i id_{V_\tau}$ $(\forall i)$. Therefore,

$$\phi = id_{V_\tau} \otimes \sum_{i=1}^{m} c_i \psi_i.$$

This means

$$Hom_G(V_\pi, V_\tau \otimes Hom_H(V_\pi|H, W)) \simeq Hom_H(V_\pi|H, W)$$

□

Proposition 6.8.6. *The followings are settings and assumptions.*

- *(i)* H *is a topological group.*
- *(iii)* (π, W) *is a finite dimensional continuous representation of* H*.*
- *(iv)* (τ, V) *is a continuous representation of* H*.*
- *(v)* $\eta(H) := \{(h, h)|h \in H\}$.

Then,

$$(V^* \times W)^{\eta(H)} \simeq Hom_H(V, W)$$

Proof. That can be proved from the same thought as the proof of Proposition6.8.4. □

Theorem 6.8.7 (Frobenius Reciprocity Theorem)**.** *The followings are settings and assumptions.*

- *(i)* G *is a compact Lie group.*
- *(ii)* H *is a closed subgroup of* G*.*
- *(iii)* (π, W) *is an irreducible continuous representation of* H*.*
- *(iv)* (τ, V_τ) *is an irreducible continuous representation of* G*.*

Then,

(i)

$$Hom_H(\pi|H, \tau) \simeq Hom_G(\pi, Ind_H^G \tau)$$

(ii)

$$[\pi|H : \tau] = [Ind_H^G \tau : \pi]$$

(iii)

$$Ind_H^G \tau = \oplus_{\pi \in \hat{G}} [\pi|H : \tau] \pi$$

Proof of (i). By Peter-Weyl Theorem,

$$L^2(G) \simeq \oplus_{\sigma \in \hat{G}} V_\sigma \otimes V_\sigma^*$$

Then

$$L^2(G/H, \mathcal{W})$$

by Proposition6.8.4

$$= L^2(G) \otimes W \simeq (\oplus_{\sigma \in \hat{G}} V_\sigma \otimes V_\sigma^* \otimes W)^{\iota(H)} \simeq \oplus_{\sigma \in \hat{G}} V_\sigma \otimes (V_\sigma^* \otimes W)^{\eta(H)} \simeq \oplus_{\sigma \in \hat{G}} V_\sigma \otimes Hom_H(V_\sigma, W)$$

So, by Proposition6.8.5,

$$Hom_G(\pi, L^2(G/H, \mathcal{W})) \simeq Hom_H(\tau|H, \pi)$$

□

Memo 6.8.8. *Frobenius Reciprocity Theorem can be purely algebraicly proved. The proof needs only Peter-Weyl Theorem and Shur Lemma and Expressing induced representation as tensor space.*

Chapter 7

Classification of irreducible representations of compact classical groups

7.1 Facts without proof

Proposition 7.1.1. *Here are settings and assumptions.*

(S1) $A' = \{a_{i,j}\}_{i,j} \in M(n, \mathbb{C})$.

(S2) $A = \begin{pmatrix} Re(a_{1,1}) & -Im(a_{1,1}) & ... & Re(a_{1,n}) & -Im(a_{1,n}) \\ Im(a_{1,1}) & Re(a_{1,1}) & ... & Im(a_{1,n}) & Re(a_{1,n}) \\ ... & ... & ... & ... & ... \\ Re(a_{n,1}) & -Im(a_{n,1}) & ... & Re(a_{n,n}) & -Im(a_{n,n}) \\ Im(a_{n,1}) & Re(a_{n,1}) & ... & Im(a_{n,n}) & Re(a_{n,n}) \end{pmatrix}$.

Then

$$detA = |detA'|^2$$

7.2 Complex Analysis

Proposition 7.2.1. *Here are settings and assumptions.*

(S1) $\{a_\alpha\}_{\alpha \in \mathbb{Z}^n} \subset \mathbb{C}$ *such that* $\#\{\alpha | a_\alpha \neq 0\} < \infty$.

(S2) $P(t) := \sum_\alpha a_\alpha t^\alpha$ $(t \in \mathbb{C}^n)$.

(A1) $P = 0$ *in* T^n.

Then $P = 0$ *in* $\mathbb{C}^n$.

Proof. For aiming contradiction, le us assume $a_\alpha \neq 0$ for some α. Let β the biggest index of $\{\alpha | a_\alpha \neq 0\}$. with respect to lexicographic order. We can assume $\beta_1 \neq 0$. For any $r > 0$,

$$|P(r, 1, ..., 1)| = |r^{\beta_1} P(1, ..., 1)| = 0$$

By increasing $r \to \infty$, we get $\infty = 0$. This is contradiction. □

7.3 A_{n-1} type case

7.3.1 Main theorem

The propositions shown in this section will not be presented with proofs in this subsection, but will be presented with proofs in the subsections that follow.

Definition 7.3.1 (Torus, Maximal Torus)**.** *Here are settings and assumptions.*

(S1) G *is a compact Lie group.*

Then

(i) *We say $T \subset G$ is a torus of G if T is a connected commutative closed subgroup of G.*

(i) *We say $T \subset G$ is a maximal torus of G if T is a torus and there is no torus which contains T as a proper subset.*

Notation 7.3.2 (Diagonal Matrix)**.** *We set*

$$diag(t_1, t_2, ..., t_n) := \begin{pmatrix} t_1 & 0 & ... & 0 \\ 0 & t_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & ... & 0 & t_n \end{pmatrix}$$

Notation 7.3.3 (Lexicographical order on $\mathbb{Z}^n$)**.** *We denote the lexicographical order on $\mathbb{Z}^n$ by $\prec$.*

Proposition 7.3.4 (Maximal torus of $U(n)$)**.**

$$T := \{diag(t_1, t_2, ..., t_n)| \ |t_1| = ... = |t_n| = 1\}$$

is a maximal trus of $U(n)$.

The following is clear.

Proposition 7.3.5 (Irreducible representation of maximal torus of $U(n)$)**.** *Let us $\alpha \in \mathbb{Z}n$.*

$$\chi_\alpha : T \ni diag(t_1, t_2, ..., t_n) \mapsto t_1^{\alpha_1} ... t_n^{\alpha_n} \in S^1$$

is a continuous homomoriphism.

Proposition 7.3.6 (Weight, Weight vector)**.** *We will succeed notations in Proposition7.3.5. Let*

(S1) *G is a compact Lie group.*

(S2) *(π, V) is a finite dimensional continuous representation of G.*

(S3) *For each $\lambda \in \mathbb{Z}$, we denote χ_λ component of $\pi|T$ by V_λ.*

Then

(i) *We say $\lambda \in \mathbb{Z}$ is a weight of V with respect to T if $V_\lambda \neq \{0\}$. We call an element of V_λ a weight vector for each weight λ.*

(ii) *We say $\lambda \in \mathbb{Z}$ is the highest weight of V with respect to T if λ is the maximum weight with $\prec$.We define the highest weight vector in the same way.*

(iii) *We call the multiplicity of χ_λ in V_λ the multiplicity of the weight λ.*

Notation 7.3.7 ($(\mathbb{Z}^n)_+$)**.** *We set*

$$(\mathbb{Z}^n)_+ := \{\lambda \in \mathbb{Z}^n | \lambda \text{ is monotone decreasing.}\}$$

The following is the main theorem in this section. In the last part of this section, we give a proof of this theorem.

Theorem 7.3.8 (Cartan-Weyl theorem of the highest weight)**.** *The followings hold.*

(i) *Let us assume (π, V) be a continuous irreducible unitary representation of $U(n)$ and λ be the highest weight of π. Then $\lambda \in (\mathbb{Z}^n)_+$ and the multiplicity of λ is 1.*

(ii) *Let us fix any $\lambda \in (\mathbb{Z}^n)_+$. Then there is the unique continuous irreducible unitary representation (π, V) whose highest weight is λ, ignoring isomorphism as continous unitary representation.*

7.3.2 General topics on compact Lie group

By Zorn's Lemma, the following holds.

Proposition 7.3.9 (Maximal torus of a compact Lie group)**.** *For any compact Lie group G, there is a maximal torus of G.*

Proof. We set

$$\mathfrak{T} := \{T \subset G | T \text{ is an abelian subgroup of } G\}$$

For any $\mathfrak{A}$ is any totally ordered subset of $\mathfrak{T}$, $\cup\mathfrak{A} \in \mathfrak{T}$. So, $\mathfrak{T}$ has a maximal element T. Because $\bar{T}$ is an abelian subgroup of G, $\bar{T} = T$. So T is a maximal torus of G. □

Proposition 7.3.10 (Weyl group)**.** *Let*

(S1) G is a compact Lie group.

(S2) T is a maximal torus of G.

(S3) We set

$$N_G(T) := \{g \in G | gtg^{-1} \in T \ (\forall t \in T)\}$$

(S4) We set

$$Z_G(T) := \{g \in G | gt = tg \ (\forall t \in T)\}$$

Then

(i) $N_G(T)$ is a compact subgroup of G.

(ii) $Z_G(T) = T$.

(iii) $Z_G(T)$ is a compact normal subgroup of $N_G(T)$.

We call the quotient compact group $N_G(T)/Z_G(T)$ the weyl group of G. We define the action of the weyl group on T by

$$w \cdot t := wtw^{-1} \ (w \in N_G(T)/Z_G(T), t \in T)$$

Proof of (i). Let us fix any $g_1, g_2 \in N_G(T)$ and $t \in T$. Because $g_1^{-1}tg_1 = (g_1t^{-1}g_1^{-1})^{-1}$ and $t, g_1t^{-1}g_1^{-1} \in T$, $g_1^{-1}tg_1 \in T$. So, $g_1^{-1} \in N_G(T)$. Because $(g_1g_2)^{-1}t(g_1g_2) = g_1^{-1}(g_2^{-1}tg_2)g_1^{-1}$ and $g_2^{-1}tg_2 \in T$, $(g_1g_2)^{-1}t(g_1g_2) \in T$. So, $g_1g_2 \in N_G(T)$. Consequently, $N_G(T)$ is a subgroup of G.

For each $t \in T$, we set $\sigma_t(g) = gtg^{-1}$ $(g \in G)$. σ_t is continuous for any $t \in T$. Because $N_G(T) = \cap_{t \in T}\sigma_t^{-1}(T)$, $N_G(T)$ is closed subset of G. □

Proof of (ii). Clearly $Z_G(T)$ is abelian compact subgroup of T and $T \subset Z_G(T)$. So, $T = Z_G(T)$. □

Proof of (iii). For any $g \in N_G(T)$, $gZ_G(T)g^{-1} = Z_G(T)$. So, $Z_G(T)$ is a normal subgroup of $N_G(T)$. □

Definition 7.3.11 (Flag variety)**.** *Let G be a compact Lie group and T be a maximal torus of G. We call G/T the flag variety.*

7.3.3 The maximal torus and Weyl group of $U(n)$

Proposition 7.3.12 (Maximal torus of $U(n)$)**.**

$$Z_{U(n)}(T) := \{g \in U(n) | gt = tg \ (\forall t \in T)\}$$

is equal to T. In special, T is the maximal torus of $U(n)$.

Proof. Let us fix any $g \in U(n)$. We take $t \in T$ such that $t_i \neq t_j$ $(\forall i \neq \forall j)$.Then

$$g_{i,j}t_j = g_{i,j}t_i \ (\forall i, j)$$

So, $g_{i,j} = \delta_{i,j}g_{i,i}$ $(\forall i, j)$. Then $g = diag(g_{1,1}, ..., g_{n,n})$. Because $g \in U(n)$, $g \in T$. So, $Z_{U(n)}(T) = T$. □

By the proof of Proposition7.3.12, the following holds.

Proposition 7.3.13. *We set*

$$T_{reg} := \{t \in T | t_i \neq t_j \ (\forall i \neq \forall j)\}$$

Then for every $t \in T_{reg}$, $Z_G(t) = T$.

Proposition 7.3.14 (Weyl group of $U(n)$)**.** *Let*

(S1) For compact group G and the maximal torus T, we set

$$N_G(T) := \{g \in G | gtg^{-1} \in T \ (\forall t \in T)\}$$

(S2) We set

$$\pi_0(w)(t) := (t_{w^{-1}(1)}, ..., t_{w^{-1}(n)}) \ (w \in \mathfrak{S}_n, t \in \mathbb{C}^n)$$

Here, $\mathfrak{S}_n$ is the symmetric group of degree n. We set $W := \pi_0(\mathfrak{S}_n)$.

(S3)

$$\Phi : W \ltimes T \ni (w,t) \mapsto wt \in GL(n\mathbb{C})$$

Then the followings hold.

(i) For any $\omega \in \mathfrak{S}_n$ *and* $t \in T$,

$$\pi_0(\omega)t\pi_0(\omega)^{-1} = diag(t_{\omega^{-1}(1)}, ..., t_{\omega^{-1}(n)})$$

So, $\pi_0(\omega) \in N_G(T)$.

(ii) $\Phi : W \times T \ni (\sigma,t) \mapsto \sigma t \in N_G(T)$ *is a bijection.*

(iii) W *and* $N_G(T)/T$ *are isomorphic as groups.*

Proof of (i). It is clear. □

Proof of (ii). Let us fix any $\sigma \in W$ and $t \in T$. For any $s \in T$, $\sigma ts(\sigma t)^{-1} = \sigma s\sigma^{-1} \in T$ by (i). So, $\Phi(W \times T) \subset N_G(T)$.

Let us fix any $g \in N_G(T)$. Let us fix $t \in T_{reg}$. We set $s := gtg^{-1}$.

Because s and t have the same set of eigenvalues. So, there is $\omega \in \mathfrak{S}_n$ such that

$$s = (t_{\omega^{-1}(1)}, ..., t_{\omega^{-1}(n)})$$

By (i), this means that $s = \pi_0(\omega)t\pi_0(\omega^{-1})$. So, $t = \pi_0(\omega^{-1})gtg^{-1}\pi_0(\omega)$. We set $t_1 := \pi_0(\omega^{-1})g$. By Proposition7.3.13, $t_1 \in Z_G(T)$. $t = \Phi(\pi_0(\omega), t_1)$. So, Φ is surjective.

Let us fix any $\sigma_1, \sigma_2 \in W$ and any $t_1, t_2 \in T$ such that $\sigma_1 t_1 = \sigma_2 t_2$. Then $\sigma_2^{-1}\sigma_1 = t_2 t_1^{-1} \in W \cap T = \{e\}$. This implies $\sigma_1 = \sigma_2$ and $t_1 = t_2$. □

Proof of (iii). We set $\Psi := \Phi^{-1}$ and $P : W \times T \ni (w,t) \mapsto w \in W$ and $\varphi := P \circ \Psi$. Clearly φ is surjective and $\varphi^{-1}(e) = T$. So it is enough to show φ is homomorphism. For any $\sigma_1, \sigma_2 \in W$ and any $t_1, t_2 \in T$,

$$\sigma_1 t_1 \sigma_2 t_2 = \sigma_1\sigma_2\sigma_2^{-1}t_1\sigma_2 t_2 = \Phi(\sigma_1\sigma_2, \sigma_2^{-1}t_1\sigma_2 t_2)$$

So, φ is homomorphism. □

By Shur Lemma, the following clearly holds.

Proposition 7.3.15. *Let*

(S1) G *is an abelian Lie group.*

(S2) $C := \{\varphi \in C(G, S^1) | \varphi$ *is a continuous homomorphism between groups.*$\}$

(S3) $\pi_\varphi(g)v := \varphi(g)z$ $(g \in G, z \in \mathbb{C}, \varphi \in C)$.

Then

(i) For any $\tau \in \hat{G}$, $\chi_\tau \in C$.

(ii) $\Phi : C \ni \varphi \mapsto \pi_\varphi \in \hat{G}$ *is bijective whose inverse is* $\Psi : \hat{G} \ni \pi \mapsto \chi_\pi \in C$.

Hereafter, we equate $\varphi \in \hat{G}$ *and* $\Phi(\varphi)$.

Proposition 7.3.16. *Let* T *be the maximal torus of* $U(n)$. *Then*

$$\hat{T} = \{\chi_\lambda | \lambda \in \mathbb{Z}^n\}$$

Hereafter, we equate $\lambda \in \mathbb{Z}^n$ *and* $\chi_\lambda \in \hat{G}$.

Proof. This proof is similar to the proof of Proposition5.7.35. We set

$$f(\theta_1, ..., \theta_n) := \tau(exp(i\theta_1 2\pi), ..., exp(i\theta_n 2\pi)) \ (\theta_1, ..., \theta_n \in \mathbb{R})$$

Then

$$f(\theta + he_i) = f(\theta)f(he_i) \ (\forall\theta \in \mathbb{R}^n, \forall h \in \mathbb{R}, \forall i)$$

So,

$$\frac{\partial f}{\partial \theta_i}(\theta) = \frac{\partial f}{\partial \theta_i}(\mathbf{0})f(\theta) \ (\forall\theta \in \mathbb{R}^n, \forall h \in \mathbb{R}, \forall i)$$

Because $f(\mathbf{0}) = 1$ and $Im(f) \subset S^1$, there are $\alpha_1, ..., \alpha_n \in \mathbb{R}$ such that

$$f(\theta) = exp(i\theta_1\alpha_1 2\pi)...exp(i\theta_n\alpha_n 2\pi)\ (\forall\theta \in \mathbb{R}^n)$$

Because $f(e_i) = 1$ $(\forall i)$, $\alpha_1, ..., \alpha_n \in \mathbb{Z}$. Consequently,

$$\hat{T} = \{\chi_\lambda | \lambda \in \mathbb{Z}^n\}$$

We denote the inverse of

$$\mathbb{Z}^n \ni \lambda \mapsto \chi_\lambda \in C$$

by Ψ. □

The following clearly holds.

Proposition 7.3.17. *We succeed in notations of Proposition7.3.15 and Proposition7.3.16.*

(S1) $W \subset U(n)$ is the weyl group of $U(n)$.

(S2) $(w \cdot \varphi)(t) := \varphi(w^{-1} \cdot t)$ $(w \in W, \varphi \in C, t \in T)$.

Then W continuously acts on C and

$$w \cdot \varphi = w^{-1}\Psi(\varphi)\ (\forall w \in W, \forall\varphi \in C)$$

Proposition 7.3.18. *Here are the settings and assumptions.*

(S1) T is the maximal torus of $U(n)$.

(S2) (π, V) is a continuous unitary representation of $U(n)$.

(S3) $\lambda \in \hat{U(n)}$.

Then

$$V_\lambda = \{w \in V | \pi(g)w = \chi_\lambda(g)w\ (\forall g \in T)\}$$

Proof. We denote the right side of the above equation by W. Let us fix any $w \in \sum_{A \in Hom_G(\chi_\lambda, \pi)} ImA$. Then there are $A_1, ..., A_m \in Hom_G(\chi_\lambda, \pi)$ and $v_1, ..., v_m \in V$ such that $w = \sum_{i=1}^m A_i v_i$. So, for any $g \in G$,

$$\pi(g)w = \sum_{i=1}^m \pi(g)A_i v_i = \sum_{i=1}^m A_i\chi_\lambda(g)v_i = \chi_\lambda(g)\sum_{i=1}^m A_i v_i = \chi_\lambda(g)w$$

So, $\sum_{A \in Hom_G(\chi_\lambda, \pi)} ImA \subset W$. Because W is closed, $V_\lambda \subset W$.

Let us fix any $w \in W$. We set $P_\lambda := P_{\chi_\lambda}$. By Proposition7.3.15 ,

$$P_\lambda w = \int_G \overline{\chi_\lambda(g)}\pi(g)wdg = \int_G \overline{\chi_\lambda(g)}\chi_\lambda(g)wdg = \int_G wdg = w$$

By Theorem5.7.24, $w \in V_\lambda$. □

7.3.4 Weyl Integral Formula

Notation 7.3.19 (G_{reg}, T_{reg})**.** *Here are the settings and assumptions.*

(S1) T is the maximal torus of $G := U(n)$.

Then $G_{reg} := \{g \in G | g$ has no duplicate eigenvalues.$\}$ and $T_{reg} := T \cap G_{reg}$.

Proposition 7.3.20. *Here are the settings and assumptions.*

(S1) $G := U(n)$.

(S2) T be the maximal torus of G.

(S3) $\epsilon > 0$.

(S4) $\mathfrak{g} := Lie(G)$, $\mathfrak{h} := Lie(T)$.

(S5) $\mathfrak{q}$ is a complement subspace of $\mathfrak{h}$ in $\mathfrak{g}$.

Then there are $\{g_i\}_{i=1}^{\infty} \subset G$ *and* $\{U_i\}_{i=1}^{\infty}$ *such that* U_i *is a open neighborhood of* 0_k $(\forall i)$ *and* $U_i \subset B_k(O,\epsilon) \cap \mathfrak{q}$ $(\forall i)$ *and* $\{\pi(g_i Exp(U_i)w)\}_{i\in\mathbb{N},w\in W}$ *is an open covering of* G/H *and for any* $i \in \mathbb{N}$, $w_0 \in W$ $\#\{(j,w) \in \mathbb{N}\times W | \pi(g_i Exp(U_i)w_0) \cap \pi(g_j Exp(U_j)w) \neq \phi\} < \infty$.

Proof. There is V which an open neighborhood of e in G such that $V^4 \subset Exp(B(O,\epsilon))$ and $\bar{V}$ is compact. There are $\{g_{0,i}\}_{i=1}^{N_0}$ and $\{\epsilon_{0,i}\}_{i=1}^{N_0} \subset (0,\infty)$ such that $\pi(\bar{V}^4 \cdot W) \subset \cup_{i=1}^{N_0}\pi(g_{0,i}Exp(B_k(O,\epsilon_{0,i})))$ and $g_{0,i}Exp(B_k(O,\epsilon_{0,i}) \subset Exp(B_k(O,\epsilon)g_{0,i}$ $(\forall i)$.

And for each $s \in \mathbb{N}$ there are $\{g_{s,i}\}_{i=1}^{N_s}$ and $\{\epsilon_{s,i}\}_{i=1}^{N_s} \subset (0,\infty)$ such that $\pi(\bar{V}^{4+s}W)\backslash\pi(V^{3+s}W) \subset \cup_{i=1}^{N_s}\pi(g_{s,i}Exp(B_k(O,\epsilon_{s,i})))$ and $g_{s,i}Exp(B_k(O,\epsilon_{s,i}) \subset Exp(B_k(O,\epsilon)g_{s,i}$ $(\forall i)$.

We set $\{g_i\}_{i=1}^{\infty} := \{g_{s,i} | s,i \in \mathbb{N}, 1 \le i \le N_s\}$ and $\{U_i\}_{i=1}^{\infty} := \{U_{s,i} | s,i \in \mathbb{N}, 1 \le i \le N_s\}$. We will show for any $i \in \mathbb{N}$ and $s \in \mathbb{N}$,

$$\pi(g_{s,i}) \notin \pi(V^{s+2}W)$$

For aiming contradiction, let us assume $s \in \mathbb{N}$ and $i \in \mathbb{N}$ such that $\pi(g_{s,i}) \in \pi(V^{s+2}W)$. So,

$$\pi(g_{s,i}Exp(B_k(O,\epsilon_{s,i}))) \subset \pi(Exp(B_k(O,\epsilon))g_{s,i}) \subset \pi(V^{s+3}W)$$

This contradicts with

$$\pi(g_{s,i}Exp(B_k(O,\epsilon_{s,i}))) \cap \pi(V^{s+3}W)^c \neq \phi$$

Nextly, we will show for any $i \in \mathbb{N}$ and $s \in \mathbb{N}$,

$$\pi(g_{s,i}Exp(B_k(O,\epsilon_{0,i}))W) \cap \pi(V^{s+1}W) = \phi$$

For aiming contradiction, let us assume $s \in \mathbb{N}$ and $i \in \mathbb{N}$ such that $\pi(g_{s,i}Exp(B_k(O,\epsilon_{0,i}))W) \cap \pi(V^{s+1}W) \neq \phi$. Then there is $X \in B_k(O,\epsilon)$ and $u \in V^{s+2}$ and $w_1, w_2 \in W$ such that $\pi(Exp(X)g_{s,i}w_1) = \pi(uw_2)$. So,

$$\pi(g_{s,i}) = g_{s,i}T = g_{s,i}w_1Tw_1^{-1} = Exp(-X)uw_2w_1^{-1}w_1Tw_1^{-1} = Exp(-X)uw_2w_1^{-1}T \in \pi(V^{s+2}W)$$

This is a contradiction. □

Notation 7.3.21 ($\Delta(t)$ $(t \in T_{reg})$, T_σ $(\sigma \in \mathfrak{G}_n)$)**.** *Here are the settings and assumptions.*

(S1) $G := U(n)$.

(S2) T *is the maximal torus of* G.

Then

(i) $\Delta(t) := \min\{|arg(t_i) - arg(t_j)|\ |i \neq j\}$. *Here, let us assume* $arg(z) \in [0,2\pi)$.

(ii) For $\sigma \in \mathfrak{G}_n$, *we set*

$$T_\sigma := \{t \in T_{reg} | arg(t_{\sigma(i)}) < arg(t_{\sigma(i+1)})\ (\forall i)\}$$

Theorem 7.3.22. *Here are the settings and assumptions.*

(S1) T *is the maximal torus of* $G := U(n)$.

(S2) $A : G/T \times T \ni (gT,t) \mapsto gtg^{-1} \in G$.

(S3) W *is the weyl group of* G.

Then

(i) A *is well-defined and surjective* C^ω*-class map.*

(ii) $A|G/T \times T_{reg}$ *is a surjective map onto* T_{reg}.

(iii) For each $g, g' \in G$ *and* $t, t' \in T$,

$$A(gT,t) = A(g'T,t') \iff \exists w \in W\ s.t\ g'T = gw^{-1}T \text{ and } t' = w \cdot t$$

Here, $w \cdot t_2 := wt_2w^{-1}$.

Proof of (i). Because T is commutative, if $g_1, g_2 \in G$ and $t_1, t_2 \in T$ and $(g_1T, t_1) = (g_2T, t_2)$ then

$$g_1t_1g_1^{-1} = g_2g_2^{-1}t_1(g_2g_2^{-1}g_1)^{-1} = g_2g_2^{-1}g_1t_1g_1^{-1}g_2g_2^{-1} = g_2t_2g_2^{-1}g_1g_1^{-1}g_2g_2^{-1} = g_2t_2g_2^{-1}$$

So A is well-defined. And clearly A is surjective.

We take $\{\pi(g_iExp(U_i))\}_i$ and $\{h_jExp(V_j)\}_j$ as the coverings in Proposition6.4.12. For each i, j and $X \in U_i$ and $Y \in V_j$,

$$A(g_iExp(X), h_jExp(Y)) := g_iExp(X)h_jExp(Y)Exp(-X)g_i^{-1}$$

So, A is C^ω-class. □

Proof of (ii). Because for any $g \in G$ and $t \in T$ gtg^{-1} has no duplicate eigenvalues $\iff$ t has no duplicate eigenvalues, (ii) holds. □

Proof of (iii). The $\Longleftarrow$ part is clear. We will show the $\Longrightarrow$ part. Let us fix any $g_1, g_2 \in G$ and $t_1, t_2 \in T_{reg}$ such that $g_1 t_1 g_1^{-1} = g_2 t_2 g_2^{-1}$. We set $g_3 := g_2^{-1} g_1$. Then

$$t_1 = g_3 t_2 g_3^{-1}$$

Because $t_1, t_2 \in T_{reg}$, there is $w \in W$ such that

$$t_2 = w^{-1} g_3 t_2 (w^{-1} g_3)^{-1}$$

So, $w^{-1} g_3 \in Z_G(t_2)$. By Proposition7.3.10, $t_3 := w^{-1} g_3 \in T$. So, $g_3 = w t_3$. Then

$$g_2 T = g_1 g_3^{-1} T = g_1 w^{-1} w t_3^{-1} w^{-1} T = g_1 w^{-1} T, t_1 = w t_3 t_2 t_3^{-1} w^{-1} = w t_2 w^{-1} =: w \cdot t_2$$

□

By Theorem7.3.22 and Proposition5.5.5 and Proposition5.6.1, the following holds.

Proposition 7.3.23. *Here are the settings and assumptions.*

(S1) $G := U(n)$.

(S2) T *is the maximal torus of* G.

(S3) (π_i, V_i)*(i=1,2) are two continuous finite dimensional representation of* G.

(A1) $\chi_{\pi_1}|T = \chi_{\pi_2}|T$.

Then $\pi_1 \simeq \pi_2$.

Proposition 7.3.24. *Here are the settings and assumptions.*

(S1) $G := U(n)$.

(S2) T *is the maximal torus of* G.

(S3) $\mathfrak{t} := Lie(T)$, $\mathfrak{g} := Lie(G)$.

(S4) $\mathfrak{g}_1 := \{X \in \mathfrak{g} | X_{i,i} = 0 \ (\forall i)\}$

Then

$$\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{t}$$

Proof. Clearly, $\mathfrak{g}_1 \cap \mathfrak{t} = \phi$ and $\mathfrak{g} \supset \mathfrak{g}_1 + \mathfrak{t}$ Let us fix any $X \in \mathfrak{g}$. Then

$$X = Y + diag(X_{1,1}, ..., X_{n,n})$$

Here,

$$Y_{i,j} = (1 - \delta_{i,j}) X_{i,j} \ (i, j = 1, 2, ..., n)$$

Then $Y \in \mathfrak{g}_1$. Because X is skew-Hermitian, $X_{j,j} \in i\mathbb{R}$ $(\forall j)$. So, $diag(X_{1,1}, ..., X_{n,n}) \in \mathfrak{t}$. So,

$$\mathfrak{g} \subset \mathfrak{g}_1 + \mathfrak{t}$$

□

Proposition 7.3.25. *Here are the settings and assumptions.*

(S1) $G := U(n)$.

(S2) T *is the maximal torus of* G.

Then there is $\{V_j\}_{i=1}^{\infty}$ *such that* $\{w \cdot V_j\}_{j \in \mathbb{N}, w \in W}$ *is a local finite open covering of* T_{reg} *and for any* i, j $\sup\{|arg(t_i) - arg(t_i)||t \in V_j\} \leq \frac{1}{2} \inf\{\Delta(t) | t \in V_j\}$ *and for any* $s \in \mathbb{N}$ $\#\{j | \Delta(t) \geq \frac{1}{2^s} \ (\exists t \in U_j)\} < \infty$ *and* $V_i \subset T_e$ $(\forall i)$.

Proof. We set

$$T_s := \{t \in T_{reg} | \Delta(t) \leq \frac{1}{2^s}\}, T_{s,\sigma} := T_s \cap T_\sigma \ (s \in \mathbb{N}, \sigma \in \mathfrak{S}_n)$$

Because $T_{1,e}$ is compact, there are $\{U_{1,i}\}_{i=1}^{N_1}$ which is a open covering of $T_{1,e}$ and N_1 is the minimum numver of open covering of $T_{1,e}$. Let us fix $s \in \mathbb{N} \cap [2, \infty)$. Because $T_{s,e} \setminus T_{s-1,e}^{\circ}$ is compact, there are $\{U_{s,i}\}_{i=1}^{N_s}$ which is a open covering of $T_{s,e} \setminus T_{s-1,e}^{\circ}$ and N_s is the minimum numver of cardinalities of all open coverings of $T_{s,e} \setminus T_{s-1,e}^{\circ}$. Clearly, $\cup_{w \in W} \cup_{s=1}^{\infty} \{w \cdot U_{s,i}\}_{i=1}^{N_s}$ is a local finite open covering of T_{reg} and satisfies the condition in the claim of this Proposition. □

Proposition 7.3.26. *Here are the settings and assumptions.*

(S1) $G := U(n)$.

(S2) T *is the maximal torus of* G.

(S3) $\mathfrak{g} := Lie(G)$, $\mathfrak{h} := Lie(T)$.

(S4) $\mathfrak{q}$ *is a complement subspace of* $\mathfrak{h}$ *in* $\mathfrak{g}$.

(S5) We set

$$A : G/T \times T \ni (gT, t) \mapsto gtg^{-1} \in T$$

Then there are $\{g_i\}_{i=1}^{\infty} \subset G$ *and* $\{U_i\}_{i=1}^{\infty}$ *such that* U_i *is a open neighborhood of* 0_k $(\forall i)$ *and* $U_i \subset B_k(O, \epsilon) \cap \mathfrak{q}$ $(\forall i)$ *and* $\{\pi(g_i Exp(U_i)w^{-1}) \times w \cdot V_j\}_{i\in\mathbb{N}, w\in W, j\in\mathbb{N}}$ *is a local finite open covering of* $G/H \times T_{reg}$ *and* $\{A\pi(g_i Exp(U_i)w^{-1}) \times w \cdot V_j\}_{i\in\mathbb{N}, w\in W, j\in\mathbb{N}}$ *is a local finite open covering of* G_{reg}.

Proof of the first part. We will succeed in notations of Propositions7.3.26 and Proposition7.3.20. Let us fix any $(gT, t) \in G/H \times T_{reg}$. There is $w \in W$ such that $(gwT, w^{-1} \cdot t) \in G/H \times T_e$. Then there are i, j such that $(gwT, w^{-1} \cdot t) \in \pi(g_i Exp(U_i)) \times V_j$. Then $t \in w \cdot V_j$. And there is $u \in Exp(U_i)$ such that $gwT = g_i uT$. Because $gwT = gTw$ and $g_i uw^{-1}Tw$,

$$gT = g_i uw^{-1}T$$

So, $(gT, t) \in \pi(g_i Exp(U_i)w^{-1}) \times w \cdot V_j$. Consequently, $\{\pi(g_i Exp(U_i)w^{-1}) \times w \cdot V_j\}_{i\in\mathbb{N}, w\in W, j\in\mathbb{N}}$ is an open covering of $G/H \times T_{reg}$.

Let us fix any $i_0, j_0 \in \mathbb{N}$ and $w_0 = \pi_0(\sigma_0) \in W$. Let us fix any $i, j \in \mathbb{N}$ and $w = \pi_0(\sigma) \in W$ such that

$$\pi(g_{i_0} Exp(U_{i_0})w_0^{-1}) \times w_0 \cdot V_{j_0} \cap \pi(g_i Exp(U_i)w^{-1}) \times w \cdot V_j \neq \phi$$

Because $V_{j_0}, V_j \subset T_e$, $w_0 = w$. So, $V_{j_0} \cap V_j \neq \phi$. Because $g_{i_0} uw_0^{-1}T = g_{i_0} uTw_0^{-1}$ and $g_i vw^{-1}T = g_i vTw^{-1}$ for any $u \in Exp(U_{i_0})$ and $v \in Exp(U_i)$, $\pi(g_{i_0} Exp(U_{i_0})) \cap \pi(g_i Exp(U_i)) \neq \phi$. So,

$$(i, j, w) \in B := \{(i, j, w) | \pi(g_i U_i) \cap \pi(g_{i_0} U_{i_0}) \neq \phi, w = w_0, V_j \cap V_{j_0} \neq \phi\}$$

Because B is finite, $\{\pi(g_i Exp(U_i)w^{-1}) \times w \cdot V_j\}_{i\in\mathbb{N}, w\in W, j\in\mathbb{N}}$ is a local finite open covering of T_{reg}. □

Proof of the last part. By the first part, clearly $\{A\pi(g_i Exp(U_i)w^{-1}) \times w \cdot V_j\}_{i\in\mathbb{N}, w\in W, j\in\mathbb{N}}$ is an open covering of T_{reg}. We set $X_i := g_i Exp(U_i)$ $(i \in \mathbb{N})$.

Let us fix any $i_0, j_0 \in \mathbb{N}$ and $w_0 \in W$. We set

$$W_0 := \{w \in W | \exists i, \exists j \text{ s.t } A\pi(X_{i_0} w_0^{-1}) \times w_0 \cdot V_{j_0} \cap A\pi(X_i w^{-1}) \times w \cdot V_j \neq \phi\}$$

Clearly, W_0 is a finite set.

We set

$$J_0 := \{j \in \mathbb{N} | \exists i, \exists w \text{ s.t } A\pi(X_{i_0} w_0^{-1}) \times w_0 \cdot V_{j_0} \cap A\pi(X_i w^{-1}) \times w \cdot V_j \neq \phi\}$$

and

$$\epsilon := \inf\{\Delta(t) | t \in V_{j_0}\}$$

Then

$$\Delta(t) \geq \frac{\epsilon}{2^2} \ (\forall t \in V_j, \forall j \in J_0)$$

So, from the definition of $\{V_j\}_{j\in\mathbb{N}}$, J_0 is a finite set.

We set

$$I_0 := \{i \in \mathbb{N} | \exists j, \exists w \text{ s.t } A\pi(X_{i_0} w_0^{-1}) \times w_0 \cdot V_{j_0} \cap A\pi(X_i w^{-1}) \times w \cdot V_j \neq \phi\}$$

From the definition of $\{X_i\}_{i\in\mathbb{N}}$, I_0 is a finite set. Consequently, $\{A\pi(g_i Exp(U_i)w^{-1}) \times w \cdot V_j\}_{i\in\mathbb{N}, w\in W, j\in\mathbb{N}}$ is local finite. □

Proposition 7.3.27. *Here are the settings and assumptions.*

(S1) T *is the maximal torus of* $G := U(n)$.

Then

(i) $T \setminus T_{reg}$ *is a zero set with respect to a Haar measure on* T.

(ii) $G \setminus G_{reg}$ *is a zero set with respect to a Haar measure on* G.

Proof of (i). Clearly, $T \setminus T_{reg} \subset \cup_{i,j} T_{i,j}$. Here, $T_{i,j} := \{t \in T | t_i = t_j\}$. So, it is enough to show $T_{i,j}$ is a zero set for any i, j. We can assume $i = n-1, j = n$. We set

$$\varphi : T \ni t \mapsto (t_1, ..., t_{n-1}, t_{n-1}) \in T, C := \{t \in T | rank(J\varphi(t)) < n\}$$

Clearly $C = T$ and $T_{n-1,n} \subset \varphi(C)$. By Sard's Theorem(See [23]), $\varphi(C)$ is a zero set. So, $T_{n-1,n}$ is a zero set. □

Proof of (ii). By (i), $G/T \times T \setminus T_{reg}$ is a zero set. And $A : G/T \times T \to G$ is a C^ω-class surjective and $G_{reg} = A(G/T \times T \setminus T_{reg})$. So, by a Lemma for Sard's Theorem(See [23]), G_{reg} is a zero set. □

Proposition 7.3.28. *Here are the settings and assumptions.*

(S1) T is the maximal torus of $G := U(n)$.

Then for any $f \in C(G)$

$$\int_G f(g)dg = \frac{1}{n!} \int_{G/T} \int_T f(gtg^{-1}) |det(dA_{(gT,t)})| dtd(gT)$$

Here,

$$det(dA_{(gT,t)}) := det(dL_{gt^{-1}g^{-1}} \circ dA_{(gT,t)} \circ j \circ d\tau_g \times dL_t \circ i)$$

$i : T_e(g) = \mathfrak{g}_1 \oplus \mathfrak{t} \to \mathfrak{g}_1 \times \mathfrak{t}$ is the natural isomorphism and $j : T_{gT}(G/T) \times T_t(T) \to T_{(gT,t)}(G/T \times T)$ is the natural isomorphism.

STEP1. Construction of a partition of unity. By Proposition7.3.27, it is enough to show

$$\int_{G_{reg}} f(g)dg = \frac{1}{n!} \int_{G/T} \int_{T_{reg}} f(gtg^{-1}) det(dA_{(gT,t)}) dtd(gT)$$

Let $\{\pi(g_i U_i w^{-1}) \times w \cdot V_j\}_{i,j \in \mathbb{N}, w \in W}$ be the open covering of $G/T \times T_{reg}$ and $\{f_{i,j,w}\}_{i,j \in \mathbb{N}, w \in W}$ be a partition of unity with respect to $\{\pi(g_i U_i w^{-1}) \times w \cdot V_j\}_{i,j \in \mathbb{N}, w \in W}$.

We set

$$g_{i,j,w}(A(gT,t)) := \frac{1}{n!} f_{i,j,w}(gw^{-1}T, w \cdot t) \ ((gT,t) \in \pi(g_i Exp(U_i)) \times V_j, i, j \in \mathbb{N}, w \in W)$$

We will show $g_{i,j,w}$ is well-defined. Let us fix any $g_1, g_2 \in \pi(g_i Exp(U_i))$ and $t_1, t_2 \in V_j$ and $w \in W$ and $i, j \in \mathbb{N}$. such that $A(g_1 T, t_1) = A(g_2 T, t_2)$. This means that $g_1 t_1 g_1^{-1} = g_2 t_2 g_2^{-1}$. Because $t_1, t_2 \in T_e$, by Theorem7.3.22, $t_1 = t_2$ and $g_1 T = g_2 T$. So, $w \cdot t_1 = w \cdot t_2$. And

$$g_1 w^{-1} T = g_1 T w^{-1} = g_2 T w^{-1} = g_2 w^{-1} T$$

So, $g_{i,j,w}$ is well-defined.

We will show $\{g_{i,j,w}\}_{i,j \in \mathbb{N}, w \in W}$ is a partition of unity on G_{reg} with respect to $\{A\pi(g_i U_i w^{-1}) \times w \cdot V_j\}_{i,j \in \mathbb{N}, w \in W}$. Let us fix any $x \in G_{reg}$. We set

$$I := \{(i,j) \in \mathbb{N}^2 | x \in A\pi(g_i U_i) \times V_j\}$$

Then, by Theorem7.3.22,

$$I \times W = \{(i,j,w) \in\in \mathbb{N}^2 | x \in A\pi(g_i U_i w^{-1}) \times w \cdot V_j\}$$

So,

$$\sum_{i,j \in \mathbb{N}, w \in W} g_{i,j,w}(x) = \sum_{(i,j) \in I, w \in W} g_{i,j,w}(x) = \sum_{w \in W} \sum_{(i,j) \in I} g_{i,j,w}(x)$$

Let us fix any $w = \pi_0(\sigma) \in W$. And let us fix any i_1, i_2, j_1, j_2 and $h_{i_1} \in g_{i_1} Exp(U_{i_1})$ and $h_{i_2} \in g_{i_2} Exp(U_{i_2})$ and $t_{j_1} \in V_{j_1}$ and $t_{j_2} \in V_{j_2}$ such that $x = (\pi(h_{i_1} w^{-1}), w \cdot t_{j_1}) = (\pi(h_{i_2} w^{-1}), w \cdot t_{j_2})$. Then, because $t_{i_1}, t_{i_2} \in T_e$, $t_{i_1} = t_{i_2}$. And

$$h_{i_1} w^{-1} T = h_{i_1} T w^{-1} = h_{i_2} T w^{-1} = h_{i_2} w^{-1} T$$

So, there is the unique $x_w \in G/T \times T_\sigma$ such that $Ax_w = x$ and

$$\sum_{w \in W} \sum_{(i,j) \in I} g_{i,j,w}(x) = \sum_{w \in W} \frac{1}{n!} \sum_{(i,j) \in I} f_{i,j}(x_w) = \sum_{w \in W} \frac{1}{n!} = 1$$

□

STEP2. Proof of our integeral formula. We set $W_i := g_i Exp(U_i)$ $(i \in \mathbb{N})$.

$$\int_{G_{reg}} f(g)dg = \sum_{i,j,w} \int_{A\pi(W_i w^{-1})\times w\cdot V_j} f(g)g_{i,j,w}(g)dg = \sum_{i,j,w} \int_{A\pi(W_i w^{-1})\times w\cdot V_j} f(g)g_{i,j,w}(g)dg$$
$$= \frac{1}{n!}\sum_{i,j,w} \int_{\pi(W_i w^{-1})\times w\cdot V_j} f(\pi(hw^{-1}), w\cdot t)f_{i,j,w}(\pi(hw^{-1}), w\cdot t)|det(dA_{(\pi(hw^{-1}),w\cdot t)})|dg$$
$$= \frac{1}{n!}\int_{G/T\times T} f(gT,t)|det(dA_{(\pi(hw^{-1}),w\cdot t)})|d\mu_{G/T}(gT)\mu_T(t)$$

□

The following clearly holds.

Proposition 7.3.29. *We succeed notations in Proposition7.3.24. Here are the settings and assumptions.*

(S1) $X_{i,j} = E_{i,j} - E_{j,i}$ $(i<j)$.

Then $B_0 := \{X_{i,j}\}_{i<j}$ *is a basis of the complexification of* $\mathfrak{g}_1$ *and* $B_0 \cup iB_0$ *is a basis of* $\mathfrak{g}_1$.

Lemma 7.3.30. *We succeed notations in Proposition7.3.28. Then*

(i) $det(dA_{(gT,t)}) = det(Ad(t)^{-1}|_{\mathfrak{g}_1} - id|_{\mathfrak{g}_1})$.

(ii) $det(Ad(t)^{-1}|_{\mathfrak{g}_1} - id|_{\mathfrak{g}_1}) = |D(t)|^2$.

Proof of (i). Let us fix any $X \in \mathfrak{g}_1$ and $Y \in \mathfrak{t}$. Then

$$dA_{(gT,t)} \circ j \circ d\tau_g \times dL_t \circ i(X+Y) = dA_{(gT,t)} \circ j \circ d\tau_g \times dL_t(X,0) + dA_{(gT,t)} \circ j \circ d\tau_g \times dL_t(0,Y)$$

Here,

$$dA_{(gT,t)} \circ j \circ d\tau_g \times dL_t(X,0) = \frac{d}{ds}_{|s=0} A(gexp(sX)T,t) = \frac{d}{ds}_{|s=0} gexp(sX)texp(-sX)g^{-1}$$
$$= \frac{d}{ds}_{|s=0} gtg^{-1}gt^{-1}exp(sX)texp(-sX)g^{-1} = \frac{d}{ds}_{|s=0} gtg^{-1}gexp(sAd(t^{-1})X)exp(-sX)g^{-1}$$
$$= dL_{gtg^{-1}}Ad(g)(Ad(t^{-1})X - X)$$

and

$$dA_{(gT,t)} \circ j \circ d\tau_g \times dL_t(0,Y) = \frac{d}{ds}_{|s=0} A(gT, texp(sY)) = \frac{d}{ds}_{|s=0} gtexp(sY)g^{-1}$$
$$= \frac{d}{ds}_{|s=0} gtg^{-1}gt^{-1}exp(sY)g^{-1} = dL_{gtg^{-1}}Ad(g)(Y)$$

So,

$$det(dA_{(gT,t)}) = det(Ad(g))det(F)$$

Here,

$$F : \mathfrak{g}_1 \times \mathfrak{t} \ni (X,Y) \mapsto (Ad(t^{-1})X - X, Y) \in \mathfrak{g} \times \mathfrak{t}$$

Because clearly $T\cdot\mathfrak{g}_1 \subset \mathfrak{g}_1$ and $\mathfrak{g}_1 \cdot T \subset \mathfrak{g}_1$, $Ad(t^{-1})X \in \mathfrak{g}_1$ $(\forall t \in T, \forall X \in \mathfrak{g}_1)$. So, $ImF \in \mathfrak{g}_1 \times \mathfrak{t}$. This implies that $det(F) = det(Ad(t^{-1})|\mathfrak{g}_1 - id_{\mathfrak{g}_1})$. And, by Proposition3.5.15, $det(Ad(g)) = 1$ $(\forall g \in G)$. □

Proof of (ii). It is enough to show that (ii) holds for any $t \in T_{reg}$. Let us fix any $t \in T_{reg}$. We succeed notations in Proposition7.3.29.

$$(Ad(t)^{-1} - id)X_{i,j} = (\frac{t_j}{t_i} - 1)X_{i,j} \ (\forall i < \forall j)$$

So, by Proposition3.13.3,

$$det(Ad(t)^{-1} - id) = (\Pi_{i<j}|(\frac{t_j}{t_i} - 1)|)^2$$
$$\text{by } |t_i| = 1 \text{ and } \overline{\frac{t_j}{t_i}} = \frac{t_i}{t_j} \ (\forall i < \forall j)$$
$$= (\Pi_{i<j}|(t_i - t_j)|)^2 = |D(t)|^2$$

Lemma7.3.30 and Proposition7.3.28 implies the following.

Theorem 7.3.31 (Weyl Integral Formula). *For any $f \in C(U(n))$,*

$$\int_{U(n)} f(g) d\mu_{U(n)}(g) = \frac{1}{n!} \int_{G/T} \int_{T} f(gtg^{-1}) |D(t)|^2 d\mu_T t d\mu_{G/T}(gT)$$

□

7.3.5 The highest weight of $U(n)$

Definition 7.3.32 (Multiplicity of weight). *We will succeed notations in Proposition7.3.5. Let*

(S1) G is a compact Lie group.

(S2) (π, V) is a finite dimensional continuous representation of G.

(S3) $\lambda \in \mathbb{Z}^n$.

We call $m_\lambda := dimV_\lambda$ the multiplicity of λ.

Definition 7.3.33 (Symmetric function). *Let T be the maximal torus of $U(n)$. We say $f \in C(T, \mathbb{C})$ is a symmetric function if*

$$f(x) = f(wx) \ (\forall x \in T, \forall w \in W)$$

We denote the set of all symmetric functions by $C(T)_1$.

Definition 7.3.34 (Alternating function). *Let T be the maximal torus of $U(n)$. We say $f \in C(T, \mathbb{C})$ is a altenating function if*

$$f(x) = sign(w) f(wx) \ (\forall x \in T, \forall w \in W)$$

We denote the set of all symmetric functions by $C(T)_{sgn}$.

Definition 7.3.35 (Laurant polynomial). *Let T be the maximal torus of $U(n)$. We say $f \in C(T, \mathbb{C})$ is a Laurant function if*

$$f(x) = \sum_{K \in \mathbb{Z}^n} a_K t^K \ (x \in T), \#\{K \in \mathbb{Z}^n | a_K \neq 0\} < \infty$$

We denote the set of all Laurant polynomials by $R(T)$. We set

$$R_{\mathbb{Z}}(T) := \{f \in R(T) | Every\ coefficient\ of\ f\ are\ in\ \mathbb{Z}\}$$

and

$$R_{\mathbb{Z}}(T)_1 := R_{\mathbb{Z}}(T) \cap C(T)_1$$

Proposition 7.3.36. *Here are the settings and assumptions.*

(S1) T is th maximal torus of $U(n)$.

(S2) $W := \pi_0(\mathfrak{S}_n)$.

(S3) (π, V) is a finite dimensional continuous representation of G.

(S4) $\Delta(V, T) := \{\lambda \in \hat{T} | V_\lambda \neq \{0\}\}$.

(S5) $\lambda \in \mathbb{Z}^n$ is the highest weight of (π, V).

Then

(i) For any $w \in W$ and $\lambda \in \mathbb{Z}^n$, $\pi(w)|V_\lambda$ is a bijection fo $V_{w\lambda}$.

(ii) $W \cdot \Delta(V, T) \subset \Delta(V, T)$.

(iii) For any $\sigma \in \mathbb{Z}^n$, $m_\sigma = m_{w\sigma}$.

(iv) $\Delta(V, T)$ is finite set.

(v) $V_\lambda \simeq m_\lambda \chi_\lambda$ as continuous unitary representation of T. The right side is a discrete direct sum.

(vi) $\chi_{\pi|T} = \sum_{\lambda \in \Delta(V,T)} m_\lambda \chi_\lambda$

(vii) $\chi_{\pi|T} \in R_{\mathbb{Z}}(T)_1$.

(viii) $\lambda \in (\mathbb{Z}^n)_+$.

Proof of (i). Firstly we will show $\pi(w)|V_\lambda \subset V_{w\cdot\lambda}$ $(\forall w \in W, \forall\lambda \in \hat{T})$. Let us fix any $w \in W$ and any $\lambda \in \mathbb{Z}^n$ and any $v \in V_\lambda$ and any $t \in T$.

$$\pi(t)\pi(w)v = \pi(w)\pi(w^{-1}\cdot t)v = \pi(w)\chi_\lambda(w^{-1}\cdot t)v = \chi_\lambda(w^{-1}\cdot t)\pi(w)v = \chi_{w\cdot\lambda}(t)\pi(w)v$$

So, by Proposition7.3.17, $\pi(w)v \in V_{w\cdot\lambda}$. Because $\pi(w^{-1})$ is the inverse of $\pi(w)$, $\pi(w)|V_\lambda$ is bijective. □

Proof of (ii). For any $w \in W$ and any $\lambda \in \Delta(V,T)$, by (i), $V_{w\cdot\lambda} = \pi(w)\cdot V_\lambda$. Because $\pi(w)\cdot V_\lambda \neq \{0\}$, $V_{w\cdot\lambda} \neq \{0\}$. So, $w\cdot\lambda \in \Delta(V,T)$. □

Proof of (iii). This is followed by (i). □

Proof of (iv). Because $\chi_{\lambda_1} \not\simeq \chi_{\lambda_2}$ $(\forall\lambda_1 \neq \forall\lambda_2)$, by Theorem5.7.26, $V = \oplus_{\lambda\in\mathbb{Z}^n} V_\lambda$. Because $dimV < \infty$, $\Delta(V,T)$ is a finite set. □

Proof of (v). Clearly V_λ is finite dimensional T-invariant space. Let us fix $w_1, ..., w_m$ which is the orthonormal basis of V_λ. We set

$$P_i z := zw_i \ (z \in \mathbb{C}, i \in \{1,2,...,m\})$$

By Proposition7.3.17,

$$P_i\chi_\lambda(t)z = z\chi_\lambda(t)w_i = z\pi(t)w_i = \pi(t)zw_i = \pi(t)P_i(z)$$

and $\mathbb{C}w_i$ is T-invariant. So, $P_i : (\chi_\lambda, \mathbb{C}) \to (\pi|\mathbb{C}w_i, \mathbb{C}w_i)$ is an isomorphism as continuous unitary representations of T. Consequently, (v) holds. □

Proof of (vi). (vi) is followed by (v) and Theorem5.7.26. □

Proof of (vii). By (vi), $\chi_{\pi|T} \in R_{\mathbb{Z}}(T)$. By (i), $\chi_{\pi|T} \in C(T)_1$. So, $\chi_{\pi|T} \in R_{\mathbb{Z}}(T)$. □

Proof of (viii). (viii) is followed by (i). □

Notation 7.3.37 (S_α, A_α)**.** *For $\alpha \in \mathbb{Z}^n$,*

$$S_\alpha(t) := \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} t^{\sigma\alpha}$$

$$A_\alpha(t) := \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} sign(\sigma)t^{\sigma\alpha}$$

Proposition 7.3.38.

(i) $\{S_\alpha\}_{\alpha\in\mathbb{Z}^n}$ is a basis of $R_{\mathbb{Z}}(T)_1$.

(ii) $\{A_\alpha\}_{\alpha\in\mathbb{Z}^n}$ is a basis of $R_{\mathbb{Z}}(T)_{sgn}$.

Proof of (i). Let us fix any $\frac{1}{n!}\sum_\alpha a_\alpha t^\alpha \in R_{\mathbb{Z}}(T)_1$. Let us fix any $\alpha \in \mathbb{Z}^n$ such that $\alpha_1 \geq ... \geq \alpha_n$. Then

$$a_{\sigma\alpha} = a_\alpha \ (\forall\sigma \in \mathfrak{S}_n)$$

So,

$$\frac{1}{n!}\sum_\alpha a_\alpha t^\alpha = \sum_{\alpha_1\geq...\geq\alpha_n} a_\alpha S_\alpha(t)$$

□

Proof of (ii). Let us fix any $\frac{1}{n!}\sum_\alpha a_\alpha t^\alpha \in R_{\mathbb{Z}}(T)_1$. If there are i, j such that $\alpha_i = \alpha_j$, then $a_\alpha = 0$ by the definition of the altenating function. Let us fix any $\alpha \in \mathbb{Z}^n$ such that $\alpha_1 > ... > \alpha_n$. Then

$$a_{\sigma\alpha} = sign(\alpha)a_\alpha \ (\forall\sigma \in \mathfrak{S}_n)$$

So,

$$\frac{1}{n!}\sum_\alpha a_\alpha t^\alpha = \sum_{\alpha_1\geq...\geq\alpha_n} a_\alpha A_\alpha(t)$$

□

Proposition 7.3.39.

(i) $\{S_\alpha\}_{\alpha\in\mathbb{Z}^n}$ *is a basis of* $R_{\mathbb{Z}}(T)_1$.

(ii) $\{A_\alpha\}_{\alpha\in\mathbb{Z}^n}$ *is a basis of* $R_{\mathbb{Z}}(T)_{sgn}$.

By the orthogonality of trigonometric functions, the following holds.

Proposition 7.3.40. *For* $\alpha_1 > ... > \alpha_n$ *and* $\beta_1 > ... > \beta_n$,

$$(A_\alpha, A_\beta)_{L^2(T)} = \begin{cases} n! & \alpha = \beta. \\ 0 & \alpha \neq \beta. \end{cases}$$

7.3.6 Weyl Character Formula

Theorem 7.3.41 (Weyl character formula). *Here are the settings and assumptions.*

(S1) T *is the maximal torus of* $U(n)$.

(S2) (π, V) *is a finite dimensional irreducible continuous representation of* G.

(S3) λ *is the highest weight of* π.

Then

(i)

$$\chi_\pi(t) = \frac{\sum_{\sigma\in\mathfrak{S}_n} sgn(\sigma) t^{\sigma\cdot(\lambda+\rho)}}{\Pi_{1\leq i<j\leq n}(t_i - t_j)}$$

Here, $\rho := (n-1, n-2, ..., 1, 0)$.

(ii) $dim(V_\lambda) = 1$.

Proof. We set

$$D(t) := \Pi_{1\leq i<j\leq n}(t_i - t_j) \ (t\in T)$$

Then $\chi_\pi(t)D(t)$ is an alternating laurant polynomial, there is $\{a_\alpha\}_{\alpha\in\mathbb{Z}^n}$ such that $\#\{\alpha | a_\alpha \neq 0\} < \infty$ and

$$\chi_\pi(t)D(t) = \sum_\alpha a_\alpha A_\alpha(t) \ (\forall t \in T)$$

By Proposition7.3.40,

$$1 = \sum_\alpha |a_\alpha|^2$$

By Proposition7.3.36(vii), for any α $a_\alpha \in \mathbb{Z}$. So $\exists! \alpha$ such that $|a_\alpha| = 1$. By Proposition7.2.1,

$$\chi_\pi D = A_\alpha \text{ (in } \mathbb{C}^n)$$

or

$$\chi_\pi D = -A_\alpha \text{ (in } \mathbb{C}^n)$$

Let m_λ denote the multiplicity of λ. And we can assume $\alpha_1 > ... > \alpha_n$. The maximal index of $D(t)$ with respect to lexicographic order is $(n-1, ..., 1)$. And the maximal index of χ_π with respect to lexicographic order is $m_\lambda \lambda$. So,

$$m_\lambda t^{(\lambda_1+n-1,...,\lambda_n+1)} = +t^\alpha \text{ (in } \mathbb{C}^n)$$

and

$$m_\lambda t^{(\lambda_1+n-1,...,\lambda_n+1)} = -t^\alpha \text{ (in } \mathbb{C}^n)$$

This implies that $m_\lambda = 1$ and

$$(\lambda_1 + n - 1, ..., \lambda_n + 1) = \alpha$$

□

7.3.7 Cartan-Weyl Highest Weight Theory

Theorem 7.3.42. *The followings hold.*

(i) For any $\phi \in R_{\mathbb{Z}}(T)_1$, $\Phi(\phi) := D\phi \in R_{\mathbb{Z}}(T)_{sgn}$.

(ii) $\Phi : R_{\mathbb{Z}}(T)_1 \to R_{\mathbb{Z}}(T)_{sgn}$ is surjective.

Proof of (i). It is clear. □

Proof of (ii). Let us fix any $\phi \in R_{\mathbb{Z}}(T)_{sgn}$. There is $N \in \mathbb{N}$ such that $p(t) = \sum_\alpha t^\alpha = t^{(N,...,N)}\phi \in P_{\mathbb{Z}}(T)_{sgn}$. For any $\alpha \in \mathbb{Z}^n$ such that $\alpha_1 = \alpha_2$, $a_\alpha = 0$.

For any $t \in T$ such that $t_1 = t_2$, $p(t) = 0$. By Proposition7.2.1, For any $z \in \mathbb{C}^n$ such that $z_1 = z_2$, $p(z) = 0$.

For each $\alpha \in \mathbb{Z}^n$ such that $\alpha_1 > \alpha_2$, $a_\alpha = -a_{S_{1,2}\alpha}$. Here, $S_{1,2}$ is the permutate of 1 and 2. So, there is $q \in P_{\mathbb{Z}}(T)$ such that

$$p(t) = (t_1 - t_2)q(t)$$

For any $t \in T$ such that $t_1 = t_3$,

$$q(t) = 0$$

So, by the same argument as the above, there is $r \in P_{\mathbb{Z}}(T)$ such that

$$q(t) = (t_1 - t_3)r(t)$$

By repeating this argument, we find that there is $\psi \in P_{\mathbb{Z}}(T)$ such that

$$\phi = D\psi$$

□

Theorem 7.3.43. *The followings hold.*

(i) For any $\phi \in C(U(n))^{Ad}$, $\Phi(\phi) := \phi|T \in C(T)_1$.

(ii) $\Phi : C(U(n))^{Ad} \to C(T)_1$ is surjective.

Proof of (i). It is clear. □

Proof of (ii). We set $G := U(n)$. Let us fix any $\phi \in C(T)_1$. For each $g \in G$, let denote the set of all eigenvalues of g by $\{\lambda_1(g), ..., \lambda_n(g)\}$. And

$$\psi(g) := \phi(\lambda_1(g), ..., \lambda_n(g))$$

Because ϕ is symmetric, ψ is well-defined. We will show ψ is continuous. Let us fix any $g_0 \in G$. Let denote $\lambda_1, ..., \lambda_m$ the distinct set of eigenvalues of g_0. Denote the degree of λ_i as zero point of characteristic polynomial of g by k_i.

By Rouche's Theorem(see [19]), for any $\epsilon > 0$, there is $\delta > 0$ such that g has just k_i eigenvalues(allow multiplicity) of g in $B(\lambda_i, \epsilon)$ for any $g \in B(g_0, \delta)$. So, ψ is continuous. Clearly, $\Phi(\psi) = \phi$. So, Φ is surjective. □

Theorem 7.3.44 (Cartan-Weyl Highest Weight Theory)**.** *The followings hold.*

(i) Let us assume (π, V) be a continuous irreducible unitary representation of $U(n)$ and λ be the highest weight of π. Then $\lambda \in (\mathbb{Z}^n)_+$ and the multiplicity of λ is 1.

(ii) Let us fix any $\lambda \in (\mathbb{Z}^n)_+$. Then there is the unique continuous irreducible unitary representation (π, V) whose highest weight is λ, ignoring isomorphism as continous unitary representation.

Proof of (i). (i) is from Weyl Character Formula(Theorem7.3.41) and Proposition7.3.36. □

Proof of (ii). The uniqueness is from Proposition7.3.23. We will show the existence. For aiming contradiction, let us assume that there exists $\lambda \in \mathbb{Z}^n_+$ such that λ is different from the highest weight of any irreducible continuous unitary representation of $U(n)$. We set

$$\rho := (n-1, ..., 1)$$

Because $A_{\lambda+\rho} \in R_{\mathbb{Z}}(T)_{sgn}$, by Theorem7.3.42 and Theorem7.3.43, there is $\psi \in C(U(n))^{Ad}$ such that $D(t)\psi = A_{\lambda+\rho}$. For any $\pi \in \hat{U(n)}$, by Weyl Integral Formula

$$\int_{U(n)} \chi_\tau(g)\bar{\psi}(g)dg = \int_T \chi_\tau(t)\bar{\psi}(t)|D(t)|^2 dt = \int_T A_{\alpha(\pi)+\rho}(t)\overline{A_{\lambda+\rho}(t)}dt = 0$$

Here, $\alpha(\pi)$ is the highest weight of π. By Theorem5.7.20, ψ is zero function. This is contradiction. □

7.3.8 Review

In this subsection, we show the result of classification of irreducible continuous unitary representations of $U(n)$. By Peter Weyl Theorem, it is enough to classify finite dimensional irreducible continuous unitary representation of $U(n)$.

We focus the set of all the set of all eigenvalues of $g \in U(n)$, $T := \mathbb{T}^n$. We can simplify discussions about $U(n)$ to discussions about T in some cases. In specialty, Weyl Integral Formula is really usefull.

Theorem 7.3.45 (Weyl Integral Formula)**.** *For any $f \in C(U(n))$,*

$$\int_{U(n)} f(g) d\mu_{U(n)}(g) = \frac{1}{n!} \int_{G/T} \int_T f(gtg^{-1}) |D(t)|^2 d\mu_T t d\mu_{G/T}(gT)$$

By this theorem, we can simply integral of class function on $U(n)$ to simply integral of symmetric function on T. Let recall the proof of Weyl Integral Formula.

$$A : G/T \times T \ni (gT, t) \mapsto gtg^{-1} \in G$$

is $n!$-th covering map of G and $\mathfrak{S}_n$ acts on $A^{-1}(g)$ for each $g \in G$. That implies

$$\int_{U(n)} f(g) d\mu_{U(n)}(g) = \frac{1}{n!} \int_{G/T} \int_T f(gtg^{-1}) |det(dA_{(gT,t)})| d\mu_T t d\mu_{G/T}(gT)$$

In the proof of this equation, we need take a good partition of unity of $U(n)$. By focusing the decomposition

$$\mathfrak{u}(n) = \mathfrak{u}(n)_1 \oplus \mathfrak{t}$$

and action on $\mathfrak{u}(n)_1$ and $\mathfrak{t}$, we get

$$det(dA_{(gT,t)}) = det(Ad(t^{-1})|\mathfrak{u}(n)_1 - id|\mathfrak{u}(n)_1)$$

Here,

$$\mathfrak{u}(n)_1 = \{X \in \mathfrak{u}(n) | X_{i,i} = 0 \ (\forall i)\}$$

By complexifying $\mathfrak{u}(n)_1$ and showing $E_{i,j}$ are eigenvector of the complexification of $Ad(t^{-1})|\mathfrak{u}(n)_1 - id|\mathfrak{u}(n)_1$ with respect to $(\frac{t_j}{t_i} - 1)$ $(\forall i \neq \forall j)$, we get

$$det(Ad(t^{-1})|\mathfrak{u}(n)_1 - id|\mathfrak{u}(n)_1) = |D(t)|^2$$

Consequently, we get Weyl Integral Formula. By Weyl Integral Formula and Shur Orthogonality Relation, we can simplify the classification of continuous finite dimensional irreducible unitary representations of $U(n)$ to the classification of $\{\chi_\pi|T|\pi$ is a continuous finite dimensional irreducible unitary representations of $U(n)\}$.

We focus the fact $D\chi_\pi|T$ is an alternating Laurant polynomial on T with $\mathbb{Z}$-coefficients. We can show $\{A_\alpha\}_{\alpha_1 > ... > \alpha_n}$ is an orthonomal system of $L^2(T)$ and a basis of $R_{\mathbb{Z}}(T)_{sgn}$. Here,

$$A_\alpha = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} sign(\sigma) t^{\sigma \cdot \alpha}, R_{\mathbb{Z}}(T)_{sgn} := \{p | p \text{ is an alternating Laurant polynomial on } T \text{ with } \mathbb{Z}\text{-coefficients.}\}$$

It is important that the decompositions of $D\chi_\pi$ with $\{A_\alpha\}_{\alpha_1 > ... > \alpha_n}$ correspoinds to the decompositions of $\pi|T$ as continuous unitary representation of T. The last decomposition is called a branching rule. Thanks to these insight, we can classify $\hat{U(n)}$ by the highest weight of each $\pi \in \hat{U(n)}$. In specialty, we get the following Weyl character formula.

Theorem 7.3.46 (Weyl character formula)**.** *Here are the settings and assumptions.*

(S1) T is the maximal torus of $U(n)$.

(S2) (π, V) is a finite dimensional irreducible continuous representation of G.

(S3) λ is the highest weight of π.

Then

(i)

$$\chi_\pi(t) = \frac{\sum_{\sigma \in \mathfrak{S}_n} sgn(\sigma) t^{\sigma \cdot (\lambda + \rho)}}{\Pi_{1 \leq i < j \leq n} (t_i - t_j)}$$

Here, $\rho := (n-1, n-2, ..., 1, 0)$.

(ii) $dim(V_\lambda) = 1$.

Inversely, for each $\lambda \in (\mathbb{Z})^n_+ := \{\alpha \in (\mathbb{Z}) | \alpha_1 \geq ... \geq \alpha\}$, there is $\psi \in C(U(n))^{Ad}$ such that $(\psi|T)D = A_{\lambda+\rho}$. Here $\rho := (n-1, ..., 0)$. That facts from the correspondance

$$U(n) \ni g \mapsto \frac{A_{\lambda+\rho}(\lambda_1(g), ..., \lambda_n(g))}{D(\lambda_1(g), ..., \lambda_n(g))} \in \mathbb{C}$$

By completeness of character about $\hat{U(n)}$, we can show there is $\pi \in \hat{U(n)}$ such that the highest weight of π is λ.

7.4 Noncompact Lie Group⋆

For noncompact connected simple Lie groups, there non trivial continuous irreducible unitary representations are infinite dimensional[3]. In [1], the answer of the excerise12.3 shows a proof for $SL(n, \mathbb{R})$. I guess a full proof is followed by the result of classification of simple Lie algebras.

Chapter 8

Examples of Lie groups, Homogeneous spaces, Representations

8.1 Lie Groups

8.1.1 $\mathbb{R}^\times, \mathbb{C}^\times$

Example 8.1.1 ($\mathbb{R}^\times$)**.** *Clearly $\mathbb{R}^\times$ is Linear Liegroup of $GL(1,\mathbb{C})$. So $\mathbb{R}^\times$ is Lie subgroup of $GL(1,\mathbb{C})$. And clearly $Lie(\mathbb{R}^\times) = \mathbb{R}$.*

Example 8.1.2 ($\mathbb{C}^\times$)**.** *Clearly $\mathbb{C}^\times$ is Linear Liegroup of $GL(1,\mathbb{C})$ and $Lie(\mathbb{C}^\times) = \mathbb{C}$.*

Clearly $G := \{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} | a, b \in \mathbb{R}$ such that $a^2 + b^2 \neq 0\}$ is Lie subgroup $GL(2,\mathbb{R})$ and G is isomorphic to $\mathbb{C}^\times$ and $Lie(G) = \{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} | a, b \in \mathbb{R}\}$. Clearly the right side is subset of the left side. We will show the proof of the inverse in below.

Proof. Let us fix any $X \in Lie(G)$.

$$exp(tX) = E + tX + O(t^2) \ (t \to 0)$$

We define

$$M(t) := \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} := exp(tX) - tX$$

So there is $C > 0$ such that $||M(t)|| \leq C|t|^2$ for any $t \in \mathbb{R}$. We assume $|x_{1,1} - x_{2,2}| \neq 0$.
We pick $t \neq 0$ such that

$$|t| < \frac{|x_{1,1} - x_{2,2}|}{2(C+1)}$$

Because $X \in Let(G)$

$$|t(x_{1,1} - x_{2,2})| = |a(t) - d(t)|$$

Because for any $t \in [-1,1]$ $|a(t) - d(t)| \leq 2C|t|^2 < |t||x_{1,1} - x_{2,2}|$,

$$|t(x_{1,1} - x_{2,2})| < |t||x_{1,1} - x_{2,2}|$$

So $1 < 1$. It implies contradiction. □

8.1.2 $\mathbb{R}, \mathbb{C}$

Example 8.1.3 ($\mathbb{R}$)**.** *Because $i : \mathbb{R} \ni t \mapsto exp(t) \in (0,\infty)$ is isomorphism of topological groups, $\mathbb{R}$ is a Lie group. Clearly $Lie(\mathbb{R}) = \{a + n\pi i | a \in \mathbb{R}, n \in \mathbb{Z}\}$.*

Example 8.1.4 ($\mathbb{C}$)**.** *By inverse function theorem about holomorphic function, $i : \mathbb{R} \times (-\pi, \pi) \ni (a,b) \mapsto exp(a)exp(ib)\mathbb{R}$ is isomorphism of topological spaces. Clearly $i|\mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$ is isomorphism in Definition3.1.1. So $\mathbb{C}$ is a Lie group. Clearly $Lie(\mathbb{C}) = \mathbb{C}$.*

8.1.3 $\mathbb{T} = S^1 = SO(2)$

Example 8.1.5 (Unitary, Cyclic, not Irreducible Representation of $\mathbb{T}$)**.** *The followings are settings.*

(S1) For $m \in \mathbb{Z}$,

$$\pi_m(z)w := z^m w \ (z \in \mathbb{T}, w \in \mathbb{C}) \tag{8.1.1}$$

Then for any $m \neq n \in \mathbb{Z}$,

$$\pi_{m,n} := \pi_m \times \pi_n \tag{8.1.2}$$

is a cyclic, not irreducible, continuous representation of $\mathbb{T}$.

8.1.4 $O(n), SO(n)$

Example 8.1.6 ($O(n), SO(n)$)**.** *The followings hold.*

(i)

$$\mathfrak{so}(n) := Lie(SO(n)) = Lie(O(n)) = \{X \in M(n, \mathbb{R}) | X^T = -X\} \tag{8.1.3}$$

(ii)

$$\dim O(n) = \frac{1}{2}(n-1)n$$

(iii) $SO(n)$ is normal subgroup of $SO(n)$ and

$$O(n)/SO(n) \simeq \{\pm 1\}$$

(iv) $SO(n)$ is path connected.

(v) $O(n), SO(n)$ are reductive Lie group.

(vi) $\mathfrak{so}(3)$ is a simple Lie algebra.

proof of (i). Let us fix any $X \in M(n, \mathbb{R})$ such that $X^T = -X$. Then for any $t \in \mathbb{R}$,

$$exp(tX)exp(tX)^T = exp(tX)exp(tX^T) = exp(tX)exp(-tX) = E$$

So the right side is subset of $Lie(O(n))$. Nextly let us fix any $X \in Lie(O(n))$. Because for any $t \in \mathbb{R}$ $exp(tX) \in M(n, \mathbb{R})$. By the argument similar to Example8.1.2, $X \in M(n, \mathbb{R})$. By Proposition3.2, $E = exp(tX)exp(tX)^T = exp(t(X+X^T)+O(t^2))$. By the argument similar to Example8.1.2, $X + X^T = O$. □

Proof if (iii).

$$O(n) \ni g \mapsto det(g) \in \{\pm 1\}$$

is a surjective continuous homomorphism and the kernel is clearly $SO(n)$. Therefore, (iii) holds. □

Proof if (iv). From Example8.2.1 later,

$$S^{n-1} \simeq SO(n)/SO(n-1)$$

For any $n \in \mathbb{N}$, S^n is connected. And $SO(2) = S^1$ is connected. Therefore, by Proposition6.2.1 and mathematical induction, for any $n \in \mathbb{N}$ $SO(n)$ is connected. From Proposition3.4.6, $SO(n)$ is path-connected. □

proof of (v). It is followed from (iv) and ${}^t\overline{g} \in O(n)$ $(\forall g \in O(n))$. □

proof of (vi). We set

$$A_{i,j} := E_{i,j} - E_{j,i} \ (i < j)$$

Then

$$[A_{1,2}, A_{1,3}] = -A_{2,3}, [A_{1,2}, A_{2,3}] = A_{1,3}, [A_{1,3}, A_{2,3}] = -A_{1,2}$$

Let us fix any $\mathfrak{h}$ which is a nonzero ideal of $\mathfrak{so}(3)$. Since $\mathfrak{h}$ is nonzero, either $ad(A_{1,2}), ad(A_{2,3}), ad(A_{1,3})$ has nonzero eigenvalue. First, let us assume $ad(A_{1,2})$ has a nonzero eigenvalue α and an eigenvector

$$A := a_1 A_{1,2} + a_2 A_{1,3} + a_3 A_{2,3} \in \mathfrak{h} \tag{8.1.4}$$

Then

$$ad(A_{1,2})A = a_3 A_{1,3} - a_2 A_{2,3}$$

and

$$\alpha A = \alpha a_1 A_{1,2} + \alpha a_2 A_{1,3} + \alpha a_3 A_{2,3}$$

This implies

$$a_1 = 0, a_2 = \alpha a_3$$

So, $a_3 \neq 0$ and

$$A := \alpha a_3 A_{1,3} + a_3 A_{2,3}$$

Since $\frac{-1}{a_3} ad(A_{1,3}) = A_{1,2}$, by (8.1.4), $\mathfrak{h} = \mathfrak{so}(3)$. In other cases, we can prove $\mathfrak{h} = \mathfrak{so}(3)$ by the same way. □

8.1.5 $U(n), SU(n)$

Example 8.1.7 ($U(n)$)**.** *The followings hold.*

(i)

$$\mathfrak{so}(n) := Lie(SU(n)) = Lie(U(n)) = \{X \in M(n, \mathbb{C}) | X^* = -X\}$$

(ii)

$$\dim U(n) = (n-1)n$$

(iii) $SU(n)$ *is a normal subgroup of* $U(n)$ *and*

$$U(n)/SU(n) \simeq \mathbb{T}$$

(iv) $U(n), SU(n)$ *are path connected.*

(v) $U(n), SU(n)$ *are reductive Lie group.*

(vi) $\mathfrak{su}(3)$ *is a simple Lie algebra.*

(vii)

$$\mathfrak{su}_{\mathbb{C}}(n) = \mathfrak{sl}(n), \mathfrak{u}_{\mathbb{C}}(n) = \mathfrak{gl}(n)$$

(viii) $\mathfrak{su}(n)$ *is a real form of* $\mathfrak{sl}(n)$ *and* $\mathfrak{u}(n)$ *is a real form of* $\mathfrak{gl}(n)$*.* $SU(n)$ *is a real form of* $SL(n, \mathbb{C})$ *which is a complex Lie group.* $U(n)$ *is a real form of* $GL(n, \mathbb{C})$ *which is a complex Lie group.*

proof of (i). It is similar to the proof of (8.1.3). □

proof of (ii). (ii) is clear from (i). □

proof of (iii).

$$U(n) \ni g \mapsto det(g) \in \mathbb{T}$$

is a surjective continuous homomorphism and the kernel is clearly $SU(n)$. Therefore, (iii) holds. □

proof of (iv). From Proposition3.4.6 and Proposition6.2.1 the fact that $\mathbb{T}$ is connected, it is enough to show $SU(n)$ is path connected.

Let us fix any $g \in SU(n)$. Then there is $h \in SU(n)$ and $\theta_1, ..., \theta_n \in \mathbb{R}$ such that

$$g = hdiag(\exp(i\theta_1), ..., \exp(i\theta_n))h^{-1}, \sum_{i=1}^{n} \theta_i = 0$$

We set

$$\varphi : [0,1] \ni t \mapsto hdiag(\exp(it\theta_1), ..., \exp(it\theta_n))h^{-1} \in U(n)$$

Then $\varphi \in C([0,1], SU(n))$ and $\varphi(0) = e$ and $\varphi(1) = g$. Therefore, $SU(n)$ is path-connected. □

proof of (v). It is followed from (iv) and ${}^t\bar{g} \in U(n)$ $(\forall g \in U(n))$. □

proof of (vi). The proof is similar to the proof when the Lie algebra is $\mathfrak{so}(3)$. □

proof of (vii). Racall

$$\mathfrak{su}_{\mathbb{C}}(n) = \{X + iY |\ X, Y \in \mathfrak{su}(n)\}$$

And

$$\mathfrak{su}(n) = \mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 = \{X \in diag(i\mathbb{R}, .., i\mathbb{R}) | tr(X) = 0\}, \mathfrak{g}_2 = \{X \in M(n, \mathbb{C}) | X_{ii} = 0\ (\forall i), X^* = -X\}$$

Clearly $\mathfrak{g}_1 + i\mathfrak{g}_1 = \{X \in diag(\mathbb{C}, .., \mathbb{C}) | tr(X) = 0\}$. For any $i < j$, $E_{ij} - E_{ji}, i(E_{ij} + E_{ji}), \in \mathfrak{g}_2$. Therefore, $E_{ij}, E_{ji} \in \mathfrak{g}_2 + i\mathfrak{g}_2$. That means

$$\mathfrak{g}_2 + i\mathfrak{g}_2 = \{X \in M(n, \mathbb{C}) | X_{ii} = 0\ (\forall i)\}$$

Consequently $\mathfrak{su}_{\mathbb{C}}(n) = \mathfrak{sl}(n)$. From the same argument, we can show $\mathfrak{u}_{\mathbb{C}}(n) = \mathfrak{gl}(n)$. □

Proposition 8.1.8.

$$SU(2) = \{\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} | |\alpha|^2 + |\beta|^2 = 1\}$$

Proof. Clearly

$$\{\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} | |\alpha|^2 + |\beta|^2 = 1\} \subset SU(2)$$

Let us fix any $g := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(n)$. Since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$|a|^2 + |b|^2 = 1, |c|^2 + |d|^2 = 1$$

and

$$(a, b) \perp (c, d)$$

Then there exists $\gamma \in \mathbb{C}$ such that

$$(c, d) = \gamma(-\bar{b}, \bar{a})$$

Since $\det(g) = 1$, $\gamma = 1$. □

Proof of (viii). That is from (vii) and

$$\mathfrak{su}(n) \cap i\mathfrak{su}(n) = \{0\}, \mathfrak{u}(n) \cap i\mathfrak{u}(n) = \{0\}$$

□

Proposition 8.1.9. *The followings are settings.*

(S1) Let $P[z, w]$ denote the set of all polynomials with two variables z, w.

(S2) We set

$$(\begin{pmatrix} a & b \\ c & d \end{pmatrix} f)(z, w) := f((\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} z \\ w \end{pmatrix})^T)\ (\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C}), f \in P[z, w])$$

Then $GL(2, \mathbb{C})$ acts on $P[z, w]$.

Proof. Let us fix any $g_1, g_2 \in GL(2, \mathbb{C})$. Then for any $f \in P[z, w]$,

$$(g_1 g_2 \cdot f)(z, w) = f((z, w) g_1 g_2) = f(((z, w) g_1) g_2) = (g_2 \cdot f)((z, w) g_1) = (g_1 \cdot (g_2 \cdot f))(z, w)$$

□

The following clearly holds.

Proposition 8.1.10 (π_l)**.** *The following is the definition.*

(i) $l \in \mathbb{N}_{\geq 1}$.

(ii)

$$V_l := \{f \in P[z, w] | f(z, w) = \sum_{k=0}^{l} a_k z^{l-k} z^l, a_k \in \mathbb{C}\ (k = 0, 1, ..., l)\}$$

(iii)

$$\pi_l(g)f := g \cdot f \ (g \in SU(2), f \in V_l)$$

Then (π_l, V_l) is a $(l+1)$ dimensional continuous representation of $SU(2)$.

Proposition 8.1.11 ($\widehat{SU(2)}$)**.** *The followings are settings and assumptions.*

(S1) We set

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The followings hold.

(i)

$$d\pi_l(H) = z\frac{\partial}{\partial z} - w\frac{\partial}{\partial w}, d\pi_l(E) = z\frac{\partial}{\partial w}, d\pi_l(F) = w\frac{\partial}{\partial z}$$

(ii) For any l, k,

$$d\pi_l(H)z^{l-k}w^k = (l-2k)z^{l-k}w^k$$

(iii) If $V \subset V_l$ is $d\pi_l(H)$-invariant, there is k such that V contains $w^{l-k}z^k$.

(iv) For any $k \geq 1$,

$$d\pi_l(E)z^{l-k}w^k = z^{l-(k-1)}w^{k-1}$$

For any $k \leq (l-1)$,

$$d\pi_l(F)z^{l-k}w^k = z^{l-(k+1)}w^{k+1}$$

(v) For any $l \in \mathbb{Z}_{\geq 1}$, π_l is irreducible.

Proof of (i). First,

$$d\pi_l(H)f = \frac{d}{dt}|_{t=0}f((z,w)\exp(tH)) = \frac{d}{dt}|_{t=0}f((z\exp(t), w\exp(-t))) = z\frac{\partial}{\partial z}f - w\frac{\partial}{\partial w}f$$

Next,

$$d\pi_l(E)f = \frac{d}{dt}|_{t=0}f((z,w)\exp(tE)) = \frac{d}{dt}|_{t=0}f((z,w)\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}) = \frac{d}{dt}|_{t=0}f((z, zt+w))) = z\frac{\partial}{\partial w}f$$

and

$$d\pi_l(F)f = \frac{d}{dt}|_{t=0}f((z,w)\exp(tE)) = \frac{d}{dt}|_{t=0}f((z,w)\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}) = \frac{d}{dt}|_{t=0}f((z+tw, w))) = w\frac{\partial}{\partial z}f$$

□

Proof of (ii).

$$d\pi_l(H)z^{l-k}w^k = (z\frac{\partial}{\partial z}z^{l-k}w^k - w\frac{\partial}{\partial w}z^{l-k}w^k) = (l-k)z^{l-k}w^k - kz^{l-k}w^k = (l-2k)z^{l-k}w^k$$

□

Proof of (iii). Let us fix any $k_1 \neq k_2$ such that $v := a_{k_1}z^{l-k_1}w^{k_1} + a_{k_2}z^{l-k_2}w^{k_2} + \sum_{k \neq k_1,k_2} a_k z^{l-k}w^k \in V$ such that $a_{k_1} \neq 0, a_{k_2} \neq 0$. It is enough to show that there are $\{c_k\}$ such that $z^{l-k_1}w^{k_1} + \sum_{k\neq k_1,k_2} c_k z^{l-k}w^k \in V$. From (i), there are $\{c_k\}$ such that

$$(l-2k_2)v - d\pi_l(H)v = (l-2k_2)a_{k_1}z^{l-k_1}w^{k_1} + \sum_{k \neq k_1,k_2} c_k z^{l-k}w^k \in V$$

By dividing with $(l-2k_2)a_{k_1}$ on the both sides, we can shot that there are $\{c'_k\}$ such that

$$z^{l-k_1}w^{k_1} + \sum_{k\neq k_1,k_2} c'_k z^{l-k}w^k \in V$$

□

Proof of (iv). It is clear. □

Proof of (v). Let us fix any invariant subspace $V \neq \{0\}$. Then V is $d\pi_l$ invariant. From (iii) and (iv), we get $V = V_l$. □

8.1.6 $SL(n,\mathbb{R}),\ SL(n,\mathbb{C})$

Example 8.1.12 ($SL(n,\mathbb{R}),\ SL(n,\mathbb{C})$)**.** *The followings hold.*

(i) By Proposition3.2.9,

$$Lie(SL(n,\mathbb{R})) = \{X \in M(n,\mathbb{R}) | tr(X) = 0\}$$

and

$$Lie(SL(n,\mathbb{C})) = \{X \in M(n,\mathbb{C}) | tr(X) = 0\}$$

So,

$$dim(SL(n,\mathbb{R})) = n^2 - 1$$

and

$$dim(SL(n,\mathbb{C})) = 2n^2 - 2$$

(ii) $SL(n,\mathbb{C})$ is a complex Lie group and $Lie(SL(n,\mathbb{R}))$ is a real form of $Lie(SL(n,\mathbb{C}))$.

(iii) $SL(n,\mathbb{R}), SL(n,\mathbb{C})$ are path connected.

(iv) $SL(n,\mathbb{R}), SL(n,\mathbb{C})$ are reductive Lie groups.

(v) $\mathfrak{sl}(2,\mathbb{R})$ is a simple Lie algebra.

(vi) $\mathfrak{sl}(2,\mathbb{C})$ is a simple Lie algebra.

Proof of (ii). It is clear. □

Proof of (iii). By the same argument as the proof of $SU(n)$, (iii) can be proven. □

Proof of (iv). It is followed from (iii) and ${}^t\bar{g} \in SL(n,F)$ $(\forall g \in SL(n,F))$ $(F = \mathbb{R}$ or $\mathbb{C})$. □

Proof of (v). We set

$$X := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Y_1 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y_2 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Then X, Y_1, Y_2 is the basis of $\mathfrak{sl}(2,\mathbb{R})$ and

$$[X, Y_1] = 2Y_1, [X, Y_2] = -2Y_1, [Y_1, Y_2] = X$$

Let us fix any $\mathfrak{h}$ which is a nonzero ideal of $\mathfrak{sl}(2,\mathbb{R})$ and $aX + b_1Y_1 + b_2Y_2 \in \mathfrak{h} \setminus \{0\}$. If $b_1 = b_2 = 0$, by multiplying it by $\frac{1}{2a}Y_1$ and $\frac{1}{-2a}Y_2$ respectively, we get $X, Y_1, Y_2 \in \mathfrak{h}$, that implies $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{R})$.

If $b_1 \neq 0$, by multiplying it by X, after that by multiplying it by Y_2, we get $X, Y_1, Y_2 \in \mathfrak{h}$. So, by the same argument as the above, we get $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{R})$.

If $b_2 \neq 0$, by the same argument as the above, we get $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{R})$. □

Proof of (vi). It can be shown by the same way as (v). □

8.1.7 General Linear Group $GL(n,\mathbb{R}), GL(n,\mathbb{C})$

Example 8.1.13 ($GL(n,\mathbb{R}),\ GL(n,\mathbb{C})$)**.** *The followings hold.*

(i)

$$\mathfrak{gl}(n,\mathbb{R}) := Lie(GL(n,\mathbb{R})) = M(n,\mathbb{R})$$

and

$$\mathfrak{gl}(n,\mathbb{C}) := Lie(GL(n,\mathbb{C})) = M(n,\mathbb{C})$$

and

$$dim(GL(n,\mathbb{R})) = n^2$$

and

$$dim(GL(n,\mathbb{C})) = 2n^2$$

(ii) $GL(n,\mathbb{C})$ is a complex Lie group and $Lie(GL(n,\mathbb{R}))$ is a real form of $Lie(GL(n,\mathbb{C}))$.

(iii) $GL(n,\mathbb{R}), GL(n,\mathbb{C})$ are path connected.

(iv) $GL(n,\mathbb{R}), GL(n,\mathbb{C})$ *are reductive Lie groups.*

(v)

$$\mathfrak{gl}(n,\mathbb{R}) = \mathbb{R}E \oplus \mathfrak{sl}(n,\mathbb{R})$$

$\mathfrak{gl}(n,\mathbb{R})$ *is a not simple Lie algebra.*

(vi)

$$\mathfrak{gl}(n,\mathbb{C}) = \mathbb{C}E \oplus \mathfrak{sl}(n,\mathbb{C})$$

$\mathfrak{gl}(n,\mathbb{C})$ *is a not simple Lie algebra.*

Proof of (i). It is clear. □

Proof of (ii). It is clear. □

Proof of (iii). By the same argument as the proof of $SU(n)$, (iii) can be proven. □

Proof of (iv). It is followed from (iii) and ${}^t\overline{g} \in GL(n,F)$ $(\forall g \in GL(n,F))$ $(F = \mathbb{R}$ or $\mathbb{C})$. □

Proof of (v). For any $A \in \mathfrak{gl}(n,\mathbb{R})$,

$$A = \frac{tr(A)}{n}E + (A - \frac{tr(A)}{n}E), \frac{tr(A)}{n}E \in \mathbb{R}E, (A - \frac{tr(A)}{n}E) \in \mathfrak{sl}(n,\mathbb{R})$$

Since $\mathbb{R}E$ is nonzero and is contained in the center of $\mathfrak{gl}(n,\mathbb{R})$, $\mathfrak{gl}(n,\mathbb{R})$ is not semisimple Lie algebra. □

8.1.8 Heisenberg group H_1

Example 8.1.14 (H_1)**.** *Here are the settings.*

(S1)

$$H_1 := \{\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} | a, b, c \in \mathbb{R}\}$$

We call H_1 *Heisenberg group.*

Then the followings hold.

(i)

$$Lie(H_1) = \{\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} | a, b, c \in \mathbb{R}\}$$

$Lie(H_1)$ *is nilpotent.*

Proof of (i). It is followed from the argument similar to Example8.1.2. □

8.1.9 Affine Group $A(1,\mathbb{R}) := ax + b$

Example 8.1.15 (A$(1,\mathbb{R})$)**.** *Here are the settings.*

(S1)

$$A(1,\mathbb{R}) := \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} | a, b \in \mathbb{R}, a > 0\}$$

We call $A(1,\mathbb{R})$ *one dimensional Affine group.*

Then the followings hold.

(ii)

$$Lie(A(1,\mathbb{R})) = \{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} | x, y \in \mathbb{R}\}$$

$Lie(A(1,\mathbb{R}))$ *is solvable and not nilpotent.*

Proof of (i). It is followed from the argument similar to Example8.1.2. □

Proof of (ii). It is clear that $Lie(A(1,\mathbb{R}))$ is not nilpotent. For any $x, y, z, w \in \mathbb{R}$, there is $a \in \mathbb{R}$,

$$[\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} z & w \\ 0 & 0 \end{pmatrix}] = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$$

That implies $Lie(A(1,\mathbb{R}))$ is solvable. □

8.2 Homogeneous Spaces

8.2.1 Unit Sphere $S^n \simeq SO(n+1)/SO(n)$

From Theorem6.1.3, the following holds.

Example 8.2.1. *The followings are settings and assumptions.*

(S1) $S^n := \{x \in \mathbb{R}^{n+1} | \ ||x||^2 = 1\}$.

(S2) For each $g \cdot SO(n+1)$ *and* $x \in S^n$,

$$g \cdot x := gx$$

Then

(i) $SO(n+1)$ *continuously acts on* S^n.

(ii)

$$\{g \in SO(n+1) | g \cdot e_{n+1} = e_{n+1}\} = H := \{\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} | h \in SO(n)\}$$

Hereafter, we see H *as* $SO(n)$.

(iii) S^n *is homeomorphism to* $SO(n+1)/SO(n)$.

8.2.2 Poincare Upper Half Plane $\mathbb{H} \simeq SL(2,\mathbb{R})/SO(2)$

In this section, we introduce Poincare metric on upper half plane $\mathbb{H}$ using Lie group and Representation theory. Poincare metric P is the metric that has the representation below with Euclidian coordinate (x, y)

$$\begin{pmatrix} P_{(x,y)}(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) & P_{(x,y)}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \\ P_{(x,y)}(\frac{\partial}{\partial y}, \frac{\partial}{\partial x}) & P_{(x,y)}(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) \end{pmatrix} = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$$

All geodesics of the Poincare Plane are semicircles whose center in the real axis. We show example of geodesics of the Poincare Plane in Figure8.1.

Poincare metric has many applications to complex analysis[16] and recently it is used for machine learning[29]. It is well known that any auto holomorphic map on Riemann sphere can be represented by a linear fractional transformation from $SL(2,\mathbb{C})$ ([16]). So, let us start linear fractional transformations from $SL(2,\mathbb{C})$.

In Proposition8.2.3, it is stated that $\mathbb{H}$ is a homogeneous space $SL(2,\mathbb{R})/SO(2)$ and Poincare metric is the "unique" $SL(2,\mathbb{R})$ invariant riemaniann metric. And the representation of Poincare metric with Euclidian coordinate is given in Proposition8.2.3.

Notation 8.2.2. *In this section, we set*

$$\mathbb{H} := \{x + iy | x, y \in \mathbb{R}, y > 0\}$$

and call $\mathbb{H}$ *upper half plane.*

Proposition 8.2.3. *The followings are settings and assumptions.*

(i) $G := SL(2,\mathbb{R})$ *and* $\mathfrak{g} := Lie(G)$.

(ii) $H := SO(2)$ *and* $\mathfrak{h} := Lie(H)$.

Then

(i) G *transitively acts on* $\mathbb{H}$ *by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d} \ (z \in \mathbb{H})$$

and the isotropy subgroup regarding i *is* H.

(ii)

$$\mathfrak{sl}(2,\mathbb{R})/\mathfrak{so}(2) = \{[\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}] | a, b \in \mathbb{R}\}$$

Hereafter, let us consider $(a, b)^t$ *as* $[\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}]$ *unless we specify otherwise.*

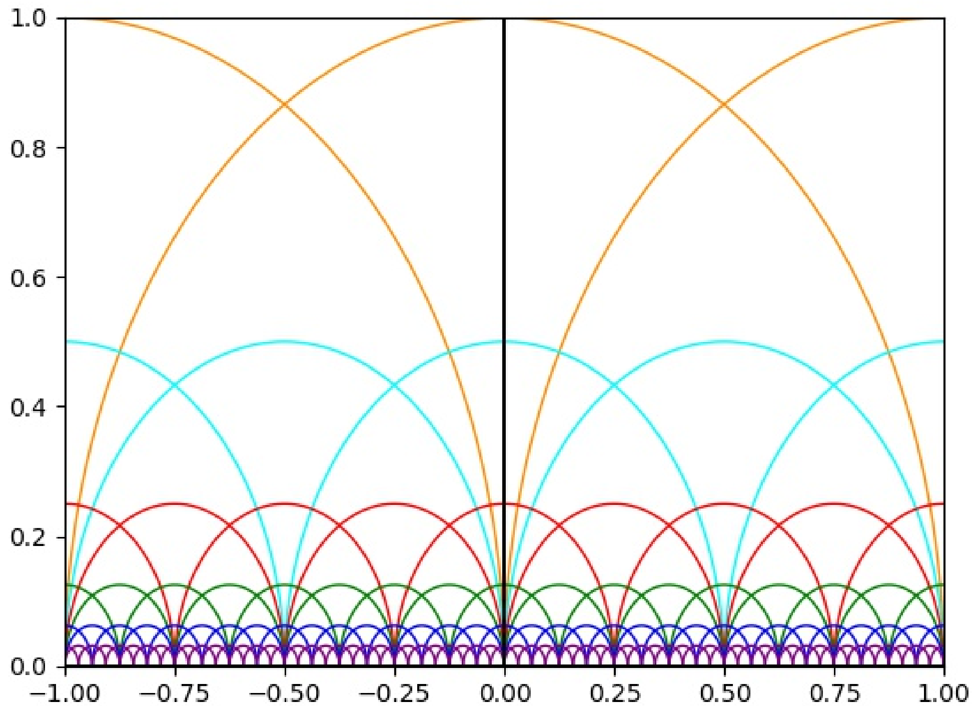

Figure 8.1: The Poincare Upper-Half Plane Model

(iii) *For* $h(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$*, the representation matrix of* $Ad_{\mathfrak{g}/\mathfrak{h}}(h(\theta))$ *is*

$$\begin{pmatrix} \cos(2\theta) & \frac{1}{2}\sin(2\theta) \\ 2\sin(2\theta) & \cos(2\theta) \end{pmatrix}$$

(iv)

$$\frac{1}{2}\int_0^{2\pi} \begin{pmatrix} \cos(2\theta) & -\frac{1}{2}\sin(2\theta) \\ 2\sin(2\theta) & \cos(2\theta) \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(2\theta) & \frac{1}{2}\sin(2\theta) \\ 2\sin(2\theta) & \cos(2\theta) \end{pmatrix} d\theta = \frac{5}{8}\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

(v)

$$\mathcal{P}_+(\mathfrak{g}/\mathfrak{h})^H = \mathbb{R}\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

(vi)

$$\exp(\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}) = \begin{pmatrix} \exp(a) & \frac{b}{2a}(\exp(a) - \exp(-a)) \\ 0 & \exp(-a) \end{pmatrix}$$

(vii) *We set*

$$\phi(a,b) := \exp(\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}) \cdot i \ (a, b \in \mathbb{R})$$

Then

$$\phi(a,b) := \frac{b}{2a}(\exp(2a) - 1) + \exp(2a)i \ (\forall a, b \in \mathbb{R})$$

(viii) *We set*

$$\psi(x+iy) := (\frac{1}{2}\log(y), x\frac{\log(y)}{y-1}) \ (\forall x+iy \in \mathbb{H})$$

Then ψ *is the inverse map of* ϕ*.*

(ix)

$$J\psi(x+iy) := \begin{pmatrix} 0 & \frac{1}{2y} \\ \frac{\log(y)}{y-1} & \frac{x}{(y-1)^2}(\frac{y-1}{y} - \log(y)) \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \frac{1}{2}\exp(-2a) \\ \frac{2a}{\exp(2a)-1} & \frac{b}{2a\exp(2a)}(\frac{\exp(-2a)+2a-1}{\exp(-2a)-1}) \end{pmatrix}$$
$$(\forall x+iy = \exp\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \in \mathbb{H})$$

(x) For any $a, b \in \mathbb{R}$,

$$\frac{\partial \phi}{\partial a}(a,b) = \frac{d}{dt}|_{t=0} \exp(\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}) \exp(t\begin{pmatrix} 1 & \frac{b}{2a^2}(\exp(-2a)+2a-1) \\ 0 & -1 \end{pmatrix}) \cdot i$$

and

$$\frac{\partial \phi}{\partial b}(a,b) = \frac{d}{dt}|_{t=0} \exp(\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}) \exp(t\begin{pmatrix} 0 & \frac{1}{2a}(1-\exp(-2a)) \\ 0 & 0 \end{pmatrix}) \cdot i$$

(xi) Let B denote the riemannian metric on $\mathbb{H}$ from $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \in \mathcal{P}_+(\mathfrak{g}/\mathfrak{h})^H$. For any $a, b \in \mathbb{R}$,

$$\begin{pmatrix} B_{\phi(a,b)}(\frac{\partial}{\partial a}, \frac{\partial}{\partial a}) & B_{\phi(a,b)}(\frac{\partial}{\partial a}, \frac{\partial}{\partial b}) \\ B_{\phi(a,b)}(\frac{\partial}{\partial b}, \frac{\partial}{\partial a}) & B_{\phi(a,b)}(\frac{\partial}{\partial b}, \frac{\partial}{\partial b}) \end{pmatrix}$$
$$= \begin{pmatrix} 4 + \frac{b^2}{4a^4}(\exp(-2a)+2a-1)^2 & -\frac{b}{4a^3}(\exp(-2a)-1)(\exp(-2a)+2a-1) \\ -\frac{b}{4a^3}(\exp(-2a)-1)(\exp(-2a)+2a-1) & \frac{1}{4a^3}(1-exp(-2a))^2 \end{pmatrix}$$

(xii) Let (x, y) denote the Euclidian coordinate on $\mathbb{H}$. Then For any $x + yi \in \mathbb{H}$,

$$\begin{pmatrix} B_{x+iy}(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) & B_{x+iy}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \\ B_{x+iy}(\frac{\partial}{\partial y}, \frac{\partial}{\partial x}) & B_{x+iy}(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) \end{pmatrix} = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$$

Proof of (i). Let us fix any $x + iy \in \mathbb{H}$. Let us solve the following equation.

$$\exp(\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}) = x + iy$$

Then

$$(a, b) = (\frac{1}{2}\log(y), x\frac{\log(y)}{y-1})$$

Therefore, the action is transitive.

Next, let us fix any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} i = i$$

Then

$$a = d, b = -c$$

This means $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SO(2)$. □

Proof of (ii). From the results in the previous section,

$$\mathfrak{sl}(2,\mathbb{R}) = \{\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} | a \in \mathbb{R}\}$$

and

$$\mathfrak{so}(2) = \{\begin{pmatrix} 0 & b \\ -b & -0 \end{pmatrix} | b \in \mathbb{R}\}$$

From that, (ii) holds. □

Proof of (iii). Let us fix $a, b, \theta \in \mathbb{R}$.

$$\begin{aligned} Ad_{\mathfrak{g}/\mathfrak{h}}(h(\theta)) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} &= \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \in \begin{pmatrix} \cos 2\theta & 2\sin 2\theta \\ 0 & -\cos 2\theta \end{pmatrix} + \mathfrak{so}(2) \end{aligned}$$

and

$$\begin{aligned} Ad_{\mathfrak{g}/\mathfrak{h}}(h(\theta)) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ = \begin{pmatrix} 0 & \cos\theta \\ 0 & \sin\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} &= \begin{pmatrix} -\cos\theta\sin 2\theta & \cos^2\theta \\ -\sin^2\theta & \cos\theta\sin\theta \end{pmatrix} \\ \in \begin{pmatrix} -\cos\theta\sin 2\theta & \cos^2\theta - \sin^2\theta \\ 0 & \cos\theta\sin\theta \end{pmatrix} + \mathfrak{so}(2) &= \begin{pmatrix} -\frac{1}{2}\sin 2\theta & \cos 2\theta \\ 0 & \frac{1}{2}\sin 2\theta \end{pmatrix} + \mathfrak{so}(2) \end{aligned}$$

Therefore, the representation matrix of $Ad_{\mathfrak{g}/\mathfrak{h}}(h(\theta))$ is

$$\begin{pmatrix} \cos 2\theta & -\frac{1}{2}\sin 2\theta \\ 2\sin 2\theta & \cos 2\theta \end{pmatrix}$$

□

Proof of (iv).

$$\begin{aligned} &\begin{pmatrix} \cos(2\theta) & -\frac{1}{2}\sin(2\theta) \\ 2\sin(2\theta) & \cos(2\theta) \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(2\theta) & -\frac{1}{2}\sin(2\theta) \\ 2\sin(2\theta) & \cos(2\theta) \end{pmatrix} \\ = &\begin{pmatrix} \cos(2\theta) & 2\sin(2\theta) \\ -\frac{1}{2}\sin(2\theta) & \cos(2\theta) \end{pmatrix} \begin{pmatrix} \cos(2\theta) & -\frac{1}{2}\sin(2\theta) \\ 2\sin(2\theta) & \cos(2\theta) \end{pmatrix} = \begin{pmatrix} \cos(2\theta) & 2\sin(2\theta) \\ -\frac{1}{2}\sin(2\theta) & \cos(2\theta) \end{pmatrix} \begin{pmatrix} \cos(2\theta) & -\frac{1}{2}\sin(2\theta) \\ 2\sin(2\theta) & \cos(2\theta) \end{pmatrix} \\ \in &\begin{pmatrix} 1 + 3\sin^2\theta & 0 \\ 0 & \frac{1}{4} + \frac{3}{4}\cos^2(2\theta) \end{pmatrix} + \mathbb{R}\cos 2\theta E_{12} + \mathbb{R}\sin 2\theta E_{12} + \mathbb{R}\cos 2\theta E_{21} + \mathbb{R}\sin 2\theta E_{21} \\ \in &\begin{pmatrix} 1 + 3\frac{1}{2} & 0 \\ 0 & \frac{1}{4} + \frac{3}{4}\frac{1}{2} \end{pmatrix} + \mathbb{R}\cos 2\theta E_{12} + \mathbb{R}\sin 2\theta E_{12} + \mathbb{R}\cos 2\theta E_{21} + \mathbb{R}\sin 2\theta E_{21} \end{aligned}$$

Therefore,

$$\frac{1}{2\pi}\int_0^{2\pi} \begin{pmatrix} \cos(2\theta) & -\frac{1}{2}\sin(2\theta) \\ 2\sin(2\theta) & \cos(2\theta) \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(2\theta) & -\frac{1}{2}\sin(2\theta) \\ 2\sin(2\theta) & \cos(2\theta) \end{pmatrix} d\theta = \frac{5}{8}\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

□

Proof of (v). Let us fix any $\begin{pmatrix} p & q \\ q & r \end{pmatrix} \in \mathcal{P}_+(\mathfrak{g}/\mathfrak{h})^H$.

$$\begin{aligned}
&\begin{pmatrix} \cos(2\theta) & -\frac{1}{2}\sin(2\theta) \\ 2\sin(2\theta) & \cos(2\theta) \end{pmatrix}^T \begin{pmatrix} p & q \\ q & r \end{pmatrix} \begin{pmatrix} \cos(2\theta) & -\frac{1}{2}\sin(2\theta) \\ 2\sin(2\theta) & \cos(2\theta) \end{pmatrix} \\
= &\begin{pmatrix} \cos(2\theta) & 2\sin(2\theta) \\ -\frac{1}{2}\sin(2\theta) & \cos(2\theta) \end{pmatrix} \begin{pmatrix} p & q \\ q & r \end{pmatrix} \begin{pmatrix} \cos(2\theta) & -\frac{1}{2}\sin(2\theta) \\ 2\sin(2\theta) & \cos(2\theta) \end{pmatrix} \\
= &\begin{pmatrix} \cos(2\theta)p + 2\sin(2\theta)q & \cos(2\theta)q + 2\sin(2\theta)r \\ -\frac{1}{2}\sin(2\theta)p + \cos 2\theta q & -\frac{1}{2}\sin(2\theta)q + \cos 2\theta r \end{pmatrix} \begin{pmatrix} \cos(2\theta) & -\frac{1}{2}\sin(2\theta) \\ 2\sin(2\theta) & \cos(2\theta) \end{pmatrix} \\
\in &\begin{pmatrix} \frac{1}{2}p + 2r & -\frac{1}{2}q \\ -\frac{1}{2}q & \frac{1}{8}p + \frac{1}{2}r \end{pmatrix} + \sum_{i \neq j} \mathbb{R} \cos 2\theta E_{ij} + \sum_{i \neq j} \mathbb{R} \sin 2\theta E_{ij}
\end{aligned}$$

So, $q = 0, p = 4r$. □

Proof of (vi).

$$\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}^{2n} = \begin{pmatrix} a^{2n} & 0 \\ 0 & a^{2n} \end{pmatrix} \ (\forall n \in \mathbb{N})$$

and

$$\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}^{2n-1} = \begin{pmatrix} a^{2n} & ba^{2(n-1)} \\ 0 & a^{2n} \end{pmatrix} \ (\forall n \in \mathbb{N})$$

Therefore (vi) holds. □

Proof ov (vii). From (vi), (vii) clearly follows. □

Proof of (ix). For any $x + iy = \exp \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \in \mathbb{H}$,

$$\begin{aligned}
J\psi(x+iy) : \ = &\begin{pmatrix} 0 & \frac{1}{2y} \\ \frac{\log(y)}{y-1} & \frac{x}{(y-1)^2}(\frac{y-1}{y} - \log(y)) \end{pmatrix} \\
= &\begin{pmatrix} 0 & \frac{1}{2}\exp(-2a) \\ \frac{2a}{\exp(2a)-1} & \frac{b(\exp(2a)-1)}{2a} \frac{1}{(\exp(2a)-1)^2}(\frac{\exp(2a)-1}{\exp(2a)} - 2a) \end{pmatrix} \\
= &\begin{pmatrix} 0 & \frac{1}{2}\exp(-2a) \\ \frac{2a}{\exp(2a)-1} & \frac{b}{2a}(\frac{1}{\exp(2a)} - \frac{2a}{\exp(2a)-1}) \end{pmatrix} \\
= &\begin{pmatrix} 0 & \frac{1}{2}\exp(-2a) \\ \frac{2a}{\exp(2a)-1} & \frac{b}{2a\exp(2a)}(1 - \frac{2a}{1-\exp(-2a)}) \end{pmatrix} \\
= &\begin{pmatrix} 0 & \frac{1}{2}\exp(-2a) \\ \frac{2a}{\exp(2a)-1} & \frac{b}{2a\exp(2a)}(\frac{\exp(-2a)+2a-1}{\exp(-2a)-1}) \end{pmatrix}
\end{aligned}$$

□

Calculation $\frac{\partial \phi}{\partial a}(a,b)$ in (x). Let us assume $x, y \in \mathbb{R}$ satisfies

$$\frac{\partial \phi}{\partial a}(a,b) = \frac{d}{dt}|_{t=0} \exp(\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}) \exp(t\begin{pmatrix} x & y \\ 0 & -x \end{pmatrix}) \cdot i$$

Then clearly $x = 1$. And

$$\exp(\begin{pmatrix} a+t & b \\ 0 & -a-t \end{pmatrix}) = \begin{pmatrix} \exp(a+t) & \frac{b}{2(a+t)}(\exp(a+t) - \exp(-(a+t))) \\ 0 & -\exp(a+t) \end{pmatrix}$$

and

$$\exp(\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}) \exp(t\begin{pmatrix} 1 & y \\ 0 & -1 \end{pmatrix}) = \begin{pmatrix} \exp(a) & \frac{b}{2a}(\exp(a) - \exp(-(a))) \\ 0 & -\exp(a) \end{pmatrix} \begin{pmatrix} \exp(t) & \frac{ty}{2t}(\exp(t) - \exp(-(t))) \\ 0 & \exp(-t) \end{pmatrix}$$

$$= \begin{pmatrix} \exp(a+t) & \{\exp(a)\frac{y}{2}(exp(t) - exp(-t)) + \frac{b}{2a}(\exp(a) - \exp(-(a)))exp(-t)\} \\ 0 & -\exp(a+t) \end{pmatrix}$$

Then

$$\begin{aligned} & \frac{d}{dt}_{t=0} \frac{b}{2(a+t)}(\exp(a+t) - \exp(-(a+t))) \\ = \ & \frac{b}{2}[\frac{-1}{(a+t)^2}(\exp(a+t) - \exp(-(a+t))) + \frac{1}{(a+t)}(\exp(a+t) + \exp(-(a+t)))]_{t=0} \\ = \ & \frac{b}{2a}\{\frac{-1}{a}(\exp(a) - \exp(-a)) + (\exp(a) + \exp(-a))\} \end{aligned}$$

and

$$\begin{aligned} & \frac{d}{dt}_{t=0}\{\exp(a)\frac{y}{2}(exp(t) - exp(-t)) + \frac{b}{2a}(\exp(a) - \exp(-(a)))exp(-t)\} \\ = \ & \exp(a)y - \frac{b}{2a}(\exp(a) - \exp(-a)) \end{aligned}$$

Therefore,

$$\begin{aligned} y &= \exp(-a)\{\frac{b}{2a}(\exp(a) - \exp(-a)) + \frac{b}{2a}\{\frac{-1}{a}(\exp(a) - \exp(-a)) + (\exp(a) + \exp(-a))\}\} \\ &= \exp(-a)\{\frac{b}{2a}\{\frac{-1}{a}(\exp(a) - \exp(-a)) + 2\exp(a)\}\} \\ &= \frac{b}{2a}\{\frac{1}{a}(\exp(-2a) - 1) + 2\} = \frac{b}{2a^2}(\exp(-2a) + 2a - 1) \end{aligned}$$

□

Calculation $\frac{\partial \phi}{\partial b}(a,b)$ *in (x).* Let us assume $x, y \in \mathbb{R}$ satisfies

$$\frac{\partial \phi}{\partial b}(a,b) = \frac{d}{dt}|_{t=0} \exp(\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}) \exp(t\begin{pmatrix} x & y \\ 0 & -x \end{pmatrix}) \cdot i$$

Then clearly $x = 0$.

$$\exp(\begin{pmatrix} a & b+t \\ 0 & -a \end{pmatrix}) = \begin{pmatrix} \exp(a) & \frac{b+t}{2a}(\exp(a) - \exp(-a)) \\ 0 & -\exp(a) \end{pmatrix}$$

and

$$\exp(\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}) \exp(t\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} \exp(a) & \frac{b}{2a}(\exp(a) - \exp(-(a))) \\ 0 & -\exp(a) \end{pmatrix} \begin{pmatrix} 1 & ty \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \exp(a) & t\exp(a)y + \frac{b}{2a}(\exp(a) - \exp(-(a))) \\ 0 & -\exp(a) \end{pmatrix}$$

So,

$$\begin{aligned} y &= \exp(-a)(\frac{1}{2a}(\exp(a) - \exp(-a))) \\ &= \frac{1}{2a}(1 - \exp(-2a)) \end{aligned}$$

□

Proof of (xi). We set

$$A := \frac{b}{2a^2}(\exp(-2a)+2a-1), B := \frac{1}{2a}(1-\exp(-2a))$$

$$\begin{pmatrix}1 & A\\ 0 & B\end{pmatrix}\begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ A & B\end{pmatrix} = \begin{pmatrix}4 & A\\ 0 & B\end{pmatrix}\begin{pmatrix}1 & 0\\ A & B\end{pmatrix} = \begin{pmatrix}4+A^2 & AB\\ AB & B^2\end{pmatrix}$$

$$= \begin{pmatrix} 4+\frac{b^2}{4a^4}(\exp(-2a)+2a-1)^2 & -\frac{b}{4a^3}(\exp(-2a)-1)(\exp(-2a)+2a-1) \\ -\frac{b}{4a^3}(\exp(-2a)-1)(\exp(-2a)+2a-1) & \frac{1}{4a^2}(1-exp(-2a))^2 \end{pmatrix}$$

□

Proof of (xii). We set

$$J\psi = \begin{pmatrix}0 & r\\ q & s\end{pmatrix}, \begin{pmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix} = \begin{pmatrix} 4+\frac{b^2}{4a^4}(\exp(-2a)+2a-1)^2 & -\frac{b}{4a^3}(\exp(-2a)-1)(\exp(-2a)+2a-1) \\ -\frac{b}{4a^3}(\exp(-2a)-1)(\exp(-2a)+2a-1) & \frac{1}{4a^2}(1-exp(-2a))^2 \end{pmatrix}$$

Then

$$\begin{pmatrix}0 & r\\ q & s\end{pmatrix}\begin{pmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}\begin{pmatrix}0 & q\\ r & s\end{pmatrix} = \begin{pmatrix} rg_{21} & rg_{22}\\ qg_{11}+sg_{21} & qg_{12}+sg_{22}\end{pmatrix}\begin{pmatrix}0 & q\\ r & s\end{pmatrix} = \begin{pmatrix} r^2g_{22} & r(qg_{12}+sg_{22})\\ r(qg_{12}+sg_{22}) & q^2g_{11}+2qsg_{12}+s^2g_{22}\end{pmatrix}$$

Let us remind

$$J\psi := \begin{pmatrix} 0 & \frac{1}{2}\exp(-2a) \\ \frac{2a}{\exp(2a)-1} & \frac{b}{2a\exp(2a)}(\frac{\exp(-2a)+2a-1}{\exp(-2a)-1}) \end{pmatrix}$$

First, we will show $qg_{12}+sg_{22}=0$.

$$qg_{12} = -\frac{b}{8a^3}\exp(-2a)(\exp(-2a)-1)(\exp(-2a)+2a-1))$$

and

$$sg_{22} = \frac{b}{8a^3}\exp(-2a)(\exp(-2a)-1)(\exp(-2a)+2a-1))$$

So,

$$r(qg_{12}+sg_{22}) = 0$$

Next, we will show $r^2g_{22} = \frac{1}{4y^2}$.

Remind

$$a = \frac{1}{2}log(y)$$

So,

$$r^2g_{22} = \frac{4a^2}{\exp(4a)(1-\exp(-2a)^2}\frac{(1-\exp(-2a))}{4a^2} = \frac{1}{y^2}$$

Last, we will show $q^2g_{11}+2qsg_{12}+s^2g_{22} = \frac{1}{y^2}$.

$$q^2g_{11} = \frac{1}{4\exp(4a)}(4+\frac{b^2}{4a^4}(\exp(-2a)+2a-1)^2) = \frac{1}{y^2}+\frac{b^2}{16a^4\exp(4a)}(\exp(-2a)+2a-1)^2$$

and

$$\begin{aligned} 2qsg_{12} &= \frac{b(\exp(-2a)+2a-1)}{2a\exp(4a)(\exp(-2a)-1)}(-1)\frac{b}{4a^3}(\exp(-2a)-1)(\exp(-2a)+2a-1) \\ &= -\frac{2b^2}{16a^4\exp(4a)}(\exp(-2a)+2a-1)^2 \end{aligned}$$

and

$$s^2 g_{22} = \frac{b^2}{4a^2\exp(4a)}\frac{\exp(-2a)+2a-1}{\exp(-2a)-1}\frac{1}{4a^2}(1-exp(-2a))^2 = \frac{b^2}{16a^4\exp(4a)}(\exp(-2a)+2a-1)(exp(-2a)-1)$$

So, $q^2 g_{11} + 2qs g_{12} + s^2 g_{22} = \dfrac{1}{y^2}$. □

8.2.3 Advanced topics⋆

Currently, there are some researches on the problem to clarify the space of all invariant Einstein metrics on the given reductive homogeneous space, like [31], [30] and others. Einstein metric is a kind of riemannian metric. Based on the introduction of [31], I will summarize the results on the topic and open problems. First, it was shown in [32] that if G/H is a reductive homogeneous space and furthermore is an an isotropy irreducible homogeneous space then G/H has a unique G-invariant metric, which is also Einstein, up to scalar. Remark that in any isotropy irreducible homogeneous space the isotropy representation is irreducible. It is known that any irreducible symmetric space is an isotropy irreducible homogeneous space. And, in [33] it is proved that a non compact irreducible homogeneous space is symmetric. There is a conjecture stating that the space is finite if the isotropy representation of G/H consists of pairwise inequivalent irreducible components[34].

Chapter 9

Weyl Unitary Trick

Theorem 9.0.1 (Weyl Unitary Trick(Simply connected case)). *The followings are settings and assumptions.*

(S1) $G_{\mathbb{C}}$ *is a complex Lie group.*

(S2) G *is a Lie group which is a real form of* $G_{\mathbb{C}}$.

(A1) $G_{\mathbb{C}}$ *is simply connected.*

(S3) $HR(G_{\mathbb{C}})_f$ *is a set of equivalent classes of finite dimensional continuous represenation of* $G_{\mathbb{C}}$ *which is holomorphic.*

(S4) $R(G)_f$ *is a set of equivalent classes of finite dimensional continuous represenation of* G.

(S5) $R(Lie(G_{\mathbb{C}}))_f$ *is a set of equivalent classes of finite dimensional continuous represenation of* $Lie(G_{\mathbb{C}})$.

(S6) $R(Lie(G))_f$ *is a set of equivalent classes of finite dimensional continuous represenation of* $Lie(G)$.

Then

(i) The followings are all bijective.

$$\begin{aligned}
&\Phi_1 : R(G_{\mathbb{C}})_f \ni [(\pi, V)] \mapsto [(\pi|G, V)] \in R(G)_f \\
&\Phi_2 : R(G)_f \ni [(\pi, V)] \mapsto [(d\pi, V)] \in R(Lie(G))_f \\
&\Phi_3 : R(Lie(G))_f \ni [(\rho, V)] \mapsto [(\rho \otimes \mathbb{C}, V)] \in R(Lie(G_{\mathbb{C}}))_f \\
&\Phi_4 : R(Lie(G_{\mathbb{C}}))_f \ni [(\rho, V)] \mapsto [(L(\rho), V)] \in R(G_{\mathbb{C}})_f
\end{aligned}$$

Here, $L(\rho)$ *is the lifting of* ρ*(See Thereom3.14.8).*

(ii) Each bijection of (i) preserves irreducibility.

(iii) Each bijection of (i) preserves subrepresentations.

STEP1.1 Proof of that Φ_1 *is surjective in (i).* We set $\Psi_1 := \Phi_4 \circ \Phi_3 \circ \Phi_2$. First, we will show

$$\Psi_1 \circ \Phi_1 = id_{R(G_{\mathbb{C}})_f}$$

Let us fix any $(\pi, V) \in R(G_{\mathbb{C}})_f$ and $(\pi', V) := \Psi_1 \circ \Phi_1((\pi, V))$. We set $\rho := d(\pi|G)$. For any $X \in Lie(G)$ and $v \in V$,

$$\pi'(exp(X))v = exp(\rho_{\mathbb{C}}(X))v = exp(\rho(X))v = exp(d(\pi|G)(X))v = \pi(exp(X))v$$

That implies $d\pi'|Lie(G) = d\pi|Lie(G)$. Since $Lie(G_{\mathbb{C}}) = Lie(G) \otimes \mathbb{C}$, $d\pi'|Lie(G_{\mathbb{C}}) = d\pi|Lie(G_{\mathbb{C}})$. From Corollary3.14.10, $\pi' = \pi$. So, $\Psi_1 \circ \Phi_1 = id_{R(G_{\mathbb{C}})_f}$ holds. □

STEP1.2 in Proof of (i): Φ_1 *is injective.* Next, we will show

$$\Phi_1 \circ \Psi_1 = id_{R(G)_f}$$

Let us fix any $(\pi, V) \in R(G)_f$ and $(\pi', V) := \Phi_1 \circ \Psi_1((\pi, V))$. We set $\rho := d\pi$. For any $X \in Lie(G)$ and $v \in V$,

$$\pi'(exp(X))v = (L(\rho_{\mathbb{C}})|G)(exp(X))v = L(\rho_{\mathbb{C}})(exp(X))v = exp(\rho_{\mathbb{C}}(X))v = exp(\rho(X))v = exp(d\pi(X))v = \pi(exp(X))v$$

Since G is connected, from Proposition3.4.6,

$$\pi' = \pi$$

□

STEP2.1 in Proof of (i): Φ_2 is surjective. We set $\Psi_2 := \Phi_1 \circ \Phi_4 \circ \Phi_3$. We will show

$$\Psi_2 \circ \Phi_2 = id_{R(G)_f}$$

Since $\Psi_2 \circ \Phi_2 = \Phi_1 \circ \Psi_1$, from STEP1.2, $\Psi_2 \circ \Phi_2 = id_{R(G)_f}$. □

STEP2.2 in Proof of (i): Φ_2 is injective. We will show

$$\Phi_2 \circ \Psi_2 = id_{R(Lie(G))_f}$$

Let us fix any $\rho \in R(Lie(G))_f$ and $v \in V$. And we set $\rho' := \Phi_2 \circ \Psi_2(\rho)$. Then

$$\rho(X)v = \rho_{\mathbb{C}}(X)v = \frac{d}{dt}|_{t=0} L(\rho_{\mathbb{C}})(Exp(tX))v = \frac{d}{dt}|_{t=0} L(\rho_{\mathbb{C}}|G)(Exp(tX))v = dL(\rho_{\mathbb{C}}|G)(X)v = \rho'(X)v$$

That means $\rho = \rho'$. □

STEP3.1 in Proof of (i): Φ_3 is surjective. We set $\Psi_3 := \Phi_2 \circ \Phi_1 \circ \Phi_4$. We will show

$$\Psi_3 \circ \Phi_3 = id_{R(Lie(G))_f}$$

Since $\Psi_3 \circ \Phi_3 = \Phi_2 \circ \Psi_2$, from STEP2.2, $\Psi_2 \circ \Phi_2 = id_{R(G)_f}$. □

STEP3.2 in Proof of (i): Φ_3 is injective. We will show

$$\Phi_3 \circ \Psi_3 = id_{R(Lie(G_{\mathbb{C}}))_f}$$

Let us fix any $\rho_{\mathbb{C}} \in R(Lie(G_{\mathbb{C}}))_f$ and $X \in Lie(G)$ and $v \in V$. And we set $\rho'_{\mathbb{C}} := \Phi_2 \circ \Psi_2(\rho_{\mathbb{C}})$. Then

$$\rho_{\mathbb{C}}(X)v = \frac{d}{dt}|_{t=0} L(\rho_{\mathbb{C}})(Exp(tX))v = \frac{d}{dt}|_{t=0} L(\rho_{\mathbb{C}}|G)(Exp(tX))v = dL(\rho_{\mathbb{C}}|G)(X)v = (\rho'_{\mathbb{C}}|Lie(G))(X)v$$

That menas $\rho_{\mathbb{C}}|Lie(G) = \rho'_{\mathbb{C}}|Lie(G)$. That implies $\rho_{\mathbb{C}} = \rho'_{\mathbb{C}}$. □

STEP4.1 in Proof of (i): Φ_4 is surjective. We set $\Psi_4 := \Phi_3 \circ \Phi_2 \circ \Phi_1$. We will show

$$\Psi_4 \circ \Phi_4 = id_{R(Lie(G_{\mathbb{C}}))_f}$$

Since $\Psi_4 \circ \Phi_4 = \Phi_3 \circ \Psi_3$, from STEP3.2, $\Psi_4 \circ \Phi_4 = id_{R(Lie(G_{\mathbb{C}}))_f}$. □

STEP3.2 in Proof of (i): Φ_4 is injective. We will show

$$\Phi_4 \circ \Psi_4 = id_{R(G_{\mathbb{C}})_f}$$

Since $\Phi_4 \circ \Psi_4 = \Psi_1 \circ \Phi_1$, from STEP1.1, $\Phi_4 \circ \Psi_4 = id_{R(G_{\mathbb{C}})_f}$. □

STEP1 in Proof of (ii): Φ_1 preserves irreducibility. Let $(\pi, V) \in \hat{G_{\mathbb{C}}}_f$. Assume $\pi' := \pi|G$ is not irreducible. Then there is nonzero invariant subspace $V' \subsetneq V$. Since V is closed and

$$d\pi(X)v = \lim_{t \to 0} \frac{d}{dt}(\pi(exp(tX))v - v) \ (\forall v \in V)$$

V is $d\pi$-invarinat. Therefore, V is $d\pi \otimes \mathbb{C}$-invarinat. Since V is closed and

$$L(d\pi \otimes \mathbb{C})(exp(X))v = exp(d\pi \otimes \mathbb{C}(X))v$$

V is $L(d\pi \otimes \mathbb{C})$-invariant. Since $L(d\pi \otimes \mathbb{C}) = \pi$, π is not irreducible. That is a contradiction. Consequently, Φ_1 preserves irreducibility. □

STEP2 in Proof of (ii): Φ_2, Φ_3, Φ_4 preserves irreducibility. It can be shown by the same argument as the one in STEP1. □

Proof of (iii). (iii) is clear from the definitions of $\Phi_1, \Phi_2, \Phi_3, \Phi_4$. □

Theorem 9.0.2 (Weyl Unitary Trick(General Case))**.** *The followings are settings and assumptions.*

(S1) $G_{\mathbb{C}}$ is a complex Lie group.

(S2) *G is a Lie group which is a real form of $G_{\mathbb{C}}$.*

(A1) *$G_{\mathbb{C}}/G$ is simply connected.*

(S3) *$HR(G_{\mathbb{C}})_f$ is a set of equivalent classes of finite dimensional continuous represenation of $G_{\mathbb{C}}$ which is holomorphic.*

(S4) *$R(G)_f$ is a set of equivalent classes of finite dimensional continuous represenation of G.*

Then the following is a bijective.

(i)

$$\Phi : HR(G_{\mathbb{C}})_f \ni [(\pi, V)] \mapsto [(\pi|G, V)] \in R(G)_f$$

(ii) Φ and Φ^{-1} preserve irreducibility.

Proof of (i). Let us fix any $[(\tau_1, V)] \in R(G)_f$. From Proposition3.14.11, the universal covering group of $G_{\mathbb{C}}$, denoted by $\tilde{G}_{\mathbb{C}}$, is also a complex Lie group. Let φ denote the covering homomorphism map from $\tilde{G}_{\mathbb{C}}$ to $G_{\mathbb{C}}$. We set $\tilde{Z} := ker(\varphi)$. We set $\mathfrak{g} := Lie(G)$. We can see $\mathfrak{g} \subset Lie(G_{\mathbb{C}})$. Let $\tilde{G}$ denote the analytic subgroup of $\tilde{G}_{\mathbb{C}}$ with the Lie algebra $\mathfrak{g}$. From Proposition6.2.6, $\tilde{Z} \subset \tilde{G}$.

Then $\tau_1 \circ \varphi \in R(\tilde{G})_f$. Since $Lie(G) = Lie(\tilde{G})$, $\tilde{G}$ is a real form of $\tilde{G}_{\mathbb{C}}$. From Weyl unitary trick for simply connected complex Lie group, there is $\rho \in R(\tilde{G}_{\mathbb{C}})_f$ such that $\rho|\tilde{G} = \tau_1 \circ \varphi$.

We set

$$\tilde{\rho}(\tilde{g}\tilde{Z}) := \rho(\tilde{g}) \ (\tilde{g} \in \tilde{G}_{\mathbb{C}})$$

Since $\rho|\tilde{G} = \tau_1 \circ \varphi$ and $\tilde{Z} \subset \tilde{G}$, $\rho|\tilde{Z} = id_{\tilde{Z}}$. Then $\tilde{\rho}$ is well-defined. And clearly $\tilde{\rho}$ is continuous homomorphism.

Let α denote the homomorphism from $\tilde{G}_{\mathbb{C}}/\tilde{Z}$ to $G_{\mathbb{C}}$. We set

$$\tau_2(g) := \tilde{\rho}(\alpha^{-1}(g)) \ (g \in G_{\mathbb{C}})$$

Let us fix any $\tilde{g} \in \tilde{G}_{\mathbb{C}}$.

$$\tau_2 \circ \varphi(\tilde{g}) = \tilde{\rho}(\alpha^{-1}(\varphi([c]))) = \tilde{\rho}([c]\tilde{Z}) = \rho(\tilde{g})$$

Therefore, we get $\tau_2 \circ \varphi = \rho$. Let us fix any $g \in G$. Then thre is $\tilde{g} \in \tilde{G}$ such that $\varphi(\tilde{g}) = g$. Then,

$$\tau_2(g) = \tau_2(\varphi(\tilde{g})) = \rho(\tilde{g}) = \tau_1 \circ \varphi(\tilde{g}) = \tau_1(g)$$

Therefore, we get $\tau_2|G = \tau_1$.

Next we will show τ_2 is holomorphic. Let us fix any $g \in G_{\mathbb{C}}$. Then there is $\tilde{g} \in \tilde{G}_{\mathbb{C}}$ and an open neighborhood of $\tilde{g}$, denoted by $\tilde{U}$, such that $\varphi(\tilde{g}) = g$ and $\varphi|\tilde{U}$ is a biholomorphisim to an open neighborhood of g, denoted by U. Then, for any $x \in U$, $\tau_2(x) = \rho((\varphi|\tilde{U})^{-1}(x))$. Since $(\varphi|\tilde{U})^{-1}$ and ρ are holomorphic, τ_2 is holomorphic.

Let Ψ denote the map which maps τ_1 to τ_2. From the above discussion, $\Phi \circ \Psi = id_{R(G)_f}$. Next, we will show

$$\Psi \circ \Phi = id_{HR(G_{\mathbb{C}})_f}$$

Let us fix any $(\tau_2, V) \in HR(G_{\mathbb{C}})_f$. We set $\tau_1 := \tau_2|G$. $\tau_2 \circ \varphi$ is the extension of $\tau_1 \circ \varphi$. Therefore, from the definition of $\Psi(\tau_1)$, $\tau_2 = \Psi(\tau_1)$. Consequently, Φ is a bijective. □

Proof of (ii). We take over the notations in the proof of (i). It is clear that if $\tau_1 \in R(G)_f$ is irreducible then $\Psi(\tau_1)$ is irreducible. Let us assume any $\tau_2 \in HR(G_{\mathbb{C}})_f$ which is irreducible. We set $\tau_1 := \tau_2|G$. Then $\tau_2 \circ \varphi$ is clearly irreducible. From Weyl unitary trick for simply connected Lie group, $\tau_1 \circ \varphi$ is irreducible. Since φ is surjective, τ_1 is irreducible. □

Chapter 10

Constructing Irreducible Representations of Compact Lie group

10.1 Main Theorem

Notation 10.1.1. *In this chapter, we use these notations.*

$$G := U(n).$$
$$G_{\mathbb{C}} := GL(n, \mathbb{C}).$$

Notation 10.1.2. *The followings are settings.*

$$\lambda \in \mathbb{Z}^n.$$

In this chapter, we use these notations.

$$\chi_\lambda(\begin{pmatrix} t_1 & 0 & ... & 0 \\ * & t_2 & ... & 0 \\ ... & ... & ... & 0 \\ * & * & ... & t_n \end{pmatrix}) := t_1^{\lambda_1}...t_n^{\lambda_n} \begin{pmatrix} t_1 & 0 & ... & 0 \\ * & t_2 & ... & 0 \\ ... & ... & ... & 0 \\ * & * & ... & t_n \end{pmatrix} \in B_-.$$

$$\mathbb{C}_\lambda := \mathbb{C}.$$

$$\mathcal{L}_\lambda := G \times_H \mathbb{C}_\lambda$$

$$q_\lambda : G_{\mathbb{C}} \times_{B_-} \mathbb{C}_\lambda \ni [g, v] \mapsto gB_- \in G_{\mathbb{C}}/B_-.$$

Proposition 10.1.3 (Borel Subgroup)**.** *We call the set of all lower triagonal matrices of $G_{\mathbb{C}}$ the Borel subgroup of $G_{\mathbb{C}}$. Let denote B_- the Borel subgroup of $G_{\mathbb{C}}$. B_- is a closed subgroup of $G_{\mathbb{C}}$.*

Proposition 10.1.4. *G/B_- is a flag manifold.*

Theorem 10.1.5 (Homogeneous holomorphic vector bundle)**.** *The followings are settings are assumptions.*

(S1) G is a complex Lie group.

(S2) H is a closed subgroup which is a complex Lie group.

(S3) (π, V) is a holomorphic representation of H.

(S4) G/H is a complex manifold.

Then

$$q : G \times_H V \to G/H$$

is a holomorphic vector bundle. We call it homogeneous holomorphic vector bundle.

Proposition 10.1.6. *For any $\lambda \in \mathbb{Z}^n$, $q_\lambda : G_{\mathbb{C}} \times_{B_-} \mathbb{C}_\lambda \to G_{\mathbb{C}}/B_-$ is a homogeneous holomorphic line bundle.*

Definition 10.1.7. *For each* $\lambda \in \mathbb{Z}^n$,

$$\mathcal{O}(\mathcal{L}_\lambda) := \{s \in \Gamma(G_\mathbb{C}/B_-, \mathcal{L}_\lambda) | s \text{ is holomorphic}\}$$

From Theorem6.6.4,

$$\mathcal{O}(\mathcal{L}_\lambda) \simeq \{f \in Hol(G_\mathbb{C}, \mathbb{C}) | \chi_\lambda(b) F(gb) = F(g) \ (\forall g \in G_\mathbb{C}, \forall b \in B_-)\}$$

as purely algebraic mean.

Theorem 10.1.8 (Borel-Weil Theorem)**.** *For each* $\lambda \in \mathbb{Z}^n$ *such that* $\lambda_1 \geq ... \geq \lambda_n$,

(i) $dim\mathcal{O}(\mathcal{L}_\lambda) < \infty$.

(ii) $\mathcal{O}(\mathcal{L}_\lambda)$ *is a finite dimensional irreducible continuous representation of* G.

(iii) *The highest weight of* $\mathcal{O}(\mathcal{L}_\lambda)$ *is* λ.

10.2 Flag manifold

Notation 10.2.1. *In this chapter, we use these notations.*

$\mathfrak{g} := u(n) := Lie(G)$.

$\mathfrak{g}_\mathbb{C} := Lie(G_\mathbb{C})$.

$\mathfrak{t} := Lie(T)$. *Here, let us remind* $T := \{diag(t_1, ..., t_n) | t_i \in S^1 \ (\forall i)\}$

$\mathfrak{t}_\mathbb{C} := \{X \in M(n, \mathbb{C}) | X_{i,j} = 0 \ (\forall i \neq \forall j)\}$.

$\mathfrak{a} := \{X \in M(n, \mathbb{C}) | X_{i,j} = 0 \ (\forall i \neq \forall j), \ X_{i,i} \in \mathbb{R} \ (\forall i)\}$

$\mathfrak{n}_- := \{X \in M(n, \mathbb{C}) | X_{i,j} = 0 \ (\forall i \leq \forall j)\}$.

$\mathfrak{n}_+ := \{X \in M(n, \mathbb{C}) | X_{i,j} = 0 \ (\forall i \geq \forall j)\}$.

$\mathfrak{b} := \mathfrak{t}_\mathbb{C} + \mathfrak{n}_-$.

$$A := \{diag(a_1, .., a_n) = \begin{pmatrix} a_1 & 0 & ... & 0 \\ 0 & a_2 & ... & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & a_n \end{pmatrix} | a_1, ..., a_n > 0\}.$$

$$N_- := \{\begin{pmatrix} 1 & 0 & ... & 0 \\ x_{21} & 1 & ... & 0 \\ ... & ... & ... & 0 \\ x_{n1} & x_{n2} & ... & 1 \end{pmatrix} | x_{ij} \in \mathbb{R} \ (i > j)\}.$$

$$N_+ := \{\begin{pmatrix} 1 & x_{12} & ... & x_{1n} \\ 0 & 0 & ... & x_{2n} \\ ... & ... & ... & x_{(n-1)n} \\ 0 & 0 & ... & 1 \end{pmatrix} | x_{ij} \in \mathbb{R} \ (i < j)\}.$$

Proposition 10.2.2 (Borel Subgroup)**.** *We call the set of all lower triagonal matrices of* $G_\mathbb{C}$ *the Borel subgroup of* $G_\mathbb{C}$. *Let denote* B_- *the Borel subgroup of* $G_\mathbb{C}$. B_- *is a closed subgroup of* $G_\mathbb{C}$.

Proof. Let $B_-(n)$ denote the borel subgroup with order n and $G_\mathbb{C}(n) := GL(n, \mathbb{C})$. Clearly B_- is closed and $B_- \cdot B_- \subset G_\mathbb{C}$. We will show $(B_-(n))^{-1} \subset G_\mathbb{C}(n)$ by mathematical induction with order n. Clearly $(B_-(1))^{-1} \subset G_\mathbb{C}(1)$.

Let us assume $(B_-(n))^{-1} \subset G_\mathbb{C}(n)$. Let us fix any $g := \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in B_-(n+1)$. Here, $a \in \mathbb{C} \setminus \{0\}$, $b \in M(n, 1, \mathbb{C})$, $c \in GL(n, \mathbb{C})$. Then clearly

$$\begin{pmatrix} a^{-1} & 0 \\ -a^{-1}c^{-1}b & c^{-1} \end{pmatrix}$$

is the inverse matrix of g and is in $B_-(n)$. So, $(B_-(n+1))^{-1} \subset G_\mathbb{C}(n+1)$. □

Proposition 10.2.3. *The followings hold.*

(i)

$$\mathfrak{g}_\mathbb{C} = \mathfrak{n}_- + \mathfrak{t}_\mathbb{C} + \mathfrak{n}_+$$

(ii)

$$\mathfrak{b}_- = \mathfrak{t}_\mathbb{C} + \mathfrak{n}_- = \mathfrak{t} + \mathfrak{a} + \mathfrak{n}_-$$

(iii)
$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + \mathfrak{b}_{-}$$

(iv)
$$\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{t}_{\mathbb{C}}$$

(v)
$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{q} \oplus \mathfrak{b}$$

Proof. (i) is clear. Since $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} + \mathfrak{a}$, (ii) holds. We will show (iii). Since

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{n}_{-} + \mathfrak{t}_{\mathbb{C}} + \mathfrak{n}_{+}$$

and

$$\mathfrak{g} + \mathfrak{b}_{-} = \mathfrak{t}_{\mathbb{C}} + \mathfrak{n}_{-}$$

it is enogh to show

$$\mathfrak{n}_{+} \subset \mathfrak{g} + \mathfrak{b}_{-}$$

Since

$$\mathfrak{g} + \mathfrak{b}_{-} \supset \mathfrak{g} + \mathfrak{a}$$

and

$$X = \frac{1}{2}(X - X^*) + \frac{1}{2}(X + X^*) \in \mathfrak{g} + \mathfrak{a} \ (\forall X \in \mathfrak{n}_{+})$$

$\mathfrak{n}_{+} \subset \mathfrak{g} + \mathfrak{b}_{-}$ holds. □

Proposition 10.2.4.
$$G \cap B_{-} = T$$

Proof. Clearly, $T \subset G \cap B_{-}$. Let us fix any $g \in G \cap B_{-}$. From Proposition10.2.2, $\bar{g}^T = g^{-1} \in G \cap B_{-}$. So, g is a diagonal matrix. Since $\bar{g}^T g = E$, $g \in T$. □

Proposition 10.2.5. *N_{+} is a complex Lie group.*

Proof. Clearly N_{+} is a connected Lie group with the Lie algebra $\mathfrak{n}_{+}$. $\mathfrak{n}_{+}$ has a complex structure

$$\mathfrak{n}_{+} \ni X \mapsto iX \in \mathfrak{n}_{+}$$

and the real form
$$\{X \in \mathfrak{n}_{+} | X_{i,j} \in \mathbb{R} \ (\forall i, j)\}$$
Therefore, from Proposition3.13.10, N_{+} is a complex Lie group. □

Theorem 10.2.6. *The followings are settings and assumptions.*

(S1) We set
$$G'_{\mathbb{C}} := \{g := \{a_{i,j}\}_{1 \le i,j \le n} \in G_{\mathbb{C}} | det\{a_{i,j}\}_{1 \le i,j \le k} \neq 0, 1 \le \forall k \le n\}.$$

(S2) We set
$$\varphi : N_{+} \times B_{-} \ni (n, b) \mapsto nb \in G_{\mathbb{C}}$$

(S3) We set $p : G_{\mathbb{C}} \ni g \mapsto gB_{-} \in G_{\mathbb{C}}/B_{-}$.

The followings hold.

(i) φ is a biholomorphism from $N_{+} \times B$ to $G'_{\mathbb{C}}$.

(ii) $p|N_{+}$ is a diffeomorphism from N_{+} to $G_{\mathbb{C}}/B_{-}$.

(iii) $G_{\mathbb{C}}/B_{-}$ is a complex manifold.

(iv) $G_{\mathbb{C}}/B_{-}$ is diffeomorphic to the flag manifolod G/T.

(v) $q : G_{\mathbb{C}} \times_{B_{-}} \mathbb{C} \to G_{\mathbb{C}}/B_{-}$ is a holomorphic line bundle. Here, B_{-} acts on $\mathbb{C}$ with χ_{λ}.

Proof of (i). Let $B_{-}(n)$ denote the borel subgroup with order n, $N_+(n)$ denote N_+ with order n, φ^n denote φ with order n and $G'_{\mathbb{C}}(n) := G'_{\mathbb{C}}$ with order n. We will show (i) by mathematical induction with order n. When $n = 1$, (i) clearly holds. Let us assume (i) holds for order n. We set $\psi^n := (\psi_1^n, \psi_2^n)$ for the inverse map of φ.

Let us fix $g := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbb{C}}(n+1)$. Here, $a \in \mathbb{C} \setminus \{0\}, b \in M(1, n, \mathbb{C}), c \in M(n, 1, \mathbb{C}), d \in G_{\mathbb{C}}(n)$. We set

$$\psi_1^{n+1}(g) := \begin{pmatrix} 1 & b\psi_2^n(d)^{-1} \\ 0 & \psi_1^n(d) \end{pmatrix}, \psi_2^{n+1}(g) := \begin{pmatrix} a - bd^{-1}c & 0 \\ \psi_1^n(d)^{-1}c & \psi_2^n(d) \end{pmatrix}$$

Since

$$d = \psi_1^n(d)\psi_2^n(d), \psi_2^n(d)^{-1}\psi_1^n(d)^{-1} = d^{-1}$$

$\varphi^{n+1}(\psi_1^{n+1}(g), \psi_2^{n+1}(g)) = g$. So, $\varphi^{n+1} \circ \psi^{n+1} = id_{G_{\mathbb{C}}(n+1)}$.

Next, we will show ψ^{n+1} is surjective. Let us fix any $g_1 := \begin{pmatrix} 1 & x \\ 0 & g'_1 \end{pmatrix} \in N_+(n+1)$, $g_2 := \begin{pmatrix} y & 0 \\ z & g'_2 \end{pmatrix} \in B_{-}(n+1)$. Here, $y \in \mathbb{C} \setminus \{0\}, x \in M(1, n, \mathbb{C}), z \in M(n, 1, \mathbb{C}), g'_1, g'_2 \in G_{\mathbb{C}}(n)$.

Then there is $d \in G'_{\mathbb{C}}(n)$ such that $\psi^n(d) = g'_1$. We set

$$a := y + bd^{-1}c, b := x\psi^n(d), c := \psi^n(d)z, g := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then clearly $\psi^{n+1}(g) = (g_1, g_2)$. Therefore, ψ^{n+1} is a surjective, in result, ψ^{n+1} is a bijective.

Let us fix any $g \in N_+ \times B_-$. There is $h \in G'_{\mathbb{C}}(n+1)$ such that $g = \psi^{n+1}(h)$. So, $\varphi^{n+1}(g) = h = (\psi^{n+1})^{-1}(g)$. This means that φ^{n+1} is the inverse map of ψ^{n+1}. □

Proof of (ii). Clearly $p|N_+$ is continuous. We set

$$C := \{\{a_{i,j}\} \in B_{-} |\ 0 < |a_{i,i}| \leq 1\ (\forall i), |a_{i,j}| \leq 1\ (\forall i, j)\}$$

First, we will show

$$gC \cap G'_{\mathbb{C}} \neq \phi\ (\forall g \in G_{\mathbb{C}}) \tag{10.2.1}$$

Let us fix any $g := \begin{pmatrix} a_1 & \dots & a_{n+1} \\ b_1 & \dots & b_{n+1} \end{pmatrix} \in G_{\mathbb{C}}$. Here, $a_1, \dots a_{n+1} \in M(n, 1, \mathbb{C})$, $b_1, \dots, b_{n+1} \in \mathbb{C}$. Then there is i such that

$$a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{n+1}$$

are linear independent. If $i = n+1$, $g \in G'_{\mathbb{C}}$. So, hereafter, let us assume $i < (n+1)$. We set

$$F(\epsilon) := \sum_{j \neq i} E_{j,j} + \epsilon E_{j,j} + E_{n+1,i}$$

Then

$$gF(\epsilon) = \begin{pmatrix} a_1 & \dots & \epsilon a_i + a_{n+1} & \dots & a_n & a_{n+1} \\ b_1 & \dots & \epsilon b_i + b_{n+1} & \dots & b_n & b_{n+1} \end{pmatrix}$$

Since

$$\lim_{\epsilon \to 0} \det(a_1, \dots, \epsilon a_i + a_{n+1}, \dots, a_n) = \det(a_1, \dots, a_{n+1}, \dots, a_n) \neq 0$$

there is $\epsilon \in (0, 1)$ such that $gF(\epsilon) \in G'_n$. So, (10.2.1) has been shown.

From (10.2.1), $p|N_+$ is bijective.

We will show $p|N_+$ is an open map. Let us fix any open subset U of $G_{\mathbb{C}}$. From (i),

$$p(U \cap N_+) = p(\varphi((U \cap N_+) \times B_)) \tag{10.2.2}$$

From (i), $\varphi((U \cap N_+) \times B_)$ is open subset of $G'_{\mathbb{C}}$. Since $G'_{\mathbb{C}}$ is an open subset, $\varphi((U \cap N_+) \times B_)$ is also an open subset. Since p is an open map, $p(U \cap N_+)$ is an open subset. Therefore, $p|N_+$ is a homeomorphism from N_+ to $G_{\mathbb{C}}/B_-$.

We set

$$\tilde{\psi} : G_{\mathbb{C}}/B_{-} \ni gB_{-} \mapsto p_1(\psi(g)) \in N_+$$

Here, $g \in \mathbb{G}'_{\mathbb{C}}$, $p_1 : N_+ \times B_- \ni (n, b) \mapsto n \in N_+$. We will show $\tilde{\psi}$ is well-defined. Let us fix any $g_1 := n_1b_1, g_2 := n_2b_2 \in \mathbb{G}'_{\mathbb{C}}$ such that $g_1B_{-} = g_2B_{-}$. Here $n_1, n_2 \in N_+$ and $b_1, b_2 \in B_{-}$. Then there is $b \in B_{-}$ such that $n_1b_1 = n_2b_2b$. From (i), $n_1 = n_2$. This means $\tilde{\psi}(g_1B_{-}) = \tilde{\psi}(g_2B_{-})$. Clearly $\tilde{\psi}$ is C^ω class and $\tilde{\psi} \circ p|N_+ = id_{N_+}$. Consequently, $p|N_+$ is a diffeomorphism from N_+ to $G_{\mathbb{C}}/B_{-}$. □

Proof of (iii). From (ii) and Proposition10.2.5, (iii) holds. ☐

Proof of (iv). Clearly G smoothly acts on $G_{\mathbb{C}}/B_{-}$. And the isotropy group is $G \cap B_{-}$. From Proposition10.2.4, the isotropy group is $G \cap B_{-} = T$.

We will show the action is transitive. We set

$$o := eB_{-},\ G \cdot o := \{g \cdot o | g\ inG\}$$

We will show

$$G \cdot o = p(G_{\mathbb{C}})$$

Clearly $G \cdot o \subset p(G_{\mathbb{C}})$. We set

$$\mathfrak{q} := \{X \in \mathfrak{g} | X_{i,i} = 0\ (\forall i)\}$$

From Proposition10.2.3(iv) and the fact that G is connected Lie group,

$$G \cdot o = \cup_{m=1}^{\infty} \Pi_{i=1}^{m} \exp(\mathfrak{q}) \cdot o = p(\cup_{m=1}^{\infty} \Pi_{i=1}^{m} \exp(\mathfrak{q}))$$

From Proposition10.2.3(v) and the fact that $G_{\mathbb{C}}$ is connected Lie group,

$$p(G_{\mathbb{C}}) = p(\cup_{m=1}^{\infty} \Pi_{i=1}^{m} \exp(\mathfrak{g}_{\mathbb{C}})) = p(\cup_{m=1}^{\infty} \Pi_{i=1}^{m} \exp(\mathfrak{q}))$$

Therefore, $G \cdot o = p(G_{\mathbb{C}})$. This means that the action is transitive. By Theorem6.1.3, G/T is diffeomorphic to $\mathbb{G}_{\mathbb{C}}/B_{-}$. ☐

Proof of (v). Since $\mathbb{G}_{\mathbb{C}}/B_{-}$ is a complex manifold and B_{-} is a complex manifold and χ_{λ} is a holomorphic representation of B_{-}, q is a holomorphic line bundle. ☐

10.3 Iwasawa decomposition

Theorem 10.3.1. *The followings are settings and assumptions.*

(S1) We set

$$\Phi : G \times A \times N_{-} \ni (k, a, n_{-}) \mapsto kan_{-} \in G_{\mathbb{C}}$$

Then

(i) AN_{-} is a subgroup of $G_{\mathbb{C}}$.

(ii) $AN_{-} \cap G = \{e\}$.

(iii) Φ is a diffeomorphism. We call

$$H(g) := (log(a_1(g)), ..., log(a_n(g)))\ (g \in G_{\mathbb{C}})$$

the Iwasawa projection. Here, $g = \Phi(k(g), diag(a_1(g), ..., a_n(g)), n(g))$.

Proof of (i). Remark

$$ana^{-1} \in N_{-}\ (\forall a \in A, \forall n \in N_{-})$$

Therefore, for any $a_1, a_2 \in A$ and any $n_1, n_2 \in N_{-}$,

$$a_1 n_1 a_2 n_2 = a_1 a_2 a_2^{-1} n_1 a_2 n_2 \in AN_{-}$$

and

$$(a_1 n_1)^{-1} = a_1^{-1}(a_1 n_1^{-1} a_1^{-1}) \in AN_{-}$$

Consequently, AN_{-} is a subgroup of $G_{\mathbb{C}}$. ☐

Proof of (ii). Let us fix any $a = diag(a_1, ..., a_n) \in A$ and $n \in N_{-}$ such that $an \in G$. Since any norm of eigenvalue of an is 1 and eigenvalues of an are $a_1, ..., a_n$, $a = e$. Since N_{-} is a subgroup of $G_{\mathbb{C}}$, $n^{-1} \in N_{-}$. Since $n \in G$, $n^{-1} = \bar{n}^T \in G$. Therefore, $\bar{n}^T \in N_{-}$. That implies $n = \{e\}$. ☐

Proof of (iii): STEP1. Proof of Φ is injective. First, we will show Φ is injective. Let us fix any $k_1, k_2 \in G$ and $a_1, a_2 \in A$ and $n_1, n_2 \in N_{-}$ such that $k_1 a_1 n_1 = k_2 a_2 n_2$. Then

$$k_2^{-1} k_1 = a_2 n_2 (a_1 n_1)^{-1} \in G \cap AN_{-} = \{e\}$$

Therefore, $k_1 = k_2$. Then

$$a_2^{-1} a_1 = n_2 (n_1)^{-1} \in A \cap N_{-} = \{e\}$$

Therefore, $a_1 = a_2$ and $n_1 = n_2$. ☐

Proof of (iii): STEP2. Proof of Φ is surjective. Let us fix any $g = \begin{pmatrix} g_1 & g_2 & ... & g_n \end{pmatrix} \in G_{\mathbb{C}}$. We set

$$a_n(g) := ||g_n||, u_n := \frac{1}{a_n} g_n$$

and

$$a_j(g) := ||g_j - \sum_{i=j+1}^{n} (g_j, u_i) u_i|| \ (j = n-1, n-2, ...1)$$

and

$$u_j(g) := \frac{1}{a_j}(g_j - \sum_{i=j+1}^{n} (g_j, u_i) u_i) \ (j = n-1, n-2, ...1)$$

and

$$k := k(g) := \begin{pmatrix} u_1 & u_2 & ... & u_n \end{pmatrix}$$

and

$$a := a(g) := diag(a_1(g), ..., a_n(g))$$

Then

$$k = g \begin{pmatrix} a_1^{-1} & 0 & ... & 0 \\ x_{21} & a_2^{-1} & ... & 0 \\ ... & .... & ... & ... \\ x_{n1} & x_{n2} & ... & a_n^{-1} \end{pmatrix}$$

We set $x := \begin{pmatrix} a_1^{-1} & 0 & ... & 0 \\ x_{21} & a_2^{-1} & ... & 0 \\ ... & .... & ... & ... \\ x_{n1} & x_{n2} & ... & a_n^{-1} \end{pmatrix}$. Then $e = k^{-1} g$. That implies $a^{-1} = a^{-1} k^{-1} g x$. Therefore,

$$n(g) := a^{-1} x^{-1} = a^{-1} k^{-1} g x \in N_-$$

Then $g = kan$. We set

$$\Psi(g) := (k(g), a(g), n(g))$$

Clearly, Ψ is C^∞-class and $\Phi \circ \Psi = id_G$. Consequently, Φ is bijective and $\Phi^{-1} = \Psi$. □

Proposition 10.3.2. *$G_{\mathbb{C}}/G$ is diffeomorphic to $N_- \times A$. In specialty, $G_{\mathbb{C}}/G$ is simply connected.*

Proof. We set

$$\phi : N \times A_- \ni (n, a) \mapsto naG \in G_{\mathbb{C}}/G$$

and

$$\psi : G_{\mathbb{C}}/G \ni gG \mapsto (n^{-1}, a^{-1}) \in A \times N_-$$

Here, $a \in A, n \in N_-, k \in G$ such that $g^{-1} = kan$. From Theorem10.3.1, ψ is C^∞-class. And clearly $\phi^{-1} = \psi$. □

10.4 A proof of Borel-Weil Theorem

Proposition 10.4.1. *The followings are settings and assumptions.*

(S1) $f \in Hol(G_{\mathbb{C}}, \mathbb{C})$

$$f(gan) = \chi_\lambda(a)^{-1} f(g) \ (\forall g \in G, \forall a \in A, \forall n \in N_-)$$

(S2) $C := \sup_{k \in U(n)} |f(k)|$.

Then

$$|F(g)| \leq C |\chi_\lambda(exp(H(g))^{-1}| \ (\forall g \in G_{\mathbb{C}})$$

Proof. Let us fix any $g \in G_{\mathbb{C}}$. From Iwasawa decomposition, there are $k \in U(n)$ and $a \in A$ and $n \in N_-$ such that $g = kan$. Then

$$|f(g)| = |f(kan)| = |\chi_\lambda(an)|^{-1} \cdot |f(k)| \leq C |\chi_\lambda(an)|^{-1} = C |\chi_\lambda(a)|^{-1} = C |\chi_\lambda(exp(H(g))|^{-1}$$

□

Proposition 10.4.2. *The followings are settings and assumptions.*

(S1) We set

$$f^{(k)} := (1, ..., 1, 0, ..., 0) = \sum_{i=1}^{k} e_i \ (k = 1, 2, ..., n)$$

(S2) We set

$$P_k(n_+) := exp(-2(H(n_+), f^{(k)}) \ (k = 1, 2, ..., n)$$

Then

$$|F(g)| \leq C|\chi_\lambda(exp(H(g))^{-1}| \ (\forall g \in G_{\mathbb{C}})$$

Part II

Applications

Chapter 11

Probability and Statistics

11.1 Basic Notations

Notation 11.1.1 (The set of all probability measures on (R)). *Denote the set of all borel sets on* $\mathbb{R}$ *by* $\mathscr{B}(\mathbb{R})$. *Denote the set of all probability measures on* $\mathscr{B}(\mathbb{R})$ *by* $\mathscr{P}(R)$.

Notation 11.1.2 (order relation in $\mathbb{R}^n$). *Let* $x, y \in \mathbb{R}^n$. *Denote* $x \le y$ ($x < y$) *if* $x_i \le y_i$ ($x_i < y_i$) ($\forall i$).

Definition 11.1.3 (A distribution of random variables). *Let* $(\Omega, \mathscr{F}, P)$ *be a probability space and let* $X = (X_1, X_2, ..., X_n)$ *be random variables on* Ω. *We define* $P_X : \mathscr{B}(\mathbb{R}^n) \ni A \mapsto P(X^{-1}(A)) \in [0,1]$. *We denote the distribution of* X *by* P_X.

Definition 11.1.4 (A distribution function of a probability measure). *Let* $\mu \in \mathscr{R}(\mathbb{R}^n)$. *We define* $F_\mu : \mathbb{R}^n \ni x \mapsto \mu((-\infty, x_1] \times (-\infty, x_2] ... \times (-\infty, x_n]) \in \mathbb{R}$ *and we call* F_μ *the destribution function of* μ.

Notation 11.1.5 (Fourier transform). *Let* $f \in L^1(\mathbb{R}^n)$. *Denote fourier transformation of f by* $\mathscr{F}(f)$ *and denote fourier inverse transformation of f by* $\mathscr{F}^{-1}(f)$.

Definition 11.1.6 (Weakly convergence of probability measures). *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S2) Let* $\{\mu_n\}_{n=1}^\infty \in \mathcal{P}(\mathbb{R}^N)$.
- *(S3) Let* $\mu \in \mathcal{P}(\mathbb{R}^N)$.

$\{\mu_n\}_{n=1}^\infty$ *is weakly converges to* μ *if* $\lim_{n\to\infty} F_{\mu_n}(x) = F_\mu(x)$ *for any point* x *at which* F_μ *is continuous. Denote this by* $\mu_n \Longrightarrow \mu$ $(n \to \infty)$

Definition 11.1.7 (Characteristic function of probability measure). *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S2) Let* $\mu \in \mathcal{P}(\mathbb{R}^n)$.

then call $\varphi_\mu : \mathbb{R}^n \ni t \mapsto \int_{\mathbb{R}^n} exp(itx) d\mu(x) \in \mathbb{C}$ *is the characteristic function of* μ. *Bellow, assume the characteristic function of* μ *denotes* φ_μ *unless otherwise noted.*

Definition 11.1.8 (Characteristic function of random variables). *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S2) Let* $X = (X_1, X_2, ..., X_n)$ *be a vector of random variables on* $(\Omega, \mathscr{F}, P)$.

then call $\varphi_X : \mathbb{R} \ni t \mapsto \int_\Omega exp(itX) dP \in \mathbb{C}$ *is the characteristic function of* X. *Bellow, assume the characteristic function of* X *denotes* φ_X *unless otherwise noted.*

Definition 11.1.9 (Tightness of probability measures). *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S2) Let* $\{\mu_n\}_{n=1}^\infty \in \mathcal{P}(\mathbb{R}^N)$.

$\{\mu_n\}_{n=1}^\infty$ *is tight if for any* $\epsilon > 0$ *there is a* $M > 0$ *such that*

$$\mu_n(\{x \in \mathbb{R}^N | |x| \le M\}) \ge 1 - \epsilon \tag{11.1.1}$$

Definition 11.1.10 (Weakly compactness of probability measures). *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S2) Let* $\{\mu_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathbb{R}^N)$.

$\{\mu_n\}_{n=1}^{\infty}$ *is weakly compact if for any subsequence* $\{\mu_{\alpha(n)}\}_{n=1}^{\infty}$ *of* $\{\mu_n\}_{n=1}^{\infty}$ *there is a subsequence of* $\{\mu_{\alpha(n)}\}_{n=1}^{\infty}$ *which weakly comverges to a probability measure.*

Definition 11.1.11 (Outer measure). *Let*

- *(S1)* X *is a set.*

$\Gamma : 2^X \to [0, \infty]$ *is an outer measure on* X *if the followings hold.*

- *(i)* $\Gamma(\phi) = 0$
- *(ii) If* $A \subset B$ *then* $\Gamma(A) \leq \Gamma(B)$
- *(iii) If* $\{A_i\}_{i=1}^{\infty} \subset 2^X$ *then* $\Gamma(\cup_{i=1}^{\infty} A_i) \leq \Sigma_{i=1}^{\infty} \Gamma(A_i)$

11.2 Finite measures on metric space

We introduce several definitions and propositions for only Section11.6.2.

11.3 several facts on metric space

The following three definitions are from [37].

Definition 11.3.1 (Elementary function family). *Let*

- *(S1)* (X, d) *is a metric space.*

$\mathscr{E} \subset Map(X, [0, \infty))$ *is called a family of elementary functions if the followings holds.*

- *(i) if* $f, g \in \mathscr{E}$ *then* $f + g \in \mathscr{E}$.
- *(ii) if* $f, g \in \mathscr{E}$ *and* $f \geq g$ *then* $f - g \in \mathscr{E}$.
- *(iii) if* $f, g \in \mathscr{E}$ *then* $min\{f, g\} \in \mathscr{E}$.

Definition 11.3.2 (Elementary integral). *Let*

- *(S1)* (X, d) *is a metric space.*
- *(S2)* $\mathscr{E} \subset Map(X, [0, \infty))$ *is a elementary function family.*

$l : \mathscr{E} \mapsto [0, \infty]$ *is an elementary integral on* $\mathscr{E}$ *if the followings hold.*

- *(i) if* $f, g \in \mathscr{E}$ *then* $l(f + g) = l(f) + l(g)$
- *(ii) if* $f, g \in \mathscr{E}$ *and* $f \leq g$ *then* $l(f) \leq (g)$

Definition 11.3.3 (Complete elementary integral). *Let*

- *(S1)* (X, d) *is a metric space.*
- *(S2)* $\mathscr{E} \subset Map(X, [0, \infty))$ *is a elementary function family.*
- *(S3)* $l : \mathscr{E} \mapsto [0, \infty]$ *is an elementary integral.*

l *is a complete elementary integral if for any* $\{f_n\}_{\{n=1\}}^{\infty}$ *such that* $\lim_{n\to\infty} f_n = f$ *(pointwise convergence) and* $f_n \leq f_{n+1}$ $(\forall n \in \mathbb{R})$ *satisfies* $\lim_{n\to\infty} l(f_n) = l(f)$

Definition 11.3.4 (Functional from elementary integral). *Let*

- *(S1)* (X, d) *is a metric space.*
- *(S2)* $\mathscr{E} \subset Map(X, [0, \infty))$ *is a elementary function family.*
- *(S3)* $l : \mathscr{E} \mapsto [0, \infty]$ *is an elementary integral.*

We define

$$L:\{\varphi : X \to [0,\infty)\} \ni \varphi \mapsto inf\{\Sigma_{i=1}^{\infty} l(\varphi_i) | \varphi_i \in \mathscr{E}\ (\forall i), \varphi \leq \Sigma_{i=1}^{\infty}\varphi_i\} \in [0,\infty] \tag{11.3.1}$$

Proposition 11.3.5. *Let*

- *(S1)* (X,d) *is a metric space.*
- *(S2)* $\mathscr{E} \subset Map(X,[0,\infty))$ *is a elementary function family.*
- *(S3)* $l : \mathscr{E} \mapsto [0,\infty]$ *is an elementary integral.*
- *(A1)* $[0,\infty)\mathscr{E} \subset \mathscr{E}$.

For any $\alpha > 0$ *and* $f \in \mathscr{E}$

$$l(\alpha f) = \alpha l(f) \tag{11.3.2}$$

Proof. Let us fix $q_1 \in (\alpha,\infty)\cap\mathbb{Q}$ and $q_2 \in (0,\alpha)\cap\mathbb{Q}$. $q_2 l(f) = l(q_2 f) \leq l(\alpha f) \leq l(q_1 f) = q_1 l(f)$. So $l(\alpha f) = \alpha l(f)$ □

Proposition 11.3.6 (Outer measure from elementary integral)**.** *Let*

- *(S1)* (X,d) *is a metric space.*
- *(S2)* $\mathscr{E} \subset Map(X,[0,\infty))$ *is a elementary function family.*
- *(S3)* $l : \mathscr{E} \mapsto [0,\infty]$ *is an elementary integral.*
- *(S4)* L *is the functional in Definition11.3.4.*
- *(S5) We set* $\Gamma : 2^X \ni A \mapsto L(\chi_A)$.

then Γ *is outer measure on* X.

Proof. It is easy to show terms except (iii) in Definition11.1.11. So we will show only (iii) in Definition11.1.11. Let us fix ${A_i}_{i=1}^{\infty} \subset 2^X$.

Let us fix $\epsilon > 0$.
For each $i \in \mathbb{N}$, there are $\{\varphi_{i,j}\}_{j=1}^{\infty} \subset \mathscr{E}$ such that $\chi_{A_i} \leq \Sigma_{j=1}^{\infty}\varphi_{i,j}$ and $\Sigma_{j=1}^{\infty} l(\varphi_{i,j}) \leq \Gamma(A_i) + \frac{\epsilon}{2^i}$
So $\chi_{\cup_{i=1}^{\infty} A_i} \leq \Sigma_{i=1,j=1}^{\infty}\varphi_{i,j}$.
$\Gamma(\cup_{i=1}^{\infty} A_i) \leq \Sigma_{i=1,j=1}^{\infty} l(\varphi_{i,j}) \leq \Sigma_{i=1}^{\infty}\Gamma(A_i) + \epsilon$
Consequently, (iii) holds.

□

Proposition 11.3.7. *Let*

- *(S1)* (X,d) *is a metric space.*
- *(S2)* $\mathscr{E} \subset Map(X,[0,\infty))$ *is a elementary function family.*
- *(S3)* $l : \mathscr{E} \mapsto [0,\infty]$ *is an elementary integral.*
- *(S4)* L *is the functional in Definition11.3.4.*
- *(S5)* Γ *is the outer measure in Proposition11.3.6.*
- *(S6)* $\mathfrak{M}_\Gamma$ *is the* σ*-algebra in Proposition11.5.9.*
- *(A1)* $C_+(X) \subset \mathscr{E}$.
- *(A2) If* A, B *are borel sets and* $d(A,B) > 0$ *then* $\mu(A) + \mu(B) = \mu(A \cup B)$.

then $\mathscr{B}(X) \subset \mathfrak{M}_\Gamma$.

Proof. Because $\mathfrak{M}_\Gamma$ is σ-algebra, it is enough to show that all closed sets are contained in $\mathfrak{M}_\Gamma$.
Let us fix closed set A. Let us subset B and C such that $A \subset B$ and $C \subset A^c$.
Because A is closed set, $C \subset \{x | d(x,A) > 0\}$.
For each $n \in \mathbb{N}$ we set $C_n := \{x \in C | d(x,A) > \frac{1}{n}\}$ and $D_n := \{x \in C | \frac{1}{n-1} \geq d(x,A) > \frac{1}{n}\}$.
The followings holds.

$$C = \cup_{n=1}^{\infty} D_n \tag{11.3.3}$$

$$C_N = \cup_{n=1}^{N} D_n\ (\forall N) \tag{11.3.4}$$

We assume $\Sigma_{n=1}^{\infty}\Gamma(D_n) < \infty$. Let us fix $\epsilon > 0$.
There is n_0 such that $\Sigma_{n=n_0}^{\infty}\Gamma(D_n) < \epsilon$.

Because $d(A, C_{n_0}) > 0$,

$$\begin{aligned}\Gamma(A)+\Gamma(C) &= \Gamma(A)+\Gamma(C_{n_0}\cup\cup_{n=n_0}^{\infty}D_n)\\ &\leq \Gamma(A)+\Gamma(C_{n_0})+\epsilon\\ &\leq \Gamma(A)+\Gamma(C_{n_0})+\epsilon\\ &= \Gamma(A\cup C_{n_0})+\epsilon\\ &\leq \Gamma(A\cup C)+\epsilon \end{aligned} \tag{11.3.5}$$

So if $\Sigma_{n=1}^{\infty}\Gamma(D_n) < \infty$ then $\Gamma(A) + \Gamma(C) = \Gamma(A \cup C)$.

We assume $\Sigma_{n=1}^{\infty}\Gamma(D_n) = \infty$. Then $\Sigma_{n=1}^{\infty}\Gamma(D_{2n}) = \infty$ or $\Sigma_{n=1}^{\infty}\Gamma(D_{2n-1}) = \infty$. We assume $\Sigma_{n=1}^{\infty}\Gamma(D_{2n}) = \infty$.

If $n_1 \neq n_2$ then $d(D_{n_1}, D_{n_2}) > 0$. So $\Gamma(C) \geq \Gamma(\cup_{n=1}^{\infty}D_{2n}) \geq \Sigma_{n=1}^{\infty}\Gamma(D_{2n}) = \infty$. So if $\Sigma_{n=1}^{\infty}\Gamma(D_{2n}) = \infty$ then $\Gamma(B) + \Gamma(C) = \Gamma(A \cup C) = \infty$.

Similary, if $\Sigma_{n=1}^{\infty}\Gamma(D_{2n-1}) = \infty$ then $\Gamma(B) + \Gamma(C) = \Gamma(A \cup C) = \infty$. □

Proposition 11.3.8. *Let*

(S1) (X, d) *is a metric space.*

(S2) $\mathscr{E} \subset Map(X, [0, \infty))$ *is a elementary function family.*

(S3) $l : \mathscr{E} \mapsto [0, \infty]$ *is an elementary integral.*

(S4) $\{f_n\}_{n=1}^{\infty} \subset \mathscr{E}$ *and* $f_n \geq f_{n+1}$ *on* X $(\forall n)$*.*

(A1) There is $f \in \mathscr{E}$ *such that* $\lim_{n\to\infty} ||f_n - f||_\infty = 0$

(A2) $\mathbb{R}\mathscr{E} \subset \mathscr{E}$

then

$$\lim_{n\to\infty} l(f_n) = l(f) \tag{11.3.6}$$

Proof. $|l(f) - l(f_n)| = l(f - f_n) \leq ||f - f_n||_\infty l(1) \to 0\ (n \to \infty)$ □

Proposition 11.3.9. *Let*

(S1) (X, d) *is a metric space.*

(S2) $l : \mathscr{E} \mapsto [0, \infty]$ *is an elementary integral on* $\mathscr{E} := \{f|f$ *is nonnegative borel measurable on* X $\}$*.*

(S3) L *is the functional in Definition11.3.4.*

(S4) $h_1, h_2 \in \mathscr{E}$*.*

(A1) $d(supp(h_1), supp(h_2)) > 0$*.*

then $L(h_1 + h_2) = L(h_1) + L(h_2)$

Proof. Let us fix arbitary $\epsilon > 0$. Let us fix f and g in Proposition2.2.11.

Let us fix $\{\varphi_i\} \subset \mathscr{E}$ such that $h_1 + h_2 \leq \Sigma_{i=1}^{\infty}\varphi_i$ and $\Sigma_{i=1}^{\infty}l(\varphi_i) \leq L(h_1 + h_2) + \epsilon$.

By definition of f and g,

$$h_1 + h_2 \leq (f + g)\Sigma_{i=1}^{\infty}\varphi_i \tag{11.3.7}$$

and

$$h_1 \leq f\Sigma_{i=1}^{\infty}\varphi_i \tag{11.3.8}$$

and

$$h_2 \leq g\Sigma_{i=1}^{\infty}\varphi_i \tag{11.3.9}$$

So

$$\begin{aligned} L(h_1+h_2)+\epsilon &\geq \Sigma_{i=1}^{\infty}l(\varphi_i)\\ &\geq \Sigma_{i=1}^{\infty}(l(f\varphi_i)+\Sigma_{i=1}^{\infty}l(g\varphi_i))\\ &\geq L(h_1)+L(h_2) \end{aligned} \tag{11.3.10}$$

Consequently

$$L(h_1) + L(h_2) \leq L(h_1 + h_2) \tag{11.3.11}$$

□

Proposition 11.3.10. *Let*

(S1) (X, d) *is a metric space.*

(S2) $l : \mathscr{E} \mapsto [0, \infty]$ *is an elementary integral on* $C_+(X)$.

(S3) L *is the functional in Definition11.3.4.*

(S4) Γ *is the outer measure in Proposition11.3.6.*

(S5) $\mathfrak{M}_\Gamma$ *is the* σ*-algebra in Proposition11.5.9.*

then $\mathscr{B}(X) \subset \mathfrak{M}_\Gamma$.

Proof. Let us fix arbitary borel sets A, B such that $d(A, B) > 0$.

By Proposition11.3.9, $\Gamma(A \cup B) = L(\chi_{A\cup B}) = L(\chi_A + \chi_B) = L(\chi_A) + L(\chi_B) = \Gamma(A) + \Gamma(B)$.

By Proposition11.3.7, $\mathscr{B}(X) \subset \mathfrak{M}_\Gamma$. □

11.4 several facts on compact metric spaces

Proposition 11.4.1. *Let*

(S1) (X, d) *is a compact metric space.*

(S2) l *is an elementary integral on* $C_+(X)$*.* $C_+(X) := \{f \in C(X) | f \geq 0\}$

then there is an unique measure μ *on* $(X, \mathscr{B}(X))$ *such that for any* $f \in C_+(X)$

$$l(f) = \int_X f\mu \tag{11.4.1}$$

Existence. Let us fix $f \in C_+(X)$.

By replacing f by $||f||_\infty - f$, it is enough to show

$$\int_X f d\mu l(f) \leq l(f) \tag{11.4.2}$$

By an argument similar to one in the proof of Proposition11.6.5, there are $a_{m,i_{1\leq m\leq\infty,1\leq i\leq\varphi(m)}} \subset \mathbb{R}$ such that

$$0 = a_{m,1} \leq a_{m,2} \leq ... \leq a_{m,\varphi(m)} > ||f||_\infty \ (\forall m \in \mathbb{N}) \tag{11.4.3}$$

$$|a_{m,i} - a_{m,i+1}| \leq \frac{1}{2^m} \ (\forall m, \forall i) \tag{11.4.4}$$

$$\mu(\{f = a_{m,i}\}) = 0 \ (\forall m, \forall i) \tag{11.4.5}$$

We set

$$h_m := \Sigma_{i=1}^{\varphi(m)} a_{m,i} \chi_{[a_{m,i}, a_{m,i+1})} \ (m \in \mathbb{N}) \tag{11.4.6}$$

and

$$h_{m,n} := \Sigma_{i=1}^{\varphi(m)} a_{m,i} \chi_{(a_{m,i}+\frac{1}{n}, a_{m,i+1}-\frac{1}{n})} \ (m \in \mathbb{N},\ 1 \leq i \leq \varphi(m)) \tag{11.4.7}$$

Let us fix $\epsilon > 0$.

By Proposition11.5.10, $f \in C_u(X)$.

By (11.4.5), there is m, n such that

$$|\int_X f d\mu - \int_X h_{m,n} d\mu| < \epsilon \tag{11.4.8}$$

Because $f \in C_u(X)$, if $i \neq j$ then $d(f^{-1}((a_{m,i} + \frac{1}{n}, a_{m,i+1} - \frac{1}{n})), f^{-1}((a_{m,j} + \frac{1}{n}, a_{m,j+1} - \frac{1}{n}))) > 0$.

So

$$l(f) \geq L(h_{m,n} \geq \int_X h_{m,n} d\mu \tag{11.4.9}$$

Therefore,

$$\int_X f d\mu - \epsilon \leq l(f) \tag{11.4.10}$$

Consequently,

$$\int_X f d\mu \leq l(f) \tag{11.4.11}$$

□

Uniqueness. Let us fix arbitary $\mu_1 \in \mathscr{P}(X)$ and arbitary $\mu_2 \in \mathscr{P}(X)$ such that

$$\int_X f d\mu_1 = \int_X f d\mu_2 \ (\forall f \in C_+(X)) \tag{11.4.12}$$

We set $\mathscr{B} := A \in \mathscr{B}(X) | \mu_1(A) = \mu_2(A)$. Clearly $\mathscr{B}$ is σ-algebra.
Let us fix closed set A.
By Proposition2.2.1, there are $\{f_m\}_{m=1}^{\infty} \subset C_+(X)$ such that

$$||f_m||_\infty \leq 1 \ (\forall m) \tag{11.4.13}$$

and

$$lim_{m\to\infty} f_m = \chi_A \ (pointwize\ convergence) \tag{11.4.14}$$

By Lebesugue's convergence theorem, $\mu_1(A) = \mu_2(A)$.
So $A \in \mathscr{B}$.
Consequently $\mathscr{B} \subset \mathscr{B}(X)$.

□

11.5 Some Facts Used Without Proofs

In this note, we use the following propositions without proofs.

Proposition 11.5.1. *Let*

(S1) $(\Omega, \mathscr{F}, P)$ *is a probability space.*

(S2) X *is a* N*-dimensional vector of random variables on* $(\Omega, \mathscr{F})$*.*

(S3) Let μ_X *be a probability distribution of* X*.*

(S4) $f \in L^1(\Omega) \cup L^\infty(\Omega)$

then

$$\int_{\mathbb{R}^N} f d\mu_X = \int_\Omega f \circ X dP \tag{11.5.1}$$

Proposition 11.5.2. *For any* $\eta > 0$*,*

$$\mathscr{F}(exp(-\eta(\cdot)^2) = \frac{1}{\sqrt{2\eta}} exp(-\frac{(\cdot)^2}{4\eta}). \tag{11.5.2}$$

Proposition 11.5.3. *Let* Σ *be a positive definite symmetric matrix.*

$$\varphi_{N(0,\Sigma)}(\boldsymbol{t}) = exp(-\frac{\boldsymbol{t}^T \Sigma^2 \boldsymbol{t}}{2}) \tag{11.5.3}$$

Proposition 11.5.4. *Let*

(S1) Arbitrarily take $M > 0$ *and fix it.*

(S2) Let $f_n : \overline{D(0,M)} \ni z \mapsto (1 + \frac{z}{n})^n \in \mathbb{C}$*, where* $\overline{D(0,M)} := \{z \in \mathbb{C} | \ |z| \leq M\}$*,* $(n = 1, 2, ...)$*.*

then $\{f_n\}_{n=1}^{\infty}$ *uniformly converges to exp on* $\overline{D(0,M)}$*.*

Proposition 11.5.5. *Let*

(A1) Let $F : \mathbb{R} \mapsto \mathbb{R}$ *is monotone increasing.*

then $\{x \mid F$ *is not continuous at* $x\}$ *is at most countable.*

Proposition 11.5.6. *Let*

(S1) $(\Omega, \mathscr{F}, P)$ *is a probability space.*

(S2) Let $\{\mu_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathbb{R})$*.*

(A1) Let $\mu \in \mathscr{P}(\mathbb{R})$ *such that* $\mu_n \Longrightarrow \mu \ (n \to \infty)$*.*

then for any bounded continuous function $f : \mathbb{R} \mapsto \mathbb{R}$

$$lim_{n\to\infty} \int_{\mathbb{R}} f(x) d\mu_n(x) = \int_{\mathbb{R}} f(x) d\mu(x) \tag{11.5.4}$$

Proposition 11.5.7. *Let*

(S1) $(\Omega, \mathscr{F}, P)$ *is a probability space.*

(S2) μ *is a probability measure on* $\mathbb{R}$.

(A1) $E[\mu] = 0$ *and* $V[\mu] = 1$.

then $\varphi_\mu(s) = 1 - \frac{s^2}{2} + o(s^2)$ $(s \to 0)$

The following propositions are used for only Section11.2 and Subsection11.6.2.

Proposition 11.5.8. *Let*

(S1) (X, d) *is a metric space.*

then there is a complete metric space $(\tilde{X}, \tilde{d})$ *and an isometry mapping* $i : (X, d) \to (\tilde{X}, \tilde{d})$ *such that* $i(X)$ *is dense in* $\tilde{X}$. *We call* $(\tilde{X}, \tilde{d})$ *is a completion of* (X, d).

Proposition 11.5.9. *Let*

(S1) X *is a set.*

(S2) Γ *is an outer measure on* X.

(S3) $\mathfrak{M}_\Gamma := \{A \subset X|$ *if* $B \subset A$ *and* $C \subset A^c$ *then* $\mu(B) + \mu(C) = \mu(B \cup C)\}$.

then the followings holds.

(i) $\mathfrak{M}_\Gamma$ *is a* σ*-algebra.*

(ii) Γ *is a measure on* $\mathfrak{M}_\Gamma$.

Proposition 11.5.10. *Let*

(S1) (X, d) *is a compact metric space.*

then $C(X) \subset C_u(X)$.

Proposition 11.5.11. *Let*

(S1) $(X, d1)$ *is a compact metric space.*

(S2) $(Y, d2)$ *is a compact metric space.*

(A1) $f \in C(X, Y)$.

then $f(X)$ *is compact in* Y.

Proposition 11.5.12. $C_c(\mathbb{R}^n)$ *is dense in* $L^1(\mathbb{R}^n)$.

11.6 Weak convergence of probability distributions

11.6.1 The Case of Single Variable

Proposition 11.6.1 (Helly's selection theorem)**.** *Let*

(S1) $(\Omega, \mathscr{F}, P)$ *is a probability space.*

(S2) Let $\{\mu_n\}_{n=1}^\infty \subset \mathcal{P}(\mathbb{R})$ *and denote* F_{μ_n} *by* F_n $(n = 1, 2, 3, ...)$.

Then there is a subsequence $\{F_{\alpha(n)}\}_{n=1}^\infty$ *and* $F : \mathbb{R} \to [0, \infty)$ *such that* F *is monotone increasing and right continuous, and* $F_{\alpha(n)}(x) \to F(x)$ *for any point* x *at which* F *is continuous.*

Proof. There is $\{x_n\}_{n=1}^\infty \subset \mathbb{R}$ such that $\overline{\{x_n\}_{n=1}^\infty} = \mathbb{R}$. Let fix such $\{x_n\}_{n=1}^\infty$. Because $0 \le F_n(x_m) \le 1$ (for any m, n in $\mathbb{N}$), there is a subsequence $\{\alpha(n)\}_{n=1}^\infty \subset \mathbb{N}$ and $\{F(x_n)\}_{n=1}^\infty \subset [0, 1]$ such that $F_{\alpha(m)}(x_n) \to F(x_n)$ $(m \to \infty)$. We fix such $\{\alpha(n)\}_{n=1}^\infty$ and $F(x_n)_{n=}^\infty$ We define $F(x) := inf_{m \in \{k | x \le x_k\}} F(x_m)$. By the definition of F, F is right continuous and monotone increasing. Arbitrarily take $x \in \mathbb{R}$ at which F is continuous and fix it. Arbitrarily take $\epsilon > 0$ and fix it. Let pick $x_{\alpha(m1)}$ and $x_{\alpha(m2)}$ such that $x_{\alpha(m1)} \le x \le x_{\alpha(m2)}$ and $(F(x_{\alpha(m2)}) - F(x_{\alpha(m1)})) \le \frac{\epsilon}{8}$. There is a $n_0 \in \mathbb{N}$ such that

$|F_n(x_{\alpha(m1)}) - F(x_{\alpha(m1)})| \leq \frac{\epsilon}{8}$ and $|F_n(x_{\alpha(m2)}) - F(x_{\alpha(m2)})| \leq \frac{\epsilon}{8}$ for any $n \geq n_0$. Let fix such n_0 and $m1$ and $m2$. For any $n \geq n_0$

$$\begin{aligned} |F_n(x_{\alpha(m1)}) - F(x)| &\leq |F_n(x_{\alpha(m1)}) - F(x_{\alpha(m1)})| + |F(x_{\alpha(m1)}) - F(x)| \\ &\leq \frac{\epsilon}{4} \end{aligned} \tag{11.6.1}$$

and

$$\begin{aligned} |F_n(x_{\alpha(m2)}) - F(x)| &\leq |F_n(x_{\alpha(m1)}) - F(x_{\alpha(m1)})| + |F(x_{\alpha(m1)}) - F(x)| \\ &\leq \frac{\epsilon}{4} \end{aligned} \tag{11.6.2}$$

So for any $n \geq n_0$

$$|F_n(x_{\alpha(m1)}) - F_n(x_{\alpha(m2)})| \leq \frac{\epsilon}{2} \tag{11.6.3}$$

Arbitrarily take $n \geq n_0$ and fix it. Because $F_n(x_{m1}) \leq F_n(x) \leq F_n(x_{m2})$,

$$max\{|F_n(x_{\alpha(m1)}) - F_n(x)|, |F_n(x_{\alpha(m2)}) - F_n(x)|\} \leq \frac{\epsilon}{2} \tag{11.6.4}$$

By (11.6.1) and (11.6.2) and (11.6.4),

$$|F_n(x) - F(x)| \leq \epsilon \tag{11.6.5}$$

□

Proposition 11.6.2. *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S2) Let* $\{\mu_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathbb{R})$.

If $\{\mu_n\}_{n=1}^{\infty}$ *is tight then* $\{\mu_n\}_{n=1}^{\infty}$ *is weakly compact.*

Proof. By Proposition11.6.1, there is $F : \mathbb{R} \to [0, \infty)$ such that F is monotone increasing and right continuous, and for any point x at which F is continuous

$$F_{\alpha(n)}(x) \to F(x)\ (n \to \infty) \tag{11.6.6}$$

Here we denote F_{μ_n} by F_n. Because of tightness of $\{\mu_n\}_{n=1}^{\infty}$, $limit_{x\to\infty}(F(x) - F(-x)) = 1$. So there is a probability measure μ such that F is a distribution function of μ. By (11.6.6), $\mu_n \Longrightarrow \mu\ (n \to \infty)$. □

Proposition 11.6.3. *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S2) Let* $\{\mu_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathbb{R})$. *and* $\mu \in \mathcal{P}(\mathbb{R})$
- *(A1)* $\mu_n \Longrightarrow \mu\ (n \to \infty)$.
- *(A2) Let f be an arbitary bouded continuous function on* $\mathbb{R}$.

then

$$\lim_{n\to\infty} \int f d\mu_n(x) = \int f d\mu(x) \tag{11.6.7}$$

Proof. Let us fix arbitary $f \in C_b(\mathbb{R})$ and $\epsilon > 0$.

Because $\mu(\mathbb{R}) = 1$ and $\mathbb{R} = \cup_{a\in\mathbb{R}} a$, for each $n \in \mathbb{N}$ $\{a \in \mathbb{R} | \mu(a) > \frac{1}{n}\}$ is finite. So $\{a \in \mathbb{R} | \mu(a) > 0\}$ is at most coutable. So there is $r_1 > 0$ and $r_2 > 0$ such that

$$1 - \mu((-r_1, r_2)) < \frac{\epsilon}{3(||f||_\infty + 1)} \tag{11.6.8}$$

and $\mu(-r_1) = 0$ and $\mu(-r_2) = 0$.

Because f is uniformly continuous on X,

So there are $a_{m,i_{1\leq m\leq\infty,1\leq i\leq\varphi(m)}} \subset \mathbb{R}$ such that

$$-r_1 = a_{m,1} \leq a_{m,2} \leq ... \leq a_{m,\varphi(m)} = r_2\ (\forall m \in \mathbb{N}) \tag{11.6.9}$$

and

$$|a_{m,i} - a_{m,i+1}| \leq \frac{1}{2^m}\ (\forall m, \forall i) \tag{11.6.10}$$

and

$$\mu(\{a_{m,i}\}) = 0 \ (\forall m, \forall i) \tag{11.6.11}$$

For each $m \in \mathbb{N}$, set $f_m := \Sigma_{i=1}^{\varphi(m)} f(a_i)\chi_{[a_i,a_{i+1})}$.
Because $\lim_{m\to\infty} f_m = f$ (pointwize convergence), by Lebesugue's convergence theorem there is $m \in \mathbb{N}$ such that

$$|\int_{-r_1}^{r_2} f_m\mu - \int_{-r_1}^{r_2} f\mu| < \frac{\epsilon}{3} \tag{11.6.12}$$

Because

$$\int_{-r_1}^{r_2} f_m\mu = \Sigma_{i=1}^{\varphi(m)} f(a_i)\mu([a_i, a_{i+1})) \tag{11.6.13}$$

and

$$\int_{-r_1}^{r_2} f_m\mu_n = \Sigma_{i=1}^{\varphi(m)} f(a_i)\mu_n([a_i, a_{i+1})) \ (\forall n) \tag{11.6.14}$$

So there is n_0 such that

$$|\int_{-r_1}^{r_2} f_m\mu_n - \int_{-r_1}^{r_2} f_m\mu| < \frac{\epsilon}{3} \ (\forall n \geq n_0) \tag{11.6.15}$$

By (11.6.8) and (11.6.12) and (11.6.15),

$$|\int_{\mathbb{R}} f\mu_n - \int_{\mathbb{R}} f\mu| < \epsilon \ (\forall n \geq n_0) \tag{11.6.16}$$

□

11.6.2 The Case of Multi Variables

Definition 11.6.4 (Weak convergence(in general metric space))**.** *Let*

(S1) (X, d) is a metric space.
(S2) $\{\mu_n\}_{n=1}^{\infty} \subset \mathscr{P}(X)$.
(S3) $\mu \in \mathscr{P}(X)$.

We say $\{\mu_n\}_{n=1}^{\infty}$ weakly converges to μ if for any borel set A such that $\mu(\partial(A)) = 0$ $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ Denote $\mu_n \Longrightarrow \mu$ by weak convergence.

The following proposition gives the equivalent definition of weak convergence.

Proposition 11.6.5. *Let*

(S1) (X, d) is a metric space.
(S2) $\{\mu_n\}_{n=1}^{\infty} \subset \mathscr{P}(X)$.
(S3) $\mu \in \mathscr{P}(X)$.

then the followings are equivalent.

(i) $\mu_n \Longrightarrow \mu$.
(ii) Set $C_b(X) := \{f \in C(X) || ||f||_\infty < \infty\}$. For any $f \in C_b(X)$

$$\lim_{n\to\infty} \int f d\mu_n = \int f d\mu \tag{11.6.17}$$

(iii) Set $C_u(X) := \{f \in C(X) |\ f$ is uniformly continuous on $X\}$. For any $f \in C_b(X) \cap C_u(X)$

$$\lim_{n\to\infty} \int f d\mu_n = \int f d\mu \tag{11.6.18}$$

(iv) For any closed set A

$$\overline{\lim_{n\to\infty}} \mu_n(A) \leq \mu(A) \tag{11.6.19}$$

(v) For any closed set U

$$\underline{\lim_{n\to\infty}} \mu_n(U) \geq \mu(U) \tag{11.6.20}$$

(i) $\Longrightarrow$ *(ii):* Let fix arbitary $f \in C_b(X)$. Because $\cup_{a\in\mathbb{R}}\{f = a\} = X$ and $\mu(X) = 1$, for any $n \in \mathbb{N}$ $\{a \in \mathbb{R}|\mu(\{f = a\}) > \frac{1}{n}\}$ is a finite set. So $\{a \in \mathbb{R}|\mu(\{f = a\}) > 0\} = \cup_{n=1}^{\infty}\{a \in \mathbb{R}|\mu(\{f = a\}) > \frac{1}{n}\}$ is at most countable.

So there are $a_{m,i}{}_{1\leq m\leq\infty,1\leq i\leq\varphi(m)} \subset \mathbb{R}$ such that

$$-||f||_\infty > a_{m,1} \leq a_{m,2} \leq ... \leq a_{m,\varphi(m)} > ||f||_\infty \ (\forall m \in \mathbb{N}) \tag{11.6.21}$$

$$|a_{m,i} - a_{m,i+1}| \leq \frac{1}{2^m} \ (\forall m, \forall i) \tag{11.6.22}$$

$$\mu(\{f = a_{m,i}\}) = 0 \ (\forall m, \forall i) \tag{11.6.23}$$

For $m \in \mathbb{N}$ set

$$g_m := \Sigma_{i=1}^{\varphi(m)} a_{m,i+1}\chi_{\{a_{m,i}\leq f\leq a_{m,i+1}\}} \tag{11.6.24}$$

and

$$h_m := \Sigma_{i=1}^{\varphi(m)} a_{m,i}\chi_{\{a_{m,i}\leq f\leq a_{m,i+1}\}} \tag{11.6.25}$$

Because for any m and i $\partial\{a_{m,i} \leq f \leq a_{m,i+1}\} \subset \{f = a_{m,i}\} \cup \{f = a_{m,i+1}\}$, for any m and i

$$\mu(\partial\{a_{m,i} \leq f \leq a_{m,i+1}\}) = 0 \tag{11.6.26}$$

Let fix arbitary $\epsilon > 0$.

By Lebesugue's convergence theorem, there is $m \in \mathbb{N}$ such that $\int g_m d\mu - \int h_m d\mu \leq \epsilon$.

By (i),

$$\begin{aligned} \int f d\mu - \epsilon &\leq \int h_m d\mu \\ &= \lim_{n\to\infty} \int h_m d\mu_n \\ &\leq \underline{\lim}_{n\to\infty} \int f d\mu_n \end{aligned} \tag{11.6.27}$$

and

$$\begin{aligned} \int f d\mu + \epsilon &\geq \int g_m d\mu \\ &= \lim_{n\to\infty} \int g_m d\mu_n \\ &\geq \overline{\lim}_{n\to\infty} \int f d\mu_n \end{aligned} \tag{11.6.28}$$

Consequently, $\int f d\mu = \lim_{n\to\infty} \int f d\mu_n$.

□

(ii) $\Longrightarrow$ *(iii):* It's trivial. □

(iii) $\Longrightarrow$ *(iv):* Let fix arbitary closed set A. We set

$$f_n(x) := |1 - min(1, d(x, A))|^n \ (n \in \mathbb{N}, x \in x) \tag{11.6.29}$$

$f_n \in C_b(X) \cap C_u(X)$ $(\forall n)$ and $\lim_{n\to\infty} f_n \to \chi_A$ $(pointwiseconvergence)$ and

$$\int f_m d\mu_n \geq \mu_n(A) \tag{11.6.30}$$

By Lebesugue's convergence theorem,

$$\mu(A) \geq \overline{\lim}_{n\to\infty} \mu_n(A) \tag{11.6.31}$$

□

(iv) $\Longleftrightarrow$ *(v):* It's trivial. □

(iv) and (v) $\Longrightarrow$ (i): Let $A \in \mathscr{B}(X)$ and $\mu(\partial A) = 0$. By (iv),

$$\begin{aligned} \overline{\lim_{n\to\infty}} \mu_n(A) &\leq \overline{\lim_{n\to\infty}} \mu_n(\overline{A}) \\ &\leq \mu(\overline{A}) \\ &= \mu(\overline{A} \setminus A) + \mu(A) \\ &\leq \mu(\partial) + \mu(A) \\ &= \mu(A) \end{aligned} \tag{11.6.32}$$

In the same way as above we obtain

$$\underline{\lim_{n\to\infty}} \mu_n(A) \geq \mu(A) \tag{11.6.33}$$

Consequently

$$\lim_{n\to\infty} \mu_n(A) = \mu(A) \tag{11.6.34}$$

□

The following is the definition of a metric of $\mathscr{P}(\mathbb{R})$.

Proposition 11.6.6. *Let*

(S1) (X, d) is a compact metric space.

(S2) $\{f_n\}_{n=1}^{\infty}$ is a dense subset of (X, d). By Proposition2.2.12, such subsets always exist.

(S3) $\tau(\mu_1, \mu_2) := \Sigma_{n=1}^{\infty} |\int f_n d\mu_1 - \int f_n d\mu_2|$ $(\mu_1, \mu_2 \in \mathscr{P}(\mathbb{R}))$.

then the followings hold.

(i) τ is a metric on $\mathscr{P}(\mathbb{R})$.

(ii) for any $\{\mu_n\}_{n=1}^{\infty} \subset \mathscr{P}(\mathbb{R})$ and $\mu \in \mathscr{P}(\mathbb{R})$, $\mu_n \Longrightarrow \mu$ $(n \to \infty)$ is equivalent to $\tau(\mu_n, \mu) \to 0$ $(n \to \infty)$.

(i): Let fix $\mu_1 \in \mathscr{P}(X)$ and $\mu_2 \in \mathscr{P}(X)$ such that $\tau(\mu_1, \mu_2) = 0$. It is enough to show $\mu_1 = \mu_2$ for showing (i). By (S2), for any $f \in C_+(X)$ $\int f d\mu_1 = \int f d\mu_2$. By uniqueness in Proposition11.4.1, $\mu_1 = \mu_2$. □

(ii): Let us assume $\tau(\mu_n, \mu) \to 0$ $(n \to \infty)$. Let us fix arbitary $\epsilon > 0$. There is $m \in \mathbb{N}$ such that $||f - f_m||_\infty < \frac{\epsilon}{3}$. There is $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$

$$|\int_X f_m d\mu_n - \int_X f_m d\mu| < \frac{\epsilon}{3}. \tag{11.6.35}$$

For any $n \geq n_0$

$$\begin{aligned} |\int_X f d\mu_n - \int_X f d\mu| &< |\int_X f d\mu_n - \int_X f_m d\mu_n| \\ &\quad + |\int_X f_m d\mu_n - \int_X f_m d\mu| + |\int_X f_m d\mu - \int_X f_m d\mu| \\ &< \epsilon \end{aligned} \tag{11.6.36}$$

Consequently, $\mu_n \Longrightarrow \mu$ $(n \to \infty)$.

The inverse is clear. □

Proposition 11.6.7. *$(\mathscr{P}(X), \tau)$ is a compact metric space.*

Proof. By Proposition2.2.5, it is enough to show $(\mathscr{P}(X), \tau)$ is sequencially compact.

Let us fix arbitary $\mu_n{}_{n=1}^{\infty} \subset \mathscr{P}(X)$.

For any $m \in \mathbb{N}$, $\{\int f_m \mu_n\}_{n=1}^{\infty}$ is bounded.

For each $m \in \mathbb{N}$, there is $\{\varphi(m,n)\}_{n=1}^{\infty}$ such that $l(f_m) := \lim_{n\to\infty} \int f_m d\mu_{\varphi(m,n)}$ exists and $|l(f_m) - \int f_m d\mu_{\varphi(m,n)}| < \frac{1}{m}$ $(\forall n \geq m)$.

We set $\psi(m) := \varphi(m,m)$ $(m \in \mathbb{N})$.

By the definition of ψ, for any $m \in \mathbb{N}$ $l(f_m) = \lim_{n\to\infty} \int f_m d\mu_{\psi(n)}$.

Let us fix arbitary $f \in C_b(X)$ and $\epsilon > 0$. There is $k \in \mathbb{N}$ such that $||f - f_k|| < \frac{\epsilon}{3}$.

There is $n_0 \in \mathbb{N}$ such that for any $m \geq n_0$ and any $n \geq n_0$ $|\int f_k d\mu_{\psi(m)} - \int f_k d\mu_{\psi(m)}| < \frac{\epsilon}{3}$

So for any $m \geq n_0$ and any $n \geq n_0$ $|\int f d\mu_{\psi(m)} - \int f d\mu_{\psi(m)}| < \epsilon$.
So $l(f) := \lim_{m\to\infty} \int f d\mu_{\psi(m)}$ exists.
Clearly l is an elementary integral on $C_+(X)$.
So by Proposition11.4.1, there is $\mu \in \mathscr{P}(X)$ such that

$$l(f) = \int_X f d\mu \ (\forall f \in C_+(X)) \tag{11.6.37}$$

Clearly $\mu_{\psi(n)} \Longrightarrow \mu \ (n \to \infty)$.

□

Proposition 11.6.8. *Let*

(S1) (X, d) is a separable metric space.

(A1) $\{\mu_n\}_{n=1}^{\infty} \subset \mathscr{P}(X)$ is tight.

There is a subsequence $\mu_\varphi(n)_{\{n=1\}}^{\infty}$ and $\mu \in \mathscr{P}(X)$ such that $\mu_\varphi(n) \Longrightarrow \mu \ (n \to \infty)$.

Proof. Let $(\tilde{X}, \tilde{d})$ be a compact metric space in Proposition2.2.9 and $i : X \to \tilde{X}$ in Proposition2.2.9. By Proposition11.4.1, for each $n \in \mathbb{N}$ there is a measure $\tilde{\mu}_n$ such that for any $g \in C_+(\tilde{X})$ and $n \in \mathbb{N}$

$$\int_X g \circ i d\mu_n = \int_{\tilde{X}} g d\tilde{\mu}_n \tag{11.6.38}$$

There is an increasing sequence of compact sets $\{K_n\}_{n=1}^{\infty}$ such that

$$\mu_m(K_n) > 1 - \frac{1}{n} \tag{11.6.39}$$

$(\forall m \in \mathbb{N}, \forall n \in \mathbb{N})$
Let $K := \cup_{n=1}^{\infty} K_n$. By (11.6.39), for any $m \in \mathbb{N}$

$$\mu_m(K) = \tilde{\mu}_m(i(K)) = 1 \tag{11.6.40}$$

For $n \in \mathbb{N}$ and $x \in \tilde{X}$, $g_{m,n}(x) := (1 - min1, d(x, K_m))^n$. $\int_{\tilde{X}} g_{m,n} d\tilde{\mu}_l \geq \tilde{\mu}_m(K_m) \geq 1 - \frac{1}{m}$. By reaching $n \to \infty$, $\mu_m(K_m) = \tilde{\mu}(i(K_m)) \geq 1 - \frac{1}{m}$. By reaching $m \to \infty$,

$$\tilde{\mu}(i(K)) = 1 \tag{11.6.41}$$

By Proposition, there is a subsequence $\{\tilde{\mu}_{\varphi(n)}\}_{n=1}^{\infty}$ and $\tilde{\mu} \in \mathscr{P}(\tilde{X})$ such that $\tilde{\mu}_n \Longrightarrow \tilde{\mu} \ (n \to \infty)$.
Because for any $n \in \mathbb{N}$ $i(K_n)$ is compact, $i(K_n) \in \mathscr{B}(\tilde{X})$. So $i(K) \in \mathscr{B}(\tilde{X})$.
We will show

$$\mathscr{B}(X) \subset \mathscr{B} := \{A \subset X | i(A \cap K) \mathscr{B}(\tilde{X})\} \tag{11.6.42}$$

Because i is injective, if $\{A_n\}_{n=1}^{\infty} \subset \mathscr{B}$ then $\cup_{n=1}^{\infty} A_n \in \mathscr{B}$. And if $A\mathscr{B}$ then $i(A^c \cap K) = i(K) \cap i(A \cap K)^c \in \mathscr{B}$ So $\mathscr{B}$ is a σ-algebra. For any closed set A, $A \in \mathscr{B}$. So (11.6.42) holds.
For $A \in \mathscr{B}(X)$, we define

$$\mu(A) := \tilde{\mu}(i(A \cup K)) \tag{11.6.43}$$

By (11.6.41),

$$\mu(K) = 1 \tag{11.6.44}$$

Let me fix arbitary $f \in C_b(X) \cap C_u(X)$. Because $f \in C_u(X)$ and $i(X) isdensein \tilde{X}$, there is $\tilde{f} \in C_b(\tilde{X}) \cap C_u(\tilde{X})$ such that $\tilde{f}|i(X) = f \circ i^{-1}$.

By the definition of $\{\mu_n\}_{n=1}^{\infty}$ and μ,

$$\begin{aligned}\lim_{n\to\infty}\int_X f d\mu_n &= \lim_{n\to\infty}\int_X \tilde{f}\circ i d\mu_n \\ &= \lim_{n\to\infty}\int_{\tilde{X}} \tilde{f} d\tilde{\mu}_n \\ &= \int_{\tilde{X}} \tilde{f} d\tilde{\mu} \\ &= \int_{i(K)} \tilde{f} d\tilde{\mu} \\ &= \int_{i(K)} f\circ i^{-1} d\tilde{\mu} \\ &= \int_K f d\mu \\ &= \int_X f d\mu \end{aligned} \tag{11.6.45}$$

□

11.7 Characteristic functions of probability distribution

11.7.1 The Case of Single Variable

By Fubini's theorem, the following holds.

Proposition 11.7.1. *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S2) Let* $\mu \in \mathcal{P}(\mathbb{R})$*.*
- *(S3) Let* $f \in L^1(\mathbb{R})$*.*

then

$$\int_{\mathbb{R}} f(t)\varphi_\mu(t)dt = \int_{\mathbb{R}} \mathscr{F}^{-1}(f)(x)d\mu(x) \tag{11.7.1}$$

Proposition 11.7.2 (Uniqueness of Characteristic Function)**.** *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S2) Let* $\mu \in \mathcal{P}(\mathbb{R})$ *and* $\mu' \in \mathcal{P}(\mathbb{R})$*.*

If $\varphi_\mu = \varphi_{\mu'}$ *then* $\mu = \mu'$*.*

Proof. Let us arbitary $f \in C_c^\infty(\mathbb{R}^n)$. By Proposition2.5.8, $\mathscr{F}(f) \in L^1(\mathbb{R}^n)$. By Proposition2.5.7, $\int_{\mathbb{R}} f(x)d\mu(x) = \int_{\mathbb{R}} f(x)d\mu'(x)$. By Proposition2.5.3, $\mu = \mu'$. □

This proposition states that convergence of distributions in law is derived from each point convergence of the characteristic function.

Proposition 11.7.3 (Levy's Continuity Theorem(Single Variable Case))**.** *Let*

- *(S1)* $\{\mu_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathbb{R})$
- *(S2)* φ_n *is the characteristic function of* μ_n $(n = 1, 2, ...)$
- *(A1)* $\{\mu_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathbb{R})$ *then the followings are equivalent.*

  - *(i) There is a* φ *s.t* φ *is a measurable function on* $\mathbb{R}$ *and* φ *is continuous at* 0 *and* $\varphi(0) = 1$ *and* $\varphi_n \xrightarrow[n\to\infty]{} \varphi$ *(converge pointwise). Below, we fix such* φ*.*
  - *(ii) Then there is a* $\mu \in \mathcal{P}(\mathbb{R})$ *such that* φ *is the characteristic function of* μ *and* $\mu_n \Longrightarrow \mu$ $(n \to \infty)$*.*

(i) $\Longrightarrow$ *(ii).* The followings are strategy of the proof.
–Memo

(STEP1) Showing $\{\mu_n\}_{n=1}^{\infty}$ is tight.

(STEP2) Getting μ of the subject.

–

(STEP1)

For each $m \in \mathbb{N}$, there is a measurable function f_m such that f_m continuous at 0 and $f_m(0) = 1$ and $supp(f) \subset [\frac{-1}{m}, \frac{-1}{m}]$ is compact and $f_m \leq 1$ in $\mathbb{R}$ and $\mathcal{F}^{-1} f_m \leq 1$ in $\mathbb{R}$. $\{\chi_{[-\frac{1}{m}, \frac{1}{m}]}\}_{m=1}^{\infty}$ sutisfies the above conditions. Fix such $\{f_m\}_{m=1}^{\infty}$.

We get

$$\int_{\mathbb{R}} f_m(x)\varphi_n(x)dx = \int_{\mathbb{R}} \mathcal{F}^{-1} f_m(x) d\mu_n(x) \tag{11.7.2}$$

So

$$1 - \frac{m}{2}\int_{\mathbb{R}} f_m(x)\varphi_n(x)dx = 1 - \frac{m}{2}\int_{\mathbb{R}} \mathcal{F}^{-1} f_m(x) d\mu_n(x) \tag{11.7.3}$$

Call the left side of the above (11.7.3) $I_{m,n}$ and call the right side of the above (11.7.3) $J_{m,n}$. Fix any $\varepsilon > 0$.

(STEP1-1)

–Memo

We will show that $I_{m,n} < \varepsilon$ for sufficient large m, n. We will show this statement using the dominated convergence theorem and continuity of φ at 0

–

(STEP1-2)

–Memo

We will show that $J_{m,n} > \mu_n(\{x \in \mathbb{R}| \; |x| \geq m\})$ for sufficient large m, n. We will show this statement using the dominated convergence theorem and continuity of φ at 0

–

The following holds.

$$\mathcal{F}^{-1} f_m(x) = \frac{1}{m}\mathcal{F}^{-1} f_m(\frac{x}{m}) \tag{11.7.4}$$

So

$$\begin{aligned} J_{m,n} &= 1 - \frac{1}{2}\int_{\mathbb{R}} \mathcal{F}^{-1} f_m(\frac{x}{m}) d\mu_n(x) \\ &= \int_{\mathbb{R}} 1 - \frac{1}{2}\mathcal{F}^{-1} f_m(\frac{x}{m}) d\mu_n(x) \\ &= \int_{\{x\in\mathbb{R}| \; |x|\geq m\}} 1 - \frac{1}{2}\mathcal{F}^{-1} f_m(\frac{x}{m}) d\mu_n(x) \end{aligned} \tag{11.7.5}$$

In (11.7.5), we use statement $\mathcal{F}^{-1} f_m \leq 1$ in $\mathbb{R}$ ($\forall m \in \mathbb{N}$).

$$\begin{aligned} 1 - \frac{1}{2}\mathcal{F}^{-1} f_m(\frac{x}{m}) &\geq 1 - \frac{1}{2} max_{y\in supp(|f_m|)}|f_m(y)|\frac{m}{|x|} \\ &\geq \frac{1}{2} \end{aligned} \tag{11.7.6}$$

So

$$J_{m,n} \geq \frac{1}{2}\mu_n(\{x \in \mathbb{R}| \; |x| \geq m\}) \tag{11.7.7}$$

By (STEP1-1) and (11.7.7) for sufficient large m and n we get

$$2\epsilon \geq \mu_n(\{x \in \mathbb{R}| \; |x| \geq m\}) \tag{11.7.8}$$

So We have shown $\{\mu_n\}_{n=1}^{\infty}$ is tight.

(STEP2)

By (STEP1), there is a subsequence $\{\mu_{\psi(n)}\}_{n=1}^{\infty}$ which converges to a μ in law. It is enough to show for any subsequence of $\{\mu_n\}_{n=1}^{\infty}$ the subsequence has some subsequnece of the subsequence which converges to μ in law. Let fix any subsequence $\{\mu_{\omega(n)}\}_{n=1}^{\infty}$. There is a subsequence $\{\mu_{\omega(\alpha(n))}\}_{n=1}^{\infty}$ which converges to μ'. By increasing n to ∞ in (11.7.3) and Proposition11.6.3, $\phi_\mu = \phi$ and $\phi_{\mu'} = \phi$. By uniqueness of characteristic function, $\mu = \mu'$.

□

(ii) $\implies$ *(i)*. $\varphi_\mu : \mathbb{R} \ni t \mapsto \int_\Omega exp(itx)d\mu$. It is easy to show φ_μ is continuous at 0.

By Proposition11.6.3,

$$\int_{\mathbb{R}} exp(itx)d\mu(x) = \lim_{n\to\infty} \int_{\mathbb{R}} exp(itx)d\mu_n \ (\forall t) \tag{11.7.9}$$

□

11.7.2 The Case of Multi variables

Proposition 11.7.4 (Levy's continuity theorem(multi variate case))**.** *Let*

- *(S1)* $\{\mu_n\}_{n=1}^\infty \subset \mathcal{P}(\mathbb{R}^N)$
- *(S2)* φ_n *is the characteristic function of* μ_n $(n = 1, 2, ...)$
- *(A1)* $\{\mu_n\}_{n=1}^\infty \subset \mathcal{P}(\mathbb{R}^N)$
- *(A1) There is a* φ *s.t* φ *is a measurable function on* $\mathbb{R}^N$ *and* φ *is continuous at* 0 *and* $\varphi(0) = 1$ *and* $\varphi_n \xrightarrow[n\to\infty]{} \varphi$ *(converge pointwise). Below, we fix such* φ*.*

Then there is a $\mu \in \mathcal{P}(\mathbb{R}^N)$ *such that* φ *is the characteristic function of* μ *and* $\mu_n \Longrightarrow \mu$ $(n \to \infty)$*.*

Proof. By an argument which is similar to the proof of Proposition11.7.3, we can show that $\{\mu_n\}_{n=1}^\infty$ is tight.

By Proposition11.6.8 and uniqueness of fourier transformation in $\mathbb{R}^N$ and Proposition11.6.5, there is $\mu \in \mathscr{P}(R)^N$ such that $\mu_n \Longrightarrow \mu$ $(n \to \infty)$ and $\varphi_\mu = \varphi$.

□

11.8 Central limit theorem

11.8.1 The Case of Single Variable

Theorem 11.8.1 (Central limit theorem)**.** *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S2)* $\{X_i\}_{i=1}^\infty$ *is a sequence of random variables on* $(\Omega, \mathscr{F}, P)$*.*
- *(A1)* $\exists\mu$ *such that* $X_i \sim \mu$ $(\forall i)$*. Bellow, we fix such* μ*.*
- *(A2)* $\{X_i\}_{i=1}^N$ *are independent for any* $N \in \mathbb{N}$*.*
- *(A3)* $E[\mu] = \nu$ *and* $V[\mu] = \sigma^2$ *and* $\sigma > 0$*.*

then $P_{\sqrt{n}(\bar{X}-\nu)}$ *weakly converges to* $N(0, \sigma)$*.*

Proof. We can assume $\nu = 0$ and $\sigma = 1$. Bellow, we assume that.
Let $Y_{i,n} := \frac{X_i}{\sqrt{n}}$ $(i = 1, 2, ..., n)$ and $Y_n := \sum_{i=1}^n Y_{i,n}$ $(n = 1, 2, ...)$. By $(A1)$, $\varphi_{Y_{i,n}} = \varphi_{Y_{1,n}}$ $(\forall i, \forall n)$. Let $\varphi_n := \varphi_{Y_n}$ and $\psi_n := \varphi_{Y_{1,n}}$ $(n = 1, 2, ...)$. And let $\psi_\mu : \mathbb{R} \ni s \mapsto \int_{\mathbb{R}} exp(isx)d\mu(x)$. Then $\varphi_n = (\psi_n)^n$ and $\psi_n(t) = \psi_\mu(\frac{t}{\sqrt{n}})$ and $(\forall t \in \mathbb{R})$. We will show the following equation. By Proposition11.5.7,

$$\varphi_{Y_{1,n}}(t) = 1 - \frac{t^2}{2n} + o(\frac{1}{n})(n \to \infty) \tag{11.8.1}$$

By the above equation and Proposition11.5.4,

$$\varphi_n(t) = (1 - \frac{t^2}{2n} + o(\frac{1}{n}))^n \to exp(-\frac{t^2}{2}) \ (n \to \infty) \tag{11.8.2}$$

By Proposition11.7.3, there is a $\mu_0 \in \mathscr{P}(\mathbb{R})$ such that $P_{\sqrt{n}\bar{X}}$ converges to μ_0 in law and $\varphi_{\mu_0} = exp(-\frac{(\cdot)^2}{2})$. Because $\varphi_{N(0,1)} = exp(-\frac{(\cdot)^2}{2})$ and uniqueness of characteristic function, $P_{\sqrt{n}\bar{X}}$ converges to $N(0, 1)$ □

11.8.2 The Case of Multi Variables

Theorem 11.8.2 (Central Limit Theorem(Multi Variables Case))**.** *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S2)* $\{X_i\}_{i=1}^{\infty}$ *is a sequence of* N*-dimensional vectors of random variables on* $(\Omega, \mathscr{F}, P)$*.*
- *(A1)* $\exists \mu$ *such that* $X_i \sim \mu$ $(\forall i)$*. Bellow, we fix such* μ*.*
- *(A2)* $\{X_i\}_{i=1}^{n}$ *are independent for any* $n \in \mathbb{N}$*.*
- *(A3)* $E[\mu] = \nu$ *and* $cov[\mu] = \sigma^2$ *and* σ *is* N*-by-*N *positive definite symmetric matrix.*

then $P_{\sqrt{n}(\bar{X}-\nu)}$ *weakly converges to* $N(0, \Sigma)$*.*

Proof. Let us fix arbitary $\boldsymbol{t} \in \mathbb{R}^N$ and $s \in \mathbb{R}$. Let us set $Y_n := s\boldsymbol{t}^T(X_n - \nu)$.

The following holds.

$$\varphi_{\sqrt{n}(\bar{X}-\nu)}(s\boldsymbol{t}) = E(exp(\sqrt{n}is\boldsymbol{t}^T(\bar{X} - \nu))) = \varphi_{\sqrt{n}(\bar{Y}-\nu)}(s) \tag{11.8.3}$$

By Theorem11.8.1 and Proposition11.7.3 and Proposition11.5.3,

$$\lim_{n\to\infty} \varphi_{\sqrt{n}(\bar{Y}-\nu)}(s) = exp(-\frac{s^2\boldsymbol{t}^T\Sigma^2\boldsymbol{t}}{2}) \tag{11.8.4}$$

By setting $s = 1$,

$$\lim_{n\to\infty} \varphi_{\sqrt{n}(\bar{X}-\nu)}(s\boldsymbol{t}) = exp(-\frac{\boldsymbol{t}^T\Sigma^2\boldsymbol{t}}{2}) \tag{11.8.5}$$

By Proposition11.7.4 and Proposition11.5.3, $P_{\sqrt{n}(\bar{X}-\nu)}$ weakly converges to $N(0, \Sigma)$.

□

11.9 Law of large numbers

Proposition 11.9.1 (Weak law of large numbers)**.** *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(A1)* $\{X_i\}_{i=1}^{\infty}$ *is a sequence of independent random variables on* $(\Omega, \mathscr{F}, P)$*.*
- *(A2) There is a* $\mu \in \mathscr{P}(\mathbb{R})$ *such that* $X_i \sim \mu(\forall i)$*.*
- *(A3)* $E[\mu] = \nu$ *and* $V[\mu] = \sigma^2$ *exist.*

then the followings hold.

(i) $\{X_i\}_{i=1}^{\infty}$ *stochastic converges to* μ*, i.e., for any* $\epsilon > 0$

$$\lim_{n\to\infty} \mu(|\bar{X} - \mu| \geq \epsilon) = 0 \tag{11.9.1}$$

Hereafter we denote stochastic convergence by $\xrightarrow[N\to\infty]{\mathrm{p}}$ *or* plim.

(ii) For any $\epsilon > 0$*,*

$$\mu(|\bar{X} - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2} \tag{11.9.2}$$

A proof using Chebyshev's inequality. For any $n \in \mathbb{N}$,

$$\begin{aligned} \mu(|\bar{X} - \mu| \geq \epsilon) &= \frac{\epsilon^2\mu(|\bar{X} - \mu|^2 \geq \epsilon^2)}{\epsilon^2} \\ &\leq \frac{1}{\epsilon^2}\int_{\{|\bar{X}-\mu|^2\geq\epsilon^2\}} \epsilon^2 dP \\ &\leq \frac{1}{\epsilon^2}V[\bar{X}] = \frac{\sigma^2}{n\epsilon^2} \end{aligned} \tag{11.9.3}$$

This implies the above equation. □

A proof using Central limit theorem. By resetting $X_i \to \dfrac{X_i - \mu}{\sigma}$, we can assume $\mu = 0$ and $\sigma = 1$. Let us fix arbitary $\epsilon > 0$ and $\delta > 0$. There is $a > 0$ such that

$$N(0,1)((-\infty, -a) \cup (a, \infty)) < \delta \tag{11.9.4}$$

By Central limit theorem, there is $n_0 \in \mathbb{N}$ such that

$$\frac{a}{\sqrt{n_0}} < \delta \tag{11.9.5}$$

and for any $n \geq n_0$

$$|\mu(|\sqrt{n}\bar{X}| \geq a) - N(0,1)((-\infty, -a) \cup (a, \infty))| < \delta \tag{11.9.6}$$

So for any $n \geq n_0$

$$\begin{aligned} \mu(|\bar{X}| \geq \epsilon) &\leq \mu(|\bar{X}| \geq \frac{a}{\sqrt{n}}) = \mu(\sqrt{n}|\bar{X}| \geq a) \\ &\leq 2\delta \end{aligned} \tag{11.9.7}$$

So $\overline{\lim_{n\to\infty}} \mu(|\bar{X}| \geq \epsilon) \leq 2\delta$. Consequently, $\lim_{n\to\infty} \mu(|\bar{X}| \geq \epsilon) = 0$. □

11.10 Multivariate normal distribution

Remark 11.10.1. *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S2)* $X := (X_1, ..., X_n)$ *is a vector of random variables.*
- *(S3)* A *is a* (m, n) *matrix.*
- *(A1)* $(X_1, ..., X_n) \sim N(0, E_n)$.

then $cov(AX) = AA^T$.

The following Proposition11.16.4 is used to prove the Proposition11.16.7 discussed later.

Proposition 11.10.2. *Let*

- *(A1)* $X := (X_1, X_2, ..., X_p)^T \sim N(\gamma, BB^T)$, *where* B *is a* (p, q) *matrix.*
- *(S1) Let* $s \in [1, p-1] \cap \mathbb{N}$ *and* $X^{(1)} := (X_1, ..., X_s)$ *and* $X^{(2)} := (X_{s+1}, ..., X_p)$.
- *(A2)* $cov(X^{(1)}, X^{(2)}) = 0$.

then $X^{(1)}$ *and* $X^{(2)}$ *are independent.*

Proof. The following proof consists of two steps.
STEP1. General case

In this step, we will show that it is enough to show the Proposition when $r := rank(B) = p \leq q$. For each $i \in \mathbb{N} \cap [1, p]$, let b_i be the i-th row vector of B. Let V_1 be the vector space generated from $b_1, b_2, ..., b_s$ and let V_2 be the vector space generated from $b_{s+1}, b_{s+2}, ..., b_p$. We can take $\{b_{\sigma(i)}\}_{i=1}^{r_1}$ is a basis of V_1 and $\{b_{\tau(i)}\}_{i=1}^{r_2}$ is a basis of V_2. Since $V_1 \perp V_2$, $\{b_{\sigma(i)}\}_{i=1}^{r_1} \cap \{b_{\tau(i)}\}_{i=1}^{r_2} = \phi$ and $\{b_{\sigma(i)}\}_{i=1}^{r_1} \cup \{b_{\tau(i)}\}_{i=1}^{r_2}$ are linear independent. So it is enough to show $\{b_{\sigma(i)}\}_{i=1}^{r_1}$ and $\{b_{\tau(i)}\}_{i=1}^{r_2}$ are independent when $rank(B)$ is the number of rows of B.

STEP2. Case when $rank(B) = p \leq q$

Let W be the orthogonal complement of the vector space generated from $b_1, b_2, ..., b_p$. We can take $c_1, ..., c_{(q-p)}$ which is an orthonormal basis of W and let
$C := \begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_{(q-p)} \end{bmatrix}$, and let $D := \begin{bmatrix} B \\ C \end{bmatrix}$. By (A1), there are random variables $\{\epsilon\}_{i=1}^{p}$ on $(\Omega, \mathscr{F})$ and random variables $\{\ Y\}_{i=1}^{q-p}$ on $(\Omega, \mathscr{F})$ such that $\boldsymbol{\epsilon} := \{\epsilon\}_{i=1}^{q}$ are $i.i.d$ and $\epsilon_i \sim N(0,1)$ $(\forall i)$
and $Z := \begin{bmatrix} X \\ Y \end{bmatrix} = D\boldsymbol{\epsilon} + \gamma$ and $cov(Z) = DD^T$.

The distribution of Z has the density function $f_q : \mathbb{R}^q \ni x \mapsto c \cdot exp(x^T DD^T x) \in \mathbb{R}$, where c is a constant. By (A2) and the definition of C,

$DD^T = \begin{bmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Sigma_2 & 0 \\ 0 & 0 & E_{(q-p)} \end{bmatrix}$, where Σ_1 and Σ_2 are symmetric positive definite matrixies. So the distribution of X has the density function $f_p : \mathbb{R}^p \ni x \mapsto d \cdot exp({x^{(1)}}^T \Sigma_1 x^{(1)}) \cdot exp({x^{(2)}}^T \Sigma_1 x^{(2)}) \in \mathbb{R}$, where d is a constant and $x^{(1)} = (x_1, ..., x_s)$ and $x^{(2)} = (x_{s+1}, ..., x_p)$. By the format of f_p, $X^{(1)}$ and $X^{(2)}$ are independent. □

11.11 Popular Probability Distributions

11.11.1 General Topics on Random Variables

By the definition of independence, the following clearly holds.

Proposition 11.11.1. *Let*

(S1) $(\S_i, \mathscr{S}, P_i)$ $(i = 1, 2, ..., N)$ *is a sequence of probability spaces.*

(S2) $(\Omega, \mathscr{F}, P)$ *is the probability spaces which is direct product of* $(\S_i, \mathscr{S}, P_i)$ $(i = 1, 2, ..., N)$

(S3) X_i *is a random variable on* S_i $(i = 1, 2, ..., N)$.

(S3) We set $Y_i := X_i \circ \pi_i$ $(i = 1, 2, ..., N)$.

then $Y_1, ..., Y_N$ *is a sequence of independent random variables.*

The following clearly holds.

Proposition 11.11.2. *Let* P *is probability measure on* $(\Omega := \mathbb{N} \cup \{0\}, 2^{\Omega})$. *Then* id_Ω *is random variable on* Ω *and* $id_\Omega \sim P$.

By Fubini's theorem(see [40]), the following two propositions clearly holds.

Proposition 11.11.3 (Marginal distribution)**.** *Let*

(S1) $(\Omega_i, \mathscr{F}_i, P_i)$ *is a probability spaces* $(i = 1, 2)$.

(A1) $P_1 \times P_2$ *has a density function* f_{P_1,P_2}.

Then for almost everywhere $x \in \mathbb{R}$, $f_{P_1,P_2}(x, \cdot)$ *is measurable and*

$$f_{P_1}(x) := \int_{\mathbb{R}} f_{P_1,P_2}(x, y) dP_2(y)$$

exists and f_{P_1} *is measurable and*

$$\int_{\mathbb{R}} f_{P_1}(x) dP_1(x) = 1$$

Proposition 11.11.4 (Conditional probability density function)**.** *Let*

(S1) $(\Omega_i, \mathscr{F}_i, P_i)$ *is a probability spaces* $(i = 1, 2)$.

(A1) $P_1 \times P_2$ *has a density function* $f_{X,Y}$.

(S2) $x \in \mathbb{R}$ *such that* $f_{X,Y}(x, \cdot)$ *is measurable and* $f_X(x) > 0$.

(S3) Set

$$f_{P_2|P_1(x)}(y) := \frac{f_{P_1,P_2}(x, y)}{f_{P_1}(x)} \ (y \in \mathbb{R})$$

We call $f_{P_2|P_1(x)}$ *the conditional probability density function of* P_2 *given the occurrence of the value* x *of* P_1.

Then

$$\int_{\mathbb{R}} f_{P_2|P_1(x)}(y) dP_2(y) = 1$$

The following definitions are based on [41].

Definition 11.11.5 (Probability model, True distribution, Prior probability)**.** *The followings are settings and assumptions.*

(A1) Q *is a probability borel measure on* $\mathbb{R}^N$ *and* Q *has the density function* q. *We call* q *a true distribution.*

(S1) W *is a Borel set of* $\mathbb{R}^d$.

(A2) Φ *is a probability borel measure on* W *that has the density function* ϕ. *We call* ϕ *a prior probability.*

(A3) $Q \times \Phi$ *has the densition function* p.

(S2) We set $p(\cdot_1|\cdot_2)$ *by for* $w \in W$ *such that* $\phi(w) > 0$

$$p(x|w) := p_{Q|\Phi(w)}(x) \ (x \in \mathbb{R}^N)$$

We call $p(\cdot_1|\cdot_2)$ *the a probability model. Or, we denote* $p(\cdot_1|\cdot_2)$ *by* $p(x|w)$.

Definition 11.11.6 (Exponential family)**.** *The followings are settings and assumptions.*

(S1) (Q, q, W, Φ, ϕ, p) *is a pair of true distribution, prior probability, probability model.*

We say p *is in exponential family if there are* v, g, f *such that* f *is a borel measurable map from* W *to* $\mathbb{R}^J$ *and* g *are borel measurable maps from* $\mathbb{R}^N$ *to* $\mathbb{R}^J$ *and* v *is a borel measurable function on* W *and for any* $x \in \mathbb{R}^N$ *and any* $w \in W$ *such that* $\phi(w) > 0$

$$p(x|w) = v(x)exp(f(w) \cdot g(x))$$

Definition 11.11.7 (Conjugate prior distribution)**.** *The followings are settings and assumptions.*

(S1) $(Q, q, W, \Phi, \phi, p, v, g, f)$ *is in exponential family.*

(S2) $v \in \mathbb{R}^J$.

Then, we set

$$\varphi(u, v) := \varphi(u|v) := \frac{exp(v \cdot f(u))}{\int_W exp(v \cdot f(w))d\Phi(w)} \ (u \in W), z(v) := \int_W exp(v \cdot f(w))d\Phi(w)$$

We call $\varphi(\cdot_1|\cdot_2)$ *the conjugate prior distribution of the exponential family* $(Q, q, W, \Phi, \phi, p, v, g, f)$.

The following is clear.

Proposition 11.11.8 (Posterior Probability Distribution)**.** *The followings are settings and assumptions.*

(S1) (Q, q, W, Φ, ϕ, p) *is a probability model.*

(A2) q *is continuous.*

(S2) $X^n := \{X_i\}_{i=1}^n$ *is a sequence of* $\mathbb{R}^N$*-valued random variables such that* $X_i \sim Q$.

(A3) p *is continuous and for any* $x_1, ..., x_n \in q^{-1}((0, \infty))$ *there is* $w \in W$ *such that* $p(x_i, w) > 0$ $(\forall i \in \mathbb{N})$.

(A4) ϕ *is continuous and* $\phi > 0$ *in* W.

(S3) $\beta > 0$.

Then,

$$Z_n(\beta) := \int_W \phi(w)\Pi_{i=1}^n p(X_i|w)^\beta d\Phi(w) > 0$$

We set

$$r(w, X^n) := r(w|X^n) := \phi(w)\Pi_{i=1}^n p(X_i|w)^\beta \frac{1}{Z_n(\beta)} \ (w \in W)$$

We call $r(\cdot|X^n)$ *is the posterior distribution of* p. *And we call* β *an inverse temperature and* $Z_n(\beta)$ *the partition function, respectively.*

Proposition 11.11.9. *The followings are settings and assumptions.*

(S1) $(Q, q, W, \Phi, \phi, p, v, g, f)$ *is an exponential family.*

(A2) q *is continuous.*

(S2) $X^n := \{X_i\}_{i=1}^n$ *is a sequence of* $\mathbb{R}^N$*-valued random variables such that* $X_i \sim Q$.

(A3) p *is continuous and for any* $x_1, ..., x_n \in q^{-1}((0, \infty))$ *there is* $w \in W$ *such that* $p(x_i, w) > 0$ $(\forall i \in \mathbb{N})$.

(A4) ϕ *is continuous and* $\phi > 0$ *in* W.

(S3) $\beta > 0$ *is an inverse temperature.*

(S4) $v \in \mathbb{R}^J$.

(S5) $\hat{v} := v + \sum_{i=1}^n \beta g(X_i)$.

Then

(i) *The partiation function is represented as below.*

$$Z_n(\beta) = (\Pi_{i=1}^n v(X_i)^\beta)\frac{z(\hat{v})}{z(v)}$$

(ii) *The posterior probability distribution is represented as below.*

$$r(w|X^n) := \varphi(w|\hat{v})$$

Proof of (i).

$$\begin{aligned} Z_n(\beta) &:= \int_W \phi(w)\Pi_{i=1}^n p(X_i|w)^\beta d\Phi(w) = \int_W \varphi(w|v)\Pi_{i=1}^n p(X_i|w)^\beta d\Phi(w) \\ &= \int_W \varphi(w|v)\Pi_{i=1}^n(v(X_i)\exp(f(w)\cdot g(X_i))))^\beta d\Phi(w) = \frac{1}{z(v)}\int_W \exp(v\cdot f(w))\Pi_{i=1}^n(v(X_i)\exp(f(w)\cdot g(X_i))))^\beta d\Phi(w) \\ &= \frac{1}{z(v)}\int_W \Pi_{i=1}^n v(X_i)^\beta \exp((v+\beta\sum_{i=1}^n g(X_i))\cdot f(w))d\Phi(w) = \frac{z(\hat{v})}{z(v)}\Pi_{i=1}^n v(X_i)^\beta \end{aligned}$$

□

Proof of (ii).

$$\begin{aligned} r(w|X^n) &:= \phi(w)\Pi_{i=1}^n p(X_i|w)^\beta \frac{1}{Z_n(\beta)} = \varphi(w|v)\Pi_{i=1}^n p(X_i|w)^\beta \frac{z(v)}{z(\hat{v})\Pi_{i=1}^n v(X_i)^\beta} \\ &= \frac{\exp(v\cdot f(w))}{z(v)}(\Pi_{i=1}^n(v(X_i)\exp(f(w)\cdot g(X_i))))^\beta)\frac{z(v)}{z(\hat{v})\Pi_{i=1}^n v(X_i)^\beta} = \frac{\exp(\hat{v}\cdot f(w))}{z(\hat{v})} = \varphi(w|\hat{v}) \end{aligned}$$

□

11.11.2 Probability Generating Function

Definition 11.11.10 (Probability Generating Function)**.** *Let*

(S1) $(\Omega = \mathbb{N}\cup 0, 2^\Omega, P)$ *is a probability space.*

then we set

$$G_P(z) := \sum_{i=0}^\infty P(i)z^i \ (z\in\mathbb{C}) \tag{11.11.1}$$

Proposition 11.11.11. *The followings hold.*

(i) *Radius of convergence of* $G_P(z)$ *is not less than 1.*

(ii) *If* $G_P = G_{P'}$ *then* $P = P'$.

(iii) *If* Y *is a random variable on any probability space such that* $Y \sim P$ *then* $G_P(z) = E(z^Y)$ *for any* $z\in D(0,1)$.

(iii) *If* Y_1, Y_2 *is a random variable on any probability space such that* Y_1, Y_2 *are independent then* $G_{P_{Y_1+Y_2}} = G_{P_{Y_1}}G_{P_{Y_2}}$.

proof of (i). Because $0\le P\le 1$, (i) holds. □

proof of (ii). By (i) and definition of G_P and G'_P, (ii) holds. □

proof of (iii). Let us fix any $z\in D(0,1)$. For any $N\in\mathbb{N}$,

$$\begin{aligned} E(z^Y) &= \sum_{i=0}^N \int_{\{Y=i\}} z^Y dQ + \int_{\{Y>N\}} z^Y dQ \\ &= \sum_{i=0}^N P(i)z^i + \int_{\{Y>N\}} z^Y dQ \end{aligned} \tag{11.11.2}$$

So

$$|E(z^Y) - \sum_{i=0}^{N} P(i)z^i| \leq |\int_{Y>N} z^Y dQ| \leq Q(\{Y > N\}) \tag{11.11.3}$$

Consequently (iii) holds. □

proof of (iv). It is enough to show (iv) by (iii). □

11.11.3 Bernoulli distribution

Definition 11.11.12 (Bernoulli distribution)**.** *We call a probability distribution P on $\{0,1\}$ the Bernoulli distribution if for some $p \in [0,1]$ $P(\{1\}) = p$ and $P(\{0\}) = 1 - p$.*

Proposition 11.11.13 (Expectation and Variance of Bernoulli distribution)**.** *Let us assumel a probability distribution P on $\{0,1\}$ is the Bernoulli distribution with $P(\{1\}) = p$.*

(i) $E(P) = p$

(ii) $V(P) = p(1-p)$,

(i). It is trivial. □

(i). $V(P) = \int_{\{0,1\}} x^2 dP - E(P)^2 = \int_{\{0,1\}} x dP - p^2 = p - p^2 = p(1-p)$ □

11.11.4 Binomial distribution

Definition 11.11.14 (Binomial distribution)**.** *For some $p \in [0,1]$ and $n \in \mathbb{N}$ we call a probability distribution $B(n,p)$ on $\{0,1,...,n\}$ the Binomial distribution if $B(n,p)(\{i\}) = {}_nCip^i(1-p)^(n-i)$ $(i = 0,1,...,n)$.*

Clearly the following holds.

Proposition 11.11.15. *Let*

(S1) $(\Omega, \mathscr{F}, P)$ *is a probability space.*

(S2) $\{X_i\}_{i=1}^n$ *be independent random variables.*

(A1) The distribution of X_i is the Bernoulli distribution B with $B(\{1\}) = p$ $(\forall i)$.

then the distribution of $\Sigma_{i=1}^n X_i$ is $B(n,p)$.

By Proposition Proposition11.11.2 and Proposition11.11.1, Random variables like the one above exist. $E(B(2,p)) = 1 \cdot {}_2C_1 p(1-p) + 2 \cdot {}_2C_2 p^2 = 2p + 0 \cdot p^2 = 2p$. $E_{B(2,p)}(x^2) = 2p + 2^2p^2 - 2p^2$. $E(B(3,p)) = 1 \cdot {}_3C_1 p(1-p)^2 + 2 \cdot {}_3C_2 p^2(1-p) + 3p^3 = 3p + 0 \cdot p^2 + 0 \cdot p^3 = 3p$. $E_{B(3,p)}(x^2) = 3p + 3^3p^2 - 3p^2 + 0 \circ p^3$. We can extend these fact to the following lemma and the following proposition.

Lemma 11.11.16.

(i) $\Sigma_{k=1}^l k_l C_k (-1)^k = 0$ $(\forall l \geq 2)$.

(ii) $\Sigma_{k=1}^l k^2{}_l C_k (-1)^k = 0$ $(\forall l \geq 3)$.

(i). $L(x) := (1-x)^l = \Sigma_{k=1}^l {}_l C_k (-1)^k (-1)^k x^k$.
$L'(x) = l(1-x)^{l-1} = \Sigma_{k=1}^l k_l C_k (-1)^k (-1)^k x^{k-1}$.

So, if $l \geq 2$, then

$$\begin{aligned} 0 &= L'(1) \\ &= \Sigma_{k=1}^l k_l C_k (-1)^k (-1)^k \end{aligned} \tag{11.11.4}$$

□

(ii). $L(x) := (1-x)^l = \Sigma_{k=1}^l {}_l C_k (-1)^k (-1)^k x^k$.
$L''(x) = l(1-x)^{l-1} = \Sigma_{k=1}^l k(k-1)_l C_k (-1)^k (-1)^k x^{k-2}$.

So, if $l \geq 3$, then

$$\begin{aligned} 0 &= L''(1) \\ &= \Sigma_{k=1}^l k(k-1)_l C_k (-1)^k (-1)^k \\ &= \Sigma_{k=1}^l k^2{}_l C_k (-1)^k (-1)^k - \Sigma_{k=1}^l k_l C_k (-1)^k (-1)^k \end{aligned} \tag{11.11.5}$$

By (i), $\Sigma_{k=1}^l k_l C_k (-1)^k (-1)^k = 0$. So $\Sigma_{k=1}^l k^2{}_l C_k (-1)^k (-1)^k = 0$. □

Proposition 11.11.17 (Expectation and Variance of Binomial distribution)**.**

(i) $E(B(n,p)) = np$

(ii) $V(B(n,p)) = np(1-p)$

proof1 of (i). Let us take $\{X_i\}_{\{}i = 1, 2., , , , n\}$ in Proposition11.11.15. $E(B(n,p)) = E(\Sigma_{i=1}^n X_i) = \Sigma_{i=1}^n E(X_i) = np$ □

proof1 of (ii). Let us take $\{X_i\}_{\{}i = 1, 2., , , , n\}$ in Proposition11.11.15. $V(B(n,p)) = \Sigma_{i=1}^n V(X_i) = np(1-p)$ □

proof2 of (i).

$$\begin{aligned}
E(B(n,p)) &= \sum_{k=1}^{n} k_n C_k p^k (1-p)^{n-k} \\
&= \sum_{k=1}^{l} k_n C_k p^k \sum_{i=0}^{n-k} {}_{n-k}C_i (-1)^i p^i \\
&= \sum_{l=1}^{n} \sum_{k=1,2\dots,l,\ i=0,1,\dots,n-k,\ k+i=l} k_n C_k p^k {}_{n-k}C_i (-1)^i p^i \\
&= \sum_{l=1}^{n} p^l \sum_{k=1,2\dots,l,\ i=0,1,\dots,n-k,\ k+i=l} k_n C_{k n-k} C_i (-1)^i \\
&= \sum_{l=1}^{n} p^l \sum_{k=1}^{l} k_n C_k \cdot {}_{n-k}C_{l-k} (-1)^{l-k} \\
&= \sum_{l=1}^{n} (-1)^l p^l \sum_{k=1}^{l} k_n C_k \cdot {}_{n-k}C_{l-k} (-1)^k \\
&= \sum_{l=1}^{n} (-1)^l p^l \sum_{k=1}^{l} k \frac{{}_nP_l}{k!(l-k)!} (-1)^k \\
&= \sum_{l=1}^{n} (-1)^l p^l \sum_{k=1}^{l} k \frac{{}_nC_l \cdot l!}{k!(l-k)!} (-1)^k \\
&= \sum_{l=1}^{n} (-1)^l p^l {}_nC_l \sum_{k=1}^{l} k \frac{l!}{k!(l-k)!} (-1)^k \\
&= \sum_{l=1}^{n} (-1)^l p^l {}_nC_l \sum_{k=1}^{l} k_l C_k (-1)^k \qquad (11.11.6)
\end{aligned}$$

By Lemma11.11.16, for any $l \geq 2$, $\sum_{k=1}^{l} k_l C_k (-1)^k = 0$. So $E(B(n,p)) = np$. □

proof2 of (ii). By the proof2 of (ii),

$$E_{B(n,p)}(x^2) = \sum_{l=1}^{n} (-1)^l p^l {}_nC_l \sum_{k=1}^{l} k^2{}_l C_k (-1)^k \qquad (11.11.7)$$

By Lemma11.11.16, for any $l \geq 3$, $\sum_{k=1}^{l} k^2{}_l C_k (-1)^k = 0$.
So $E_{B(n,p)}(x^2) = \sum_{l=1}^{2} (-1)^l p^l {}_nC_l \sum_{k=1}^{l} k^2{}_l C_k (-1)^k = np(1-p) + n^2p^2$. By (i), $V(B(n,p)) = E_{B(n,p)}(x^2) - E(B(n,p))^2 = np(1-p)$. □

11.11.5 Geometric distribution

Definition 11.11.18 (Geometric distribution)**.** *Let* $p \in (0,1)$.

$$P(k) := (1-p)^{k-1} p \ (k = 1, 2, ...) \qquad (11.11.8)$$

We call P *is Geometric distribution with* p

Clearly P is a probability measure on $\{1, 2, ..., n, ...\}$.

Proposition 11.11.19. *Let P is Geometric distribution with p. Then*

$$G_P(z) = \frac{pz}{1-(1-p)z} \tag{11.11.9}$$

Proof.

$$\begin{aligned} G_P(z) &= \sum_{k=1}(1-p)^{k-1}pz^k \\ &= pz\sum_{k=1}(1-p)^{k-1}pz^{k-1} \\ &= pz\frac{1}{1-(1-p)z} \end{aligned} \tag{11.11.10}$$

□

Proposition 11.11.20. *Let P is Geometric distribution with p. Then*

$$E(P) = \frac{1}{p} \tag{11.11.11}$$

and

$$V(P) = \frac{1-p}{p^2} \tag{11.11.12}$$

proof1 of (11.11.11).

$$G'_P(z) = \frac{p(1-(1-p)z) + pz(1-p)}{(1-(1-p)z)^2}$$

So

$$\begin{aligned} E(P) &= G'_P(1) = \frac{p(1-(1-p)1) + p1(1-p)}{(1-(1-p)1)^2} \\ &= \frac{p^2+p-p^2)}{p^2} = \frac{1}{p} \end{aligned} \tag{11.11.13}$$

□

proof2 of (11.11.11).

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \tag{11.11.14}$$

By calculating the derivative,

$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^{k-1} \tag{11.11.15}$$

So

$$E(P) = p\sum_{k=1}^{\infty} k(1-(1-p))^{k-1} = p\frac{1}{(1-(1-p))^2} = \frac{1}{p} \tag{11.11.16}$$

□

proof of (11.11.12). By calculating the derivative of (11.11.17),

$$\frac{2}{(1-x)^3} = \sum_{k=2}^{\infty} k(k-1)x^{k-2} \tag{11.11.17}$$

So

$$\begin{aligned} E_P(x(x-1)) &= p\sum_{k=2}^{\infty} k(k-1)(1-p)^{k-1} \\ &= p(1-p)\sum_{k=2}^{\infty} k(k-1)(1-p)^{k-2} \\ &= p(1-p)\frac{2}{(p^3} = \frac{2(1-p)}{p^2} \end{aligned} \tag{11.11.18}$$

$$V(P) = E_P(x(x-1)) + E_P(x) - E_P(x)^2 = \frac{2(1-p)}{p^2} + \frac{p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2} \tag{11.11.19}$$

□

11.11.6 Negative binomial distribution

Definition 11.11.21 (Negative binomial distribution)**.** *We call a probability distribution P on $\{1, 2, ...\}$ the Negative binomial distribution if for some $p \in [0, 1]$ $P(\{k\}) = p_{r+k-2}C_{r-1}(1-p)^{k-1}p^{r-1}$. We denote this distribution by $NB(r, p)$.*

Proposition 11.11.22.

$$G_{NB(r,p)}(z) = \frac{p^r z}{(1-(1-p)z)^r} \tag{11.11.20}$$

Proof. Because

$$\frac{1}{1-z} = \sum_{i=0}^{\infty} z^i \tag{11.11.21}$$

the following holds by $r-1$ times derivative.

$$\frac{(r-1)!}{(1-z)^r} = \sum_{i=r-1}^{\infty} i(i-1)...(i-r+2)z^i \tag{11.11.22}$$

□

Proposition 11.11.23. *Let $X_1, ..., X_r$ are independent random variables and for any i P_{X_i} is the geometric distribution. Then the distribution of $\sum_{i=1}^{r} X_i - (r-1)$ is $N(r, p)$.*

11.12 Descriptive statistics

11.12.1 Skewness

Definition 11.12.1 (Skewness)**.** *Let*

- *(S1) $\mu \in \mathscr{P}(\mathbb{R})$.*
- *(A1) $\nu := E[\mu]$ and $\sigma^2 := V[\mu]$ exist.*

Let us call $E[\frac{(x-\nu)^3}{\sigma^3}]$ be the skewness of μ.

Proposition 11.12.2. *Let*

- *(S1) f is a probability density function on $\mathbb{R}$.*
- *(A1) $f(x) = f(-x)$ a.e $x > 0$.*
- *(A2) $\int_{\mathbb{R}} |x|^i f(x)dx < \infty$ $(i = 1, 2)$.*
- *(A3) $\int_{\mathbb{R}} xf(x)dx = 0$.*

Then the skewness of the distribution from f is zero.

Proof. We denote S by the skewness of the distribution from f.

$$\begin{aligned} S &= \int_{\mathbb{R}} x^3 f(x)dx \\ &= \int_0^{\infty} x^3 f(x)dx + \int_{-\infty}^{0} x^3 f(x)dx \\ &= \int_0^{\infty} x^3 f(x)dx + \int_{\infty}^{0} (-y)^3 f(-y)(-1)dy \\ &= \int_0^{\infty} x^3 f(x)dx - \int_0^{\infty} y^3 f(y)dy \\ &= 0 \end{aligned} \tag{11.12.1}$$

□

Proposition 11.12.3. *Let*

(S1) f *is a probability density function on* $\mathbb{R}$.

(A1) $\int_{\mathbb{R}} |x|^i f(x)dx < \infty$ $(i = 1, 2, 3)$.

(S2) $d > 0$.

(A2) *For any* $\epsilon > 0$, *there is* $A, B, a, b \in \mathbb{R}$ *such that* $1 < A < B$ *and* $0 \leq a < b$ *and* $b \leq A$ *and* $(b-a) \leq (B-A)$ *and* $\frac{1}{b-a}\int_a^b xf(-x)dx \leq \frac{1}{B-A}\int_A^B xf(x)dx$ *and* $(A^2-1)\int_A^B xf(-x)dx - (b^2-1)\int_a^b xf(-x)dx \geq d$ *and* $|\int_0^\infty x^i f(x)dx - \int_A^B x^i f(x)dx| < \epsilon$ *and* $|\int_0^\infty x^i f(-x)dx - \int_a^b x^i f(x)dx| < \epsilon$ $(i = 1, 3)$.

(S3) *We denote the skewness of the distribution from* f *by* S.

Then $S \geq d$.

Proof.

$$\begin{aligned}
\int_0^\infty x^3 f(-x)dx &\leq \int_a^b x^3 f(-x)dx + \epsilon \\
&\leq \int_a^b x^3 f(-x)dx - \int_a^b xf(-x)dx + \int_a^b xf(-x)dx + \epsilon \\
&\leq \int_a^b (x^2-1)xf(-x)dx + \int_a^b xf(-x)dx + \epsilon \\
&\leq (b^2-1)\int_a^b xf(-x)dx + \int_0^\infty xf(-x)dx + 2\epsilon \\
&\leq (A^2-1)\int_A^B xf(-x)dx - d + \int_0^\infty xf(-x)dx + 2\epsilon \\
&\leq A^2\int_A^B xf(x)dx - d - \int_A^B xf(-x)dx + \int_0^\infty xf(x)dx + 2\epsilon \\
&\leq A^2\int_A^B xf(x)dx - d - \int_0^\infty xf(-x)dx + \int_0^\infty xf(x)dx + 3\epsilon \\
&\leq \int_A^B x^3 f(x)dx - d + 3\epsilon \\
&\leq \int_0^\infty x^3 f(x)dx - d + 4\epsilon
\end{aligned} \tag{11.12.2}$$

So $S \geq d$. □

11.12.2 Kurtosis

Definition 11.12.4 (Kurtosis)**.** *Let*

(S1) $\mu \in \mathscr{P}(\mathbb{R})$.

(A1) $\nu := E[\mu]$ *and* $\sigma^2 := V[\mu]$ *exist.*

Let us call $E[\frac{(x-\nu)^4}{\sigma^4}] - 3$ *be the kurtosis of* μ *and denote it by* $Kurt(\mu)$.

Proposition 11.12.5. *The kurtosis of* $N(\mu, \sigma)$ *is 0.*

Proof. Let us denote by $C_\sigma := \dfrac{1}{\sigma\sqrt{2\pi}}$.

$$\begin{aligned}
E_{N(\mu,\sigma)}[(x-\mu)^4] &= C_\sigma \int_{-\infty}^\infty (x-\mu)^4 exp(-\frac{1}{2}(\frac{x-\mu}{\sigma})^2)dx \\
&= C_\sigma \int_{-\infty}^\infty (x-\mu)^4 exp(-\frac{1}{2}(\frac{x-\mu}{\sigma})^2)dx \\
&= C_\sigma \int_{-\infty}^\infty -\sigma^2(x-\mu)^3\{exp(-\frac{1}{2}(\frac{x-\mu}{\sigma})^2)\}'dx \\
&= 3C_\sigma \int_{-\infty}^\infty -\sigma^2(x-\mu)^2 exp(-\frac{1}{2}(\frac{x-\mu}{\sigma})^2)dx \\
&= 3\sigma^4
\end{aligned} \tag{11.12.3}$$

□

Proposition 11.12.6. *For $\tau > 0$ let us denote kurtosis of $h_\tau := \frac{1}{2\tau}\chi_{[-\tau,\tau]}$ by $k(h_\tau)$. Then $\lim_{\tau\to 0} k(h_\tau) = \infty$ and $\lim_{\tau\to\infty} k(h_\tau) = -3$.*

Proof. Because $E[xf] = 0$,

$$k(h_\tau) + 3 = \frac{E[x^4 h_\tau]}{(E[x^2 h_\tau])^2} \tag{11.12.4}$$

The followings hold.

$$E[x^4 h_\tau] = \frac{2}{5}\tau^5 \tag{11.12.5}$$

and

$$E[x^2 h_\tau] = \frac{2}{3}\tau^3 \tag{11.12.6}$$

So there is constant $C > 0$

$$k(h_\tau) + 3 \sim C\frac{\tau^5}{(\tau^3)^2} = C\frac{1}{\tau}\ (\tau \to 0 \text{ or } \tau \to \infty) \tag{11.12.7}$$

□

Proposition 11.12.7. *We set for $\epsilon > 0$ and $\delta > 0$*

$$f_{\epsilon,\delta}(x) = \begin{cases} \frac{1}{x^{(5+\delta)}} & \text{if } |x| > 1, \\ \\ \frac{1}{\epsilon}(\frac{1}{2} - \frac{1}{4+\delta}) & \text{if } |x| \le \epsilon \\ 0 & \text{otherwise} \end{cases} \tag{11.12.8}$$

Then $f_{\epsilon,\delta}$ is a probability density function. Let us denote the kurtosis of $f_{\epsilon,\delta}$ by $k(f_\delta)$. Then the followings hold.

(i) Then for any $\epsilon > 0$ $\lim_{\delta\to 0} k(f_{\epsilon,\delta}) = \infty$.

(ii) For any $\delta > 0$ $\lim_{\epsilon\to 0} k(f_{\epsilon,\delta}) = \infty$.

Proof. Because

$$\int_1^\infty \frac{1}{x^{(5+\delta)}} dx = \frac{1}{4+\delta} \tag{11.12.9}$$

$f_{\epsilon,\delta}$ is a probability density function.

Because $E[xf_{\epsilon,\delta}] = 0$,

$$k(f_{\epsilon,\delta}) + 3 = \frac{E[x^4 f_{\epsilon,\delta}]}{(E[x^2 f_{\epsilon,\delta}])^2} \tag{11.12.10}$$

The followings holds.

$$\begin{aligned} E[x^2 f_{\epsilon,\delta}] &= 2(\int_0^\epsilon x^2 f_{\epsilon,\delta}(x)dx + \int_1^\infty x^2 f_{\epsilon,\delta}(x)dx) \\ &= 2(\frac{\epsilon^3}{3}(\frac{1}{2} - \frac{1}{4+\delta}) + \int_1^\infty \frac{1}{x^{(3+\delta)}}dx) \\ &= 2(\frac{\epsilon^3}{3}(\frac{1}{2} - \frac{1}{4+\delta}) + \frac{1}{(2+\delta)})) \end{aligned} \tag{11.12.11}$$

$$\begin{aligned} E[x^4 f_{\epsilon,\delta}] &= 2(\int_0^\epsilon x^4 f_{\epsilon,\delta}(x)dx + \int_1^\infty x^4 f_{\epsilon,\delta}(x)dx) \\ &= 2(\frac{\epsilon^5}{3}(\frac{1}{2} - \frac{1}{4+\delta}) + \int_1^\infty \frac{1}{x^{(1+\delta)}}dx) \\ &= 2(\frac{\epsilon^5}{3}(\frac{1}{2} - \frac{1}{4+\delta}) + \frac{1}{\delta})) \end{aligned} \tag{11.12.12}$$

So, if we fix δ then there is constant $C > 0$

$$k(f_{\epsilon,\delta}) + 3 \sim C\frac{\epsilon^5}{(\epsilon^3)^2} = C\frac{1}{\epsilon}\ (\epsilon \to 0) \tag{11.12.13}$$

and if we fix ϵ then there is constant $C > 0$

$$k(f_{\epsilon,\delta}) + 3 \sim C\frac{1}{\delta}\ (\delta \to 0) \tag{11.12.14}$$

Then (i) and (ii) hold. □

11.13 Bayes's theorem

Theorem 11.13.1.

$$P(H_i|A) = \frac{P(H_i)P(A|H_i)}{\Sigma_{j=1}^n P(H_j)P(A|H_j)} \tag{11.13.1}$$

Proof. By the definition of conditional probability,

$$P(H_i|A) = \frac{P(H_i)P(A|H_i)}{P(A)} \tag{11.13.2}$$

and

$$P(A) = \Sigma_{j=1}^n P(A \cup H_j) = \Sigma_{j=1}^n P(H_j)P(A|H_j) \tag{11.13.3}$$

So, the above equation holds. □

11.14 Crude Monte Carlo method

Proposition 11.14.1. *Let*

(S1) $(S := \{1, 2, ..., M\}, 2^\Omega, H)$ *is a probability space.*
(S2) $(\Omega, \mathscr{F}, P)$ *is a probability space.*
(S3) $\{X_n\}_{n=1}^\infty$ *is a sequence of independet random variables on* Ω *such that* $X_n(\Omega) \subset S$ *for any* $n \in \mathbb{N}$.
(A1) $X_n \sim H$ *for any* $n \in \mathbb{N}$. $X_n \sim H)$ *means that* $P(\{X_n = i\}) = H(i)$
(S4) g *is a function on* S.
(S5) $\{Y_n\}_{n=1}^\infty$ *is a sequence of independet random variables on* Ω *such that* $Y_n(\Omega) \subset S$ *for any* $n \in \mathbb{N}$.
(A2) $Y_n \sim C$ *for any* $n \in \mathbb{N}$. *Here,* C *is the counting measure of* S.

then

$$\operatorname*{plim}_{N\to\infty} \frac{\Sigma_{i=1}^N g(X_i)}{N} = \Sigma_{s\in S}\ g(s)H(\{s\}) = \#S \operatorname*{plim}_{N\to\infty} \frac{\Sigma_{i=1}^N g(Y_i)H(\{Y_i\})}{N} \tag{11.14.1}$$

STEP1. Showing (the left side)=(the middle side) . Clearly $\{g(X_n)\}_{n=1}^\infty$ is a sequence of independet random variables on Ω. By $(A1)$,

$$\int_\Omega g(X_n)dP = \Sigma_{s\in S}\ g(s)H(\{s\}) \tag{11.14.2}$$

and

$$\int_\Omega g(X_n)^2 dP = \Sigma_{s\in S}\ g^2(s)H(\{s\}) \tag{11.14.3}$$

So by weak law of large numbers (11.14.1) holds. □

STEP2. Showing (the right side)=(the middle side) . We set

$$G : S \ni s \mapsto g(s)H(\{s\})\#S \in \mathbb{R} \tag{11.14.4}$$

By applying the method of STEP1 to G and C,

$$\begin{aligned} \operatorname*{plim}_{N\to\infty} \frac{\Sigma_{i=1}^N g(Y_i)H(\{Y_i\})\#S}{N} &= \Sigma_{s\in S}\ g(s)H(\{s\})\#SC(\{s\}) \\ &= \Sigma_{s\in S}\ g(s)H(\{s\}) \end{aligned} \tag{11.14.5}$$

□

11.15 Chi-Squared Test for Categorical Data

Proposition 11.15.1. *Let*

(S1) $(\Omega, \mathscr{F}, P)$ *is a probability space.*

(S2) $\{X_i\}_{i=1}^{\infty}$ *is a sequence of* N*-dimensional vectors of random variables on* $(\Omega, \mathscr{F}, P)$.

(A1) $\{X_i\}_{i=1}^{\infty}$ *distribution converges to* $N(0, E_N)$.

then $\{|X_i|^2\}_{i=1}^{\infty}$ *distribution converges to* $\chi^2(N)$.

Proof. Let us fix arbitary $a > 0$.

Let λ be the N-dimensional Lebesugue's measure. By (A1) and $\lambda(\partial B(X, \sqrt{a})) = 0$,

$$\begin{aligned}\mu(\{|X_i|^2 \leq a\}) &= \mu(\{X_i \in \overline{B(X, \sqrt{a})}\}) \\ &\to N(0, E_N)(\overline{B(X, \sqrt{a})}) \ (i \to \infty)\end{aligned} \tag{11.15.1}$$

By the definition of chi-squared distribution with degree of free N,

$$N(0, E_N)(\overline{B(X, a)}) = \chi^2(N)([0, a]) \tag{11.15.2}$$

So $\{|X_i|^2\}_{i=1}^{\infty}$ distribution converges to $\chi^2(N)$. □

Theorem 11.15.2. *Let*

(S1) $(\Omega, \mathscr{F}, P)$ *is a probability space.*

(S2) $\{X_i\}_{i=1}^{\infty}$ *is a sequence of* K*-dimensional vectors of random variables on* $(\Omega, \mathscr{F}, P)$.

(S3) $\{\pi_k\}_{k=1}^{K} \subset (0, 1)$ *such that* $\Sigma_{k=1}^{K} \pi_k = 1$.

(A1) $P(\{X_{i,k} = 1\}) = 1 \ (\forall i, \forall k)$.

(A2) For any k, l *such that* $k \neq l$, $\{X_{i,k} = 1\} \cup \{X_{i,l} = 1\} = \phi \ (\forall i)$.

(S4) $O_{n,k} := \Sigma_{i=1}^{n} X_{i,k} \ (n \in \mathbb{N}, k \in \mathbb{N})$.

(S5) $E_{n,k} := n\pi_k \ (n \in \mathbb{N}, k \in \mathbb{N})$.

then

$$Q(n) := \Sigma_{k=1}^{K} \frac{(O_{n,k} - E_{n,k})^2}{n\pi_i} \tag{11.15.3}$$

distribution converges to $\chi^2(K-1)$.

Proof. We set

$$Y_{n,k} := \sqrt{n}(\bar{X}_k - \pi_k) \ (n \in \mathbb{N}, k \in \mathbb{N}) \tag{11.15.4}$$

Then

$$Y_{n,K} := -\Sigma_{k=1}^{K-1} Y_{n,k} \ (\forall n) \tag{11.15.5}$$

and

$$O_{n,k} - E_{n,k} = \sqrt{n} Y_{n,k} \ (n \in \mathbb{N}, k \in \mathbb{N}) \tag{11.15.6}$$

$$Y_n := (Y_{n,1}, ..., Y_{n,K-1})^T \tag{11.15.7}$$

If we set $A := \{a_{i,j}\}_{i,j=1,...,K-1}$ by

$$a_{i,j} = \begin{cases} \frac{1}{\pi_i} + \frac{1}{\pi_K} & \text{if } (i = j), \\ \frac{1}{\pi_K} & \text{if } (i \neq j), \end{cases} \tag{11.15.8}$$

So

$$Q(n) = Y_n^T A Y_n \ (n \in \mathbb{N}) \tag{11.15.9}$$

and A is a nonnogetive definite symmetric matrix.

We set $(K-1)$-by-$(K-1)$ matrix $\Sigma := \{\sigma_{i,j}\}_{i,j=1,...,K-1}$ by $\sigma_{i,j} = cov(X_{1,i}, X_{1,j})$. Then

$$\sigma_{i,j} = \begin{cases} \pi_i(1 - \pi_i) & \text{if } (i = j), \\ -\pi_i \pi_j & \text{if } (i \neq j), \end{cases} \tag{11.15.10}$$

and

$$\sigma_{i,j} = cov(X_{n,i}, X_{n,j}) \ (\forall n, \forall i, \forall j) \tag{11.15.11}$$

By Proposition11.15.3, Σ is positive definite symmetric matrix.

By the central limit theomre(see [?]), $Y_{n\,n=1}^{\ \infty}$ distribution converges to $N(0, \Sigma)$.

By Proposition11.15.1, $\{Q(n)\}_{n=1}^{\infty}$distribution converges to $\chi^2(K-1)$.

□

Proposition 11.15.3. *Let A and B be matrixies in the proof oh Theorem11.15.2. Then* $A^{-1} = \Sigma$

Proof. For any $i \in \{1, 2, ..., K-1\}$

$$\begin{aligned}(A\Sigma)_{i,i} &= a_{i,i}\sigma_{i,i} + \Sigma_{k\neq i}a_{i,k}\sigma_{k,i} \\ &= (\frac{1}{\pi_i} + \frac{1}{\pi_K})\pi_i(1-\pi_i) + \Sigma_{k\neq i}\frac{1}{\pi_K}(-\pi_i\pi_j) \\ &= (1-\pi_i) + \pi_i\frac{(1-\pi_i) - \Sigma_{k\neq i}\pi_k}{\pi_K} \\ &= 1 \end{aligned} \tag{11.15.12}$$

For any $i \in \{1, 2, ..., K-1\}$ and any $j \in \{1, 2, ..., K-1\}$ such that $i \neq j$,

$$\begin{aligned}(A\Sigma)_{i,j} &= a_{i,i}\sigma_{i,j} + a_{i,j}\sigma_{j,j} + \Sigma_{k\neq i,j}a_{i,k}\sigma_{k,i} \\ &= (\frac{1}{\pi_i} + \frac{1}{\pi_K})(-\pi_i\pi_j) + \frac{1}{\pi_K}\pi_j(1-\pi_j) + \Sigma_{k\neq i,j}\frac{1}{\pi_K}(-\pi_k\pi_j) \\ &= (-\pi_j - \frac{\pi_j}{\pi_K}\pi_i) + (\frac{\pi_j}{\pi_K} - \frac{\pi_j}{\pi_K}\pi_j) - \frac{\pi_j}{\pi_K}\Sigma_{k\neq i,j}\pi_k \\ &= -\pi_j + \frac{\pi_j}{\pi_K} - \frac{\pi_j}{\pi_K}(1-\pi_K) \\ &= 0 \end{aligned} \tag{11.15.13}$$

□

11.16 Linear Regression

11.16.1 Preliminaries for Linear Regression

Throughout this section, we assume the following settings.

Setting 11.16.1 (Linear regression). *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S2) Let* $X := \{X_{i,j}\}_{\{1\leq i\leq N, 1\leq j\leq K\}}$ *be a* (N, K) *matrix.*
- *(A1)* X^TX *is a regular matrix of order* $(K+1)$*.*
- *(S3) Let* $\epsilon := \{\epsilon_i\}_{\{1\leq i\leq N\}}$ *be N random variables.*
- *(A2)* $\{\epsilon_i\}_{\{1\leq i\leq N\}} \overset{iid}{\sim} N(\mathbf{0}, \Sigma_{i=1}^N\sigma^2E_N)$*, where* $\sigma > 0$*.*
- *(S4) Let* $\{\beta_i\}_{\{1\leq i\leq K\}}$ *be a real K-dimension vector.*
- *(S5) Let* $y := \{y_i\}_{\{1\leq i\leq N\}}$ *be N random variables which are defined by the following equation.*

$$y = X\beta + \epsilon \tag{11.16.1}$$

Remark 11.16.2. *By (A1),*

$$rank(X) = K \tag{11.16.2}$$

Definition 11.16.3 (Least squares estimate). *Let*

$$\hat{\beta} := (X^TX)^{-1}(X^Ty) \tag{11.16.3}$$

We call $\hat{\beta}$ *the least squares estimate of (11.16.1).*
And let

$$\hat{y} := X\hat{\beta} \tag{11.16.4}$$

We call $\hat{y}$ *the predicted values of (11.16.1).*
Lastly let

$$\hat{e} := y - \hat{y} \tag{11.16.5}$$

We call $\hat{e}$ *the residual of (11.16.1).*

Remark 11.16.4. *$\hat{\beta}$ is the point which minimize $\mathbb{R}^K \ni z \mapsto |y - Xz|^2 \in [0, \infty)$. And*

$$\hat{\beta} := \beta + (X^T X)^{-1} X^T \epsilon \tag{11.16.6}$$

and for each i $\hat{\beta}_i \sim N(\beta_i, \sigma^2 \xi_i)$ and $\xi_i > 0$, where ξ_i is (i, i) component of $(X^T X)^{-1}$.

Definition 11.16.5 (Multivariate normal distribution)**.** *Let X_i be a random variable on $(\Omega, \mathscr{F})$ $(i = 1, 2, ..., N)$. $\{X_i\}_{i=1}^N \sim N(\gamma, \Sigma)$ if there is a natural number l and (N, l) matrix A and there are random variables $\{\epsilon\}_{i=1}^l$ on $(\Omega, \mathscr{F})$ such that $\boldsymbol{\epsilon} := \{\epsilon\}_{i=1}^l$ are i.i.d and $\epsilon_i \sim N(0, 1)$ $(\forall i)$ and $X = A\boldsymbol{\epsilon} + \gamma$ and $\Sigma = AA^T$.*

11.16.2 Interval estimation of regression coefficients

Proposition 11.16.6.

$$\frac{|\hat{e}|^2}{\sigma^2} \sim \chi^2(N - K) \tag{11.16.7}$$

Proof. The following holds.

$$\hat{e} = (E_N - X(X^T X)^{-1} X^T)\epsilon \tag{11.16.8}$$

Let $A := (E_N - X(X^T X)^{-1} X^T)$ then A is symmetric and idempotent. So each eigenvalue of A is 0 or 1. And $tr(A) = N - tr(X(X^T X)^{-1} X^T) = N - tr((X^T X)^{-1} X^T X) = N - K$ so $rank(A) = N - K$. So by Proposition**??**, $\frac{|\hat{e}|^2}{\sigma^2} \sim \chi^2(N - K)$. □

Proposition 11.16.7. *$\hat{\beta}$ and $\hat{e}$ are independent.*

Proof. By (11.16.6) and (11.16.8), $cov(\hat{e}, \hat{\beta}) = 0$. So by Proposition11.16.7 $\hat{\beta}$ and $\hat{e}$ are independent. □

By Remark and Proposition11.16.6 and Proposition11.16.6 and Proposition11.16.7, the folloing Proposition holds.

Proposition 11.16.8. *For each $i \in \mathbb{N} \cap [1, K]$,*

$$\frac{(\hat{\beta}_i - \beta_i)\sqrt{(N - K)}}{|\hat{e}|\sqrt{\xi_i}} \sim t(N - K) \tag{11.16.9}$$

In the above equation, t_{N-K} is the t-distribution whose degrees of freedom is $N - K$ and ξ_i is (i, i) component of $(X^T X)^{-1}$.

The following is a remark.

Proposition 11.16.9.

$$E\left(\frac{|\hat{e}|^2 \xi_i}{N - K}\right) = V(\hat{\beta}_i) \ (\forall i) \tag{11.16.10}$$

Proof. By Proposition11.16.6, $E\left(\frac{|\hat{e}|^2 \xi_i}{N - K}\right) = \sigma^2 \xi_i$. By Remark11.16.2, $V(\hat{\beta}_i) = \sigma^2 \xi_i$ □

By the above remak, $\frac{|\hat{e}|\sqrt{\xi_i}}{\sqrt{N - K}}$ is denoted by $se(\hat{\beta}_i)$.

11.16.3 Decomposition of TSS

Proposition 11.16.10.

$$(\hat{y}, \hat{e}) = 0 \tag{11.16.11}$$

Proof. By (11.16.6),

$$X^T \hat{y} = X^T X \hat{\beta} = X^T (X\beta + \boldsymbol{\epsilon}) = X^T y \tag{11.16.12}$$

So

$$\begin{aligned} (\hat{y}, \hat{e}) &= \beta^T X^T \hat{e} \\ &= \beta^T X^T (y - \hat{y}) \\ &= 0 \end{aligned}$$

□

Proposition 11.16.11. *Let*

(A1) There is a K-by-K matrix B such that the first column of XB is 1_N

then

$$\overline{\hat{y}} = \overline{y} \tag{11.16.13}$$

Proof. By (11.16.6),

$$X^T \hat{y} = X^T X \hat{\beta} = X^T (X\beta + \boldsymbol{\epsilon}) = X^T y \tag{11.16.14}$$

So the following holds.

$$B^T X^T \hat{e} = 0 \tag{11.16.15}$$

The fitst component of the $B^T X^T \hat{e}$ is $\overline{\hat{y}} - \overline{y}$. So $\overline{\hat{y}} = \overline{y}$. □

Proposition 11.16.12. *Let*

(S1) $TSS := |y - \overline{y}1_n|^2$

(S2) $RSS := |\hat{y} - \overline{y}1_n|^2$

(S3) $ESS := |y - \hat{y}|^2$

(A1) (A1) in Proposition11.16.11

then

$$TSS = RSS + ESS \tag{11.16.16}$$

Proof. Because

$$TSS = y^T (E - \frac{1}{N} 1_{N,N}) y \tag{11.16.17}$$

and

$$RSS = y^T (X^T (X^T X)^{-1} X - \frac{1}{N} 1_{N,N}) y \tag{11.16.18}$$

and

$$ESS = y^T (E - X^T (X^T X)^{-1} X) y \tag{11.16.19}$$

$TSS = RSS + ESS$. □

11.16.4 Cochran's theorem

Proposition 11.16.13. *Let*

(S1) $m \in \mathbb{N}$ and A_i:N-by-N symmetric matrix $(i = 1, 2, ..., m)$

(A1) $E_N = \Sigma_{i=1}^m A_i$

(A2) $N = \Sigma_{i=1}^m rank(A_i)$

then

$$A_i A_j = \delta_{i,j} A_i \ (\forall i, \ \forall j) \tag{11.16.20}$$

where $\delta_{i,j}$ is a Kronecker delta.

Proof. Let $V_i := A_i \mathbb{R}^N$ and $n_i := rank(A_i)$ and $\{v_{i,j}\}_{1 \le j \le n_i}$ be a basis of V_i $(i = 1, 2, ..., m)$. By (A1) and (A2), $\{v_{i,j}\}_{1 \le i \le m, 1 \le j \le n_i}$ is a basis of $\mathbb{R}^N$. and

$$\mathbb{R}^N = \bigoplus_{i=1}^m V_i \tag{11.16.21}$$

Let fix arbitary $i \in \{1, 2, ..., N\}$ and fix arbitary $x \in \mathbb{R}^N$. $A_i x = (\Sigma_{i=1}^m A_i) A_i x = (A_i)^2 x + (\Sigma_{j \ne i} A_j A_i x$. By (11.16.21), $A_i x = {A_i}^2 x$ and $A_j A_i x = 0$. □

By Proposition11.16.13 and Proposition**??** and Proposition, the following theorem holds.

Proposition 11.16.14 (Cochran's theorem)**.**
We take over (S1) and (A1) in Proposition11.16.13. And let

(S2) $(\Omega, \mathscr{F}, P)$ is a probability space.

(A1) $\boldsymbol{\epsilon} \sim N(0, E_N)$

(S3) $Q_i := \boldsymbol{\epsilon}^T A_i \boldsymbol{\epsilon}$ $(i = 1, 2, ..., m)$

then $Q_i \sim \chi^2(rank A_i)$ $(\forall i)$ and Q_i and Q_j are independent for all $(i, j) \in \{(i, j) | i \ne j\}$

11.16.5 Testing

Throughout this subsection, we assume

$$\beta = (\beta_0, 0, 0, ..., 0)^T \tag{11.16.22}$$

and

$$X = \begin{pmatrix} 1 & x_{1,1} & x_{1,2} & ... & x_{1,L} \\ 1 & x_{2,1} & x_{2,2} & ... & x_{2,L} \\ ... & ... & ... & ... & \\ 1 & x_{N,1} & x_{N,2} & ... & x_{N,L} \end{pmatrix} \tag{11.16.23}$$

Then

$$X\beta = \beta_0 1_{N,1} \tag{11.16.24}$$

So

$$\begin{aligned} \hat{y} &= X(X^TX)^{-1}X^Ty \\ &= X(X^TX)^{-1}X^T(X\beta + \epsilon) \\ &= \beta_0 1_{N,1} + X(X^TX)^{-1}X^T\epsilon \end{aligned} \tag{11.16.25}$$

And

$$\bar{y}1_{N,1} = \beta_0 \frac{1}{N} 1_{N,1} + 1_{N,N}\epsilon \tag{11.16.26}$$

Consequently,

$$RSS = \epsilon^T (X(X^TX)^{-1}X^T - \frac{1}{N} 1_{N,1})\epsilon \tag{11.16.27}$$

Because $X(X^TX)^{-1}X^T$ is symmetric, $X(X^TX)^{-1}X^T$ and $\frac{1}{N}1_{N,1}$ are commutative.
And because $X(X^TX)^{-1}X^T$ is idempotent and symmetric, $(X(X^TX)^{-1}X^T - \frac{1}{N}1_{N,1})$ is idempotent and symmetric.
$rank(X(X^TX)^{-1}X^T - \frac{1}{N}1_{N,1}) = tr(X(X^TX)^{-1}X^T - \frac{1}{N}1_{N,1}) = L$
So by Proposition11.16.14, RSS and ESS are independent and $RSS \sim \chi^2(L)$ and $ESS \sim \chi^2(N - L - 1)$.
So,

$$\frac{\frac{RSS}{L}}{\frac{ESS}{N-L-1}} \sim F(L, N - L - 1) \tag{11.16.28}$$

11.16.6 Simple linear regression

Throughout this subsection, we set

$$T_x = \sum_{i=1}^{n} x_i, \; T_y = \sum_{i=1}^{n} y_i, \; T_{x,x} = \sum_{i=1}^{n} x_i^2, \; T_{x,y} = \sum_{i=1}^{n} x_i y_i \tag{11.16.29}$$

(1) Case1: there is intercept
Throughout this subsection, we assume

$$X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ ... & ... \\ 1 & x_n \end{pmatrix} \tag{11.16.30}$$

Then

$$\begin{aligned}
\hat{\beta} &= \begin{pmatrix} \hat{\alpha} \\ \hat{\gamma} \end{pmatrix} \\
&= (X^T X)^{-1} X^T y \\
&= (\begin{pmatrix} 1 & 1 & ... & 1 \\ x_1 & x_2 & ... & x_n \end{pmatrix} \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ ... & ... \\ 1 & x_n \end{pmatrix})^{-1} X^T y \\
&= \begin{pmatrix} n & T_x \\ T_x & T_{x,x} \end{pmatrix}^{-1} X^T y \\
&= \frac{1}{nT_{x,x} - T_x^2} \begin{pmatrix} T_{x,x} & -T_x \\ -T_x & n \end{pmatrix} \begin{pmatrix} T_y \\ T_{x,y} \end{pmatrix}
\end{aligned} \tag{11.16.31}$$

So

$$\begin{aligned}
\hat{\gamma} &= \frac{nT_{x,y} - T_x T_y}{nT_{x,x} - T_x^2} \\
&= \frac{T_{x,y} - \frac{1}{n} T_x T_y}{T_{x,x} - \frac{1}{n} T_x^2} \\
&= \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}
\end{aligned} \tag{11.16.32}$$

Consequently,

$$\hat{\gamma} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \tag{11.16.33}$$

(2) Case2: there is no intercept
Throughout this subsection, we assume

$$X = (x_1, x_2, ..., x_n)^T \tag{11.16.34}$$

Then

$$\hat{\beta} = \frac{T_{x,y}}{T_{x,x}} \tag{11.16.35}$$

11.16.7 Estimation about population mean

Throughout this section, we assume $X = 1_N$ is one and we define μ by $\beta = \mu 1_1$. The followings hold.

$$X^T X = N \tag{11.16.36}$$

$$Y := X(X^T X)^{-1} X^T = \frac{1}{N} 1_{N,N} \tag{11.16.37}$$

$$\hat{e} := y - \overline{y} 1_N \tag{11.16.38}$$

$$\frac{|\hat{e}|^2}{\sigma^2} \sim \chi^2(N-1) \tag{11.16.39}$$

$$\frac{(\mu - \overline{y})\sqrt{N(N-1)}}{|y - \overline{y}|} \sim t(N-1) \tag{11.16.40}$$

11.16.8 Estimation about difference between two population means

Throughout this section, we assume

$$X = \begin{pmatrix} 1_M & 0 \\ 0 & 1_N \end{pmatrix} \tag{11.16.41}$$

and

$$\beta = \begin{pmatrix} \mu_1 1_M \\ \mu_2 1_N \end{pmatrix} \tag{11.16.42}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} := y \tag{11.16.43}$$

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} := \epsilon \tag{11.16.44}$$

Then the followings hold.

$$X^T X = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} \tag{11.16.45}$$

$$Y := \begin{pmatrix} \frac{1}{M} 1_{M,M} & 0 \\ 0 & \frac{1}{N} 1_{N,N} \end{pmatrix} \tag{11.16.46}$$

$$\mu_1 = (\hat{y_1})_1 = \overline{y_1} + \overline{\epsilon_1} \tag{11.16.47}$$

$$\mu_2 = (\hat{y_2})_1 = \overline{y_2} + \overline{\epsilon_2} \tag{11.16.48}$$

So, by reproductive property of normal distribution,

$$\mu_1 - \mu_2 - (\overline{y_1} - \overline{y_2}) \sim N(0, (\frac{1}{M} + \frac{1}{N})\sigma^2) \tag{11.16.49}$$

And the following holds.

$$|\hat{e}|^2 = |y_1 - \mu_1 1_M|^2 + |y_2 - \mu_2 1_N|^2 \tag{11.16.50}$$

By Proposition11.16.7, $(\mu_1 - \mu_2 - (\overline{y_1} - \overline{y_2}))$ and $|y_1 - \mu_1 1_M|^2 + |y_2 - \mu_2 1_N|^2$ are independent. Consequently, the following holds.

$$\frac{(\mu_1 - \mu_2 - (\overline{y_1} - \overline{y_2}))\sqrt{M+N-2}}{\sqrt{(|y_1 - \mu_1 1_M|^2 + |y_2 - \mu_2 1_N|^2)(\frac{1}{M} + \frac{1}{N})}} \sim t(M+N-2) \tag{11.16.51}$$

11.16.9 One way analysis of variance

Throughout this section we set

$$y := (y_{1,1}, ..., y_{1,n_1}, y_{2,1}, ..., y_{2,n_2}, ..., y_{K,1}, ..., y_{K,n_K})^T \tag{11.16.52}$$

$$\beta := (\mu_1, \mu_2, ..., \mu_K)^T \tag{11.16.53}$$

$$\bar{y}_{i,\cdot} := \frac{\sum_{j=1}^{n_i} y_{i,j}}{n_i} \ (i = 1, 2, ..., K) \tag{11.16.54}$$

$$X := \begin{pmatrix} 1_{n_1} & O & O & O \\ 1_{n_2} & 1_{n_2} & O & O \\ ... & ... & ... & ... \\ 1_{n_K} & O & O & 1_{n_K} \end{pmatrix} \tag{11.16.55}$$

Then

$$Y := X(X^TX)^{-1}X^T := \begin{pmatrix} \frac{1}{n_1}1_{n_1,n_1} & O & O & O \\ O & \frac{1}{n_2}1_{n_2,n_2} & O & O \\ ... & ... & ... & ... \\ O & O & O & \frac{1}{n_K}1_{n_K,n_K} \end{pmatrix} \tag{11.16.56}$$

In this subsection, hereafter, we assume there is a real number μ such that

$$\beta = \mu 1_K \tag{11.16.57}$$

Then the followings holds.

$$TSS = \boldsymbol{\epsilon}^T(E_N - \frac{1}{N}1_{N,N})\boldsymbol{\epsilon} \tag{11.16.58}$$

$$ESS = \boldsymbol{\epsilon}^T(Y - \frac{1}{N}1_{N,N})\boldsymbol{\epsilon} \tag{11.16.59}$$

$$rank(Y - \frac{1}{N}1_{N,N}) = K - 1 \tag{11.16.60}$$

$$RSS = \boldsymbol{\epsilon}^T(E_N - Y)\boldsymbol{\epsilon} \tag{11.16.61}$$

$$rank(E_N - Y) = N - K \tag{11.16.62}$$

So, by Cohchran's theorem, ESS and RSS are independent, and $ESS \sim \chi^2(K-1)$ and $RSS \sim \chi^2(N-K)$. Consequently, the following theoem holds.

Theorem 11.16.15. *Under the setting(11.16.55) and the assumption(11.16.57)*

$$(ESS/(K-1))/(RSS/(N-K)) \sim F(K-1, N-K) \tag{11.16.63}$$

And the followings hold.

$$(X^TX)^{-1} = \begin{pmatrix} \frac{1}{n_1} & 0 & ... & 0 \\ 0 & \frac{1}{n_2} & ... & 0 \\ ... & ... & ... & ... \\ 0 & ... & 0 & \frac{1}{n_K} \end{pmatrix} \tag{11.16.64}$$

$$\hat{\beta} = (\bar{y}_{1,\cdot}, \bar{y}_{2,\cdot}, ..., \bar{y}_{K,\cdot})^T \tag{11.16.65}$$

So, by Proposition11.16.8, the following theoem holds.

Theorem 11.16.16. *Under the setting(11.16.55)*

$$(\bar{y}_{i,\cdot} - \mu_i)\sqrt{\frac{(N-K)n_i}{ESS}} \sim t(N-K) \tag{11.16.66}$$

11.17 Principal Component Analysis

11.18 Kernel Method

11.18.1 Motivation

Kernel Method is a method for effectively analyzing high dimensional data which does not fit statistical linear model.

Terminology 11.18.1 (Feature Space, Feature Map)**.** *The followings are settings.*

(S1) Ω *be a set.*

(S2) $\mathcal{H}$ *be a real inner product space.*

(S3) $\Phi : \Omega \to H$*.*

We call Ω a feature space and Φ a feature map, respectively.

I imagine Ω to be a high dimensional data set like a subset of $\mathbb{R}^{10000}$. And I assume that for a given statistical problem like regression or principal component analysis or others, Ω does not fit statistical linear model like linear regression or linear principal component analysis or others. So, I hope $\Phi(\Omega)$ does fit the linear model. Since Ω is high dimensional, in many case H is also high dimensional. In general, that impose us highly costed calculation of the inner product. However, if we find $k : \Omega \times \Omega \to \mathbb{R}$ such that

$$(\Phi(X), \Phi(Y)) = k(X, Y) \ (\forall X, Y \in \Omega)$$

the inner product is easy to calculate. Here, k is called a kernel function and H is called a reproducing kernel Hilbert space. Kernel method is the method to solove a given problem using (H, k). In addition, such statistical problems are often reduced to an optimization problem in H. By the theory of the kernel method, it is shown that a solution of the optimization problem can be expressed as a linear combination of $\{\Phi(X_i)\}_{i=1}^{n}$.

$$\sum_{i=1}^{m} \alpha_i \Phi(X_i)$$

11.18.2 Positive Definite Kernel Function

Definition 11.18.2 (Real Valued Positive Definite Kernel Function)**.** *The followings are settings.*

- *(S1) Ω be a set.*
- *(S2) k be a real valued function on Ω.*

We say k is a positive definite kernel function if for any $x_1, ..., x_m \in \Omega$ $\{k(x_i, x_j)\}_{i,j=1,2,...,m}$ is a positive semi-definite symmetric matrix.

Definition 11.18.3 (Complex Valued Positive Definite Kernel Function)**.** *The followings are settings.*

- *(S1) Ω be a set.*
- *(S2) k be a complex valued function on Ω.*

We say k is a complex valued positive definite kernel function if for any $x_1, ..., x_m \in \Omega$ $\{k(x_i, x_j)\}_{i,j=1,2,...,m}$ is a positive semi-definite Hermitian matrix.

Notation 11.18.4. *The followings are settings.*

- *(S1) Ω be a set.*
- *(S2) k be a real or complex valued function on Ω.*
- *(S3) $x \in \Omega$.*

We set

$$k_x(y) := k(y, x) \ (y \in \Omega)$$

11.18.3 Reproducing Kernel Hilbert Space(RKHS)

Definition 11.18.5 (Reproducing Kernel Hilbert Space)**.** *The followings are settings.*

- *(S1) Ω is a set.*
- *(S3) H is a Hilbert space.*

We say H is a real reproducing kernel Hilbert space over Ω if

$$H \subset Map(\Omega, \mathbb{R})$$

and for each $x \in \Omega$ there exists $k_x \in H$ such that

$$(u, k_x) = u(x) \ (\forall u \in H)$$

We call a function

$$k : \Omega^2 \ni (x, y) \mapsto k_x(y) \in \mathbb{R}$$

reproducing kernel.

Proposition 11.18.6. *The followings are settings.*

(S1) H is a real reproducing kernel Hilbert space over Ω.

Then the reproducing kernel is uniquely determined and is a positive definite kernel function.

Proof. Let us fix any reproducing kernel functions $k, \bar{k}$. Since

$$\bar{k}_x(y) = (\bar{k}_x, k_y) = (k_y, \bar{k}_x) = k_y(x) = (k_y, k_x) = (k_x, k_y) = k_x(y) \ (\forall x, y \in \Omega)$$

the reproducing kernel is uniquely determined. From the above equation, it is shown $k(x, y) = k(y, x)\ \forall x, y \in \Omega$.

Next, let us fix any $a_1, ..., a_m \in \mathbb{R}$ and $x_1, ..., x_m \in \Omega$. Then

$$\sum_{i,j} k(x_i, x_j) a_i a_j = \sum_{i,j} k_{x_i}(x_j) a_i a_j = \sum_{i,j} (k_{x_i}, k_{x_j}) a_i a_j = (\sum_i a_i k_{x_i}, \sum_j a_j k_{x_j}) \geq 0$$

So, $\{k(x_i, x_j)\}_{i,j}$ is a positive semi-definite symmetric matric. □

Proposition 11.18.7. *The followings are settings.*

(S1) H is a complex reproducing kernel Hilbert space over a topological space Ω with kernel k.

(A1) $\Omega \ni x \mapsto k(x, x) \in \mathbb{C}$ is continuous.

(A2) For any $y \in \Omega$, $\Omega \ni x \mapsto Re[k(x, y)] \in \mathbb{C}$ is continuous.

Then $H \subset C(X, \mathbb{C})$.

Proof. Let us fix any $f \in H$, For any $x, y \in X$,

$$||f(x) - f(y)|| = ||(f, k(\cdot, x) - k(\cdot, y)|| \leq ||f|| \cdot ||k(\cdot, x) - k(\cdot, y)||$$

and

$$||k(\cdot, x) - k(\cdot, y)|| = k(x, x) + k(y, y) - 2Re(k(x, y))$$

Therefore f is continuous. □

Definition 11.18.8 (Universal)**.** *The followings are settings.*

(S1) Ω is a compact topological space.

(S2) k is a positive definite kernel over Ω.

We say k is universal if $\mathcal{H}_k$ is dense in $C(\Omega)$ regarding $|| \cdot ||_\infty$.

Proposition 11.18.9. *The followings are settings.*

(S1) $H \subset M(\Omega, \mathbb{C})$ is a complex Hilbert space.

Then H is a RKHS if and only if for any $x \in \Omega$

$$H \ni f \mapsto f(x) \in \mathbb{C}$$

is continuous.

Proof of 'only if' part. Let K denote the reproducing kernel on H. Then for any $x \in \Omega$ and $f \in H$, $|f(x)| = |(f, k_x)|$. So,

$$H \ni f \mapsto f(x) \in \mathbb{C}$$

is continuous. □

Proof of 'if' part. From Riez representation theorem, for each $x \in \Omega$, there exists $k_x \in H$ such that

$$f(x) = (f, k_x) \ (\forall f \in H)$$

That means H is a RKHS. □

11.18.4 Relations between Positive Definite Function and RKHS

The following theorem shows that a positive definite kernel function identify a reproducing Hilbert space.

Theorem 11.18.10 (Moore-Aronszajn). *The followings are settings.*

(S1) k is a real positive definite kernel function over Ω.

Then there is a reproducing kernel Hilbert space H over Ω such that

(i) k is a reproducing kernel of H.

(ii) For any $x \in \Omega$, $k(\cdot, x) \in H$.

(iii) $\{k(\cdot, x)\}_{x\in\Omega}$ are dense in H.

Proof. We set $H_0 := \langle \{k_{X_i}\}_{i=1}^N \rangle$. For $f := \sum_{i=1}^m a_i k_{x_i}$ and $g := \sum_{j=1}^n b_i k_{x_j}$

$$(f, g) := \sum_{i,j} a_i b_j k(x_i, x_j)$$

First, we will show the inner product is well-defined. Let us fix any $\{a_i\}_{i=1}^m, \{a'_i\}_{i=1}^m, \{b_i\}_{i=1}^n, \{b'_i\}_{i=1}^n$ such that $f := \sum_{i=1}^m a_i k_{x_i} = \sum_{i=1}^m a'_i k_{x_i}$ and $g := \sum_{i=1}^n b_i k_{x_i} = \sum_{i=1}^n b'_i k_{x_i}$. Then

$$\sum_{i,j} a_i b_j k(x_i, x_j) = \sum_j b_j \sum_i a_i k(x_i, x_j) = \sum_j b_j \sum_i a_i k(x_j, x_i) = \sum_j b_j f(x_j) = \sum_j b_j \sum_i a'_i k(x_j, x_i) = \sum_{i,j} a'_i b_j k(x_i, x_j)$$

By the same argument as the above, we get

$$\sum_{i,j} a_i b_j k(x_i, x_j) = \sum_{i,j} a'_i b'_j k(x_i, x_j)$$

Therefore, the inner product is well-defined. Since k is positive semi-definite, the inner product is positive semi-definite.

Next, we will show the inner product is positive definite. Let us any fix $f := \sum_{i=1}^m a_i k_{x_i} \in H_0$ such that $||f|| = 0$. For any $x \in \Omega$,

$$\begin{aligned} |f(x)| &= |\sum_i a_i k_{x_i}(x)| = |\sum_i a_i k_{x_i}(x)| = |\sum_i a_i k(x_i, x)| = |(f, k_x)| \\ &\text{by Cauchy-Schwartz inequality} \\ &\leq ||f|| ||k_x|| = 0 \end{aligned}$$

So, $f = 0$. Remark

$$|f(x)| = (f, k_x) \tag{11.18.1}$$

and

$$|f(x)| \leq ||f|| ||k_x|| \tag{11.18.2}$$

That means the inner product is positive definite.

Let $\tilde{H}$ denote the completion of H_0. Let us fix any $[\{f_n\}_{n\in\mathbb{N}}] \in \tilde{H}$. By (11.18.2), for each $x \in \Omega$, $\{f_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence. So, $f(x) := \lim_{n\to\infty} f_n(x)$ exists. From the definition of the completion, clearly, $f \in M(\Omega, \mathbb{R})$ is well-defined. So we get a map

$$\Phi : \tilde{H} \ni [\{f_n\}_{n\in\mathbb{N}}] \mapsto f \in M(\Omega, \mathbb{R})$$

Clearly Φ is a $\mathbb{R}$ linear map. From the definition of the completion, clearly, Φ is injective. We set

$$H := \Phi(\tilde{H}), (\Phi(u), \Phi(v)) := (u, v) \ (u, v \in \tilde{H})$$

Then H is also a Hilbert space. Since $\tilde{H}$ is the completion of H_0, $\langle \{[k_x]\}_{x\in\Omega} \rangle$ is dense in $\tilde{H}$. And clearly,

$$\Phi([k_x]) = k_x \ (\forall x \in \Omega)$$

So, $\{k_x\}_{x\in\Omega} \subset H$ and $\langle \{k_x\}_{x\in\Omega} \rangle$ is dense in H. Finally, let us fix any $\Phi([\{f_n\}_{n\in\mathbb{N}}]) \in H$ and $x \in \Omega$. We set $u := [\{f_n\}_{n\in\mathbb{N}}]$. From (11.18.1),

$$|f_n(x)| = (f_n, k_x) \ (\forall n \in \mathbb{N})$$

By reaching $n \to \infty$,

$$|\Phi(u)(x)| = (\Phi(u), k_x)$$

So, H is a reproducing kernel Hilbert space with k. ◻

The following proposition give a way to construct a RKHS from a dense subset of a Hilbert space of all self squared functions.

Proposition 11.18.11. *The followings are settings.*

(S1) (T, μ) *is a measurable space.*

(S2) Ω *is a set.*

(S3) $\{H(\cdot, x)\}_{x\in\Omega}$ *is a dense subset of* $L^2(T, \mu)$*.*

(S4) We define

$$J : L^2(T, \mu) \ni F \mapsto Map(\chi, \mathbb{C})$$

by

$$J(F)(x) := (F, H(\cdot, x))_{L^2(T,\mu)} \int_T F(t)\overline{H(t, x)}d\mu(t) \ (F \in L^2(T, \mu), x \in \Omega)$$

We set $\mathcal{H} := J(L^2(T, \mu))$*.*

(S5) We define

$$(J(f), J(g)) := (f, g) \ (f, g \in L^2(T, \mu))$$

Then

(i) J *is a continuous injective linear map.*

(ii) The inner product defined in (S5) is well defined and give a Hilbert space.

(iii) $\mathcal{H}$ *is a RKHS over* Ω *and the kernel function is below.*

$$k(x, y) := \int_T H(t, x)\overline{H(t, y)}d\mu(t) \ (x, y \in \Omega)$$

Moore-Aronzjan Theorem also gives us a good feature map.

Proposition 11.18.12. *The followings are settings.*

(S1) Ω *is a feature space.*

(S2) k *is a real positive definite kernel function over* Ω*.*

(S3) H *is a reproducing kernel space with* k*.*

(S4) We define a feature map by

$$\Phi : \Omega \ni x \mapsto k(\cdot, x) \in H$$

Then

$$(\Phi(x), \Phi(y)) = k(x, y) \ (\forall x, y \in \Omega)$$

Proof. The proposition is clear from the definition of reproducing kernel space. □

The following theorem clarify a form of a solution of optimization problems in a reproducing Hilbert space.

Theorem 11.18.13 (Representer Theorem)**.** *The followings are settings.*

(S1) Ω *is a feature space.*

(S2) Λ *is a set.*

(S3) $\{(X_i, Y_i)\}_{i=1}^N \subset \Omega \times \Lambda$*.*

(S4) $\Psi : [0, \infty) \to \mathbb{R}$ *a strictly monotone increasing function.*

(S5) H *is a reproducing kernel Hilbert space.*

(S6) $L : H^N \mapsto \mathbb{R}$*.*

(S7) $h_1, ..., h_m \in H$*.*

Then the optimization problem

$$\min_{f\in H, c\in\mathbb{R}^m} F(f, c) := (L(\{f(X_i) + \sum_{\alpha=1}^m c_\alpha h_\alpha(X_i)\}_{i=1}^N) + \Psi(||f||))$$

has solutions in $\langle\{k_{X_i}\}_{i=1}^N\rangle$*.*

Proof. We set $H_0 := \langle \{k_{X_i}\}_{i=1}^N \rangle$. Let us fix any $f \in H$ and $c \in \mathbb{R}^m$. Then there are $f_0 \in H_0$ and $f_1 \in H_0^\perp$. From this, $||f_0||^2 \leq ||f||^2$. So,

$$f(X_i) = (f, k_{X_i}) = (f_0, k_{X_i}) = f_0(X_i)$$

and

$$\Psi(||f_0||^2) \leq \Psi(||f||^2)$$

This implies $F(f_0, c) \leq F(f_0, c)$. □

11.18.5 Kernel Principal Components Analysis

Proposition 11.18.14. *The followings are settings and assumptions.*

(S1) Ω *is a feature space.*

(S2) H *is a reproducing kernel Hilbert space over* Ω *with the reproducing kernel* k.

(S3) $\Phi : \Omega \to H$ *is a feature map such that*

$$\Phi(x) = k_x \ (\forall x \in \Omega)$$

(S4) $\{X_i\}_{i=1}^N \subset \Omega$.

(S5) $\tilde{\Phi}(X_i) := \Phi(X_i) - \dfrac{1}{N} \sum_{j=1}^N \Phi(X_j)$.

(S6) We call the optimization problem

$$\max_{f \in H, ||f||=1} \frac{1}{N} \sum_{i=1}^N ((f, \Phi(X_i)) - \frac{1}{N} \sum_{j=1}^N (f, \Phi(X_j))^2$$

problemA1.

(S7) We set

$$\tilde{K}_{i,j} := (\tilde{\Phi}(X_i), \tilde{\Phi}(X_j)) = k(X_i, X_j) - \frac{1}{N} \sum_{b=1}^N k(X_i, X_b) - \frac{1}{N} \sum_{a=1}^N k(X_a, X_j) + \frac{1}{N^2} \sum_{a,b=1}^N k(X_a, X_b) \ (i, j = 1, 2, ..., N)$$

We call $\tilde{K} := \{K_{i,j}\}_{i,j=1}^N$ *the centering gram matrix. Let* $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N$ *denote all eigenvalues of* $\tilde{K}$. *For each* i, *let* u_i *denote an unit eigenvector regarding to* λ_i.

(S8) We call the optimization problem

$$\max_{a \in \mathbb{R}^N, a^T \tilde{K} a = 1} a^T \tilde{K}^2 a$$

problemB1.

Then the followings holds.

(i) A solution of the problemA1 exists in $\langle \{\tilde{\Phi}(X_i) | i = 1, 2, ..., N\} \rangle$.

(ii) For any solution of the problemB1, denoted by a, $\sum_{i=1}^m a_i \tilde{\Phi}$ *is a solution of problemA1.*

(iii) $f^1 := \sum_{i=1}^N \alpha_i^1 \tilde{\Phi}(X_i)$, $\alpha_i^1 = \dfrac{1}{\sqrt{\lambda_1}} u^1$. *Then* f^1 *is a solution of problemA1.*

(iv) $(\tilde{\Phi}(X_i), f^1) = \sqrt{\lambda_i} u_1^1$ *for any* i.

(v) We define the optimization problem

$$\max_{f \in H, ||f||=1, f \perp \langle f_1 \rangle} \frac{1}{N} \sum_{i=1}^N ((f, \Phi(X_i)) - \frac{1}{N} \sum_{j=1}^N (f, \Phi(X_j))^2$$

and we call it problemA2. By the same way, we define problemA3,...,problemAN. And $f^p := \sum_{i=1}^N \alpha_i^1 \tilde{\Phi}(X_i)$, $\alpha_i^p = \dfrac{1}{\sqrt{\lambda_p}} u^p$. *Then* f^p *is a solution of problemAp* $(p = 1, 2, ..., N)$.

(iv) $(\tilde{\Phi}(X_i), f^p) = \sqrt{\lambda_i} u_p^1$ *for any* i *and* p.

Chapter 12

Mathematical Programming

12.1 Linear Programming

Definition 12.1.1 (Standard Form of Linear Programming)**.** *The followings are settings ans assumptions.*

(S1) $A \in M(m, n; \mathbb{R})$.

(A1) $rank(A) = m < n$.

(S2) $b \in \mathbb{R}^m$.

(S3) $c \in \mathbb{R}^n$.

We call the following problem a standard form of a linear programming.

$$arg \min_{x \geq 0, Ax=b} c^T x$$

Definition 12.1.2 (Basic Feasible Solution)**.** *The followings are settings ans assumptions.*

(S1) $A = (a_1, ..., a_n) \in M(m, n; \mathbb{R})$.

(A1) $rank(A) = m < n$.

(S2) $b \in \mathbb{R}^m$.

(S3) $c \in \mathbb{R}^n$.

(S4) $\sigma \in S_n$.

(S5) $B = (a_{i_1}, a_{i_2}, ..., a_{i_m})$*, where* $i_1 < i_2 < ... < i_m$ *and* $rank(B) = m$.

(S6) $Ax = b$ *and* $x_i = 0$ $(\forall i \notin \{i_1, ..., i_2\})$.

We call x *a basic feasible solution of the standard form.*

Theorem 12.1.3 (Foundamental Theorem for Linear Programming)**.** *The followings are settings ans assumptions.*

(S1) $A \in M(m, n; \mathbb{R})$.

(A1) $rank(A) = m < n$.

(S2) $b \in \mathbb{R}^m$.

(S3) $c \in \mathbb{R}^n$.

(i) If the standard form has an optimal solution, the standard form has a basic optimal solution.

(ii) If the standard form has an optimal solution, there are $i_1 < i_2 < ... < i_m$ *such that*

$$\bar{c}_N := c_N - (B^{-1}N)^T c_B \geq 0$$

where,

$$B := (a_{i_1}, ..., a_{i_m}), N := (a_{j_1}, ..., a_{j_{n-m}})$$

We call the inequality the optimality criterion.

Proof of (i). Let us assume that the standard form has an optimal solution. We set

$$f(x) := c^T x \ (x \in \mathbb{R}^n)$$

and

$$S := \{(x, (i_1, ..., i_m)) | x \text{ is an optimal solution}, i_1 < ... < i_m, a_{i_1}, ..., a_{i_m} \text{ are linear independent.}\}$$

From (A1), $S \neq \phi$. For each $s := (x, (i_1, ..., i_m)) \in S$,

$$n(s) := \#\{j | j \notin \{i_1, ..., i_m\}, x_j \neq 0\}$$

We fix $s := (x, (i_1, ..., i_m)) \in S$ such that

$$n(s) = \min\{n(s') | s' \in S\}$$

For aiming contradiction,

$$n(s) > 0$$

Without generality, we can assume that $(i_1, ..., i_m) = (1, 2, ..., m)$ and $x_{m+1} \neq 0$. And we can assume

$$x_{m+1} := \inf\{x' \geq 0 | (x_B, x', x_{m+1}, ..., x_n) \text{ is an optimal solution}\} \tag{12.1.1}$$

We set

$$B := (a_1, ..., a_m), N := (a_{m+1}, ..., a_n)$$

and

$$x_B := (x_1, ..., x_m), x_N := (x_{m+1}, ..., x_n)$$

Since $b = Ax = Bx_B + Nx_N$,

$$x_B = B^{-1}b - B^{-1}Nx_N$$

We set

$$(d_{m+1}, ..., d_n) := B^{-1}N$$

If $d_{m+1} = 0$, for any $x'_{m+1} \in [0, \infty)$, $(x_B, x'_{m+1}, x_{m+2}, ..., x_n)$ is a feasible solution. Since x is an optimal solution, $c_{m+1} = 0$ and for any $x'_{m+1} \in [0, \infty)$, $(x_B, x'_{m+1}, x_{m+2}, ..., x_n)$ is an optimal solution. Therefore, $(x_B, 0, x_{m+2}, ..., x_n)$ is an optimal solution. That contradicts with the minimality of $n(s)$. So, $d_{m+1} \neq 0$.

Next, we will show that there is i such that $d_{m+1,i} \neq 0$ and $x_i = 0$. Let us assume for any i such that $d_{m+1,i} \neq 0$, $x_i \neq 0$. Then, there is $\epsilon \in (0, x_{m+1})$ such that for any $x' \in [x_{m+1} - \epsilon, x_{m+1} + \epsilon]$, $(x'_B, x', x_{m+1}, ..., x_n)$ is a feasible solution. Here, $x'_B := B^{-1}b - B^{-1}N(x'_{m+1}, x_{m+2}, ..., x_n)$. If $c_{m+1} \neq 0$,

$$\min\{f((x_B, x_{m+1} - \epsilon, x_{m+1}, ..., x_n)), f((x_B, x_{m+1} + \epsilon, x_{m+1}, ..., x_n))\} < f(x)$$

That is a contradiction. So, $c_{m+1} = 0$. Therefore, there is $x'_{m+1} \in (x_{m,m+1} + \epsilon]$ such that $(x'_B, x', x_{m+1}, ..., x_n)$ is a feasible solution and there is i such that $d_{m+1,i} \neq 0$ and $x'_i = 0$.

We will show $a_2, ..., a_m, a_{m+1}$ are linear independent. For aiming contradiction, let us assume $a_2, ..., a_m, a_{m+1}$ are linear dependent. Then there are $\alpha_1, ..., \alpha_m$ such that

$$a_{m+1} = (a_2, ..., a_m) \begin{pmatrix} \alpha_1 \\ ... \\ \alpha_m \end{pmatrix}$$

By multiplying B^{-1},

$$d_{m+1} = (e_2, ..., e_m) \begin{pmatrix} \alpha_1 \\ ... \\ \alpha_m \end{pmatrix}$$

The right side has zero 1'th row, while the left side has nonzero 1'th row. That is a contradiction. □

Proof of (ii). From (i), there is a basic optimal solution $x := (x_B, 0)$. We can assume $i_1 = 1, i_2 = 2, ..., i_m = m$. Then $Bx_B = b$. That implies

$$x_B = B^{-1}b$$

Let us fix any solution $x' := (x'_B, x'_N) \geq 0$. Then $c_B^T x'_B + c_N^T x'_N \geq c_B^T x_B$. That implies

$$c_B^T x'_B + c_N^T x'_N \geq c_B^T B^{-1} b$$

Since $Bx'_B + Nx'_N = b$, $x'_B = B^{-1}b - B^{-1}Nx'_N$. Therefore,

$$c_B^T B^{-1}b - c_B^T B^{-1}Nx'_N + c_N^T x'_N \geq c_B^T B^{-1}b$$

That implies $(c_N - (B^{-1}N)^T c_B)^T x'_N \geq 0$. That means

$$c_N - (B^{-1}N)^T c_B \geq 0$$

□

Definition 12.1.4 (Dual Problem). *We take over the notations in Definition12.1.1 We call the following problem the dual problem of the original problem with standard form.*

$$arg \max_{y \geq 0, A^T y \leq c} b^T y$$

We call the original problem the primary problem.

Theorem 12.1.5 (Weakly Duality Theorem). *The followings are settings ans assumptions.*

(S1) $A \in M(m, n; \mathbb{R})$.

(A1) $rank(A) = m < n$.

(S2) $b \in \mathbb{R}^m$.

(S3) $c \in \mathbb{R}^n$.

Then the followings hold.

(i) Let us fix x which is any feasible solution of the primal problem and fix y which is any feasible solution of the dual problem. Then

$$b^T y \leq c^T x$$

(ii) Let us assume there are x and y such that x is a feasible solution of the primary problem and y is a feasible solution of the dual problem and

$$b^T y = c^T x$$

Then x is an optimal solution of the primal and y is an optimal solution of the dual.

Proof of (i). Since $y^T A \leq c^T$,

$$b^T y = y^T b = y^T Ax \leq c^T x$$

□

Proof of (ii). It is clear from (i). □

Theorem 12.1.6 (Duality Theorem). *The followings are settings ans assumptions.*

(S1) $A \in M(m, n; \mathbb{R})$.

(A1) $rank(A) = m < n$.

(S2) $b \in \mathbb{R}^m$.

(S3) $c \in \mathbb{R}^n$.

(A2) The primary problem has a basic optimal solution $x = (x_B, 0)$.

(S4) $y := (B^T)^{-1} c_B$. *We call it a simplex multiplier.*

Then y is an optimal solution of the dual and

$$b^T y = c^T x$$

Proof.

$$A^T y = \begin{pmatrix} B^T \\ N^T \end{pmatrix} (B^T)^{-1} c_B = \begin{pmatrix} c_B \\ (B^{-1}N)^T c_B \end{pmatrix}$$

From optimality criterion,

$$A^T y \leq \begin{pmatrix} c_B \\ (B^{-1}N)^T c_B \end{pmatrix} = c$$

Therefore, y is a feasible solution of the dual. And,

$$b^T y = y^T b = c_B^T B^{-1} b = c_B^T B^{-1} B x_B = c^T x$$

From weakly duality theorem, y is an optimal solution of the dual. □

Proposition 12.1.7. *The followings are settings ans assumptions.*

(S1) $A \in M(m, n; \mathbb{R})$.

(A1) $rank(A) = m < n$.

(S2) $b \in \mathbb{R}^m$.

(S3) $c \in \mathbb{R}^n$.

(S4) We call the following problem the original problem.

$$arg \min_{Ax \geq b, x \geq 0} (c^T x)$$

(S5) We call the following problem the original problem with standard form.

$$arg \min_{Ax - z = b, x, z \geq 0} (c^T x)$$

Then

(i) The dual problem of the original problem with standard form is equivalent to the following problem.

$$arg \max_{A^T y \leq c, y \geq 0} (b^T y)$$

(ii) We call the following problem the dual problem with standard form.

$$arg \min_{-A^T y - z = -c, y, z \geq 0} (-b^T y)$$

Then the dual problem of it is equivalent to the original problem.

Proof of (i). We set

$$\bar{c} := \begin{pmatrix} c & 0_m \end{pmatrix}, \bar{A} := \begin{pmatrix} A & -E \end{pmatrix}$$

Then the dual problem is the following.

$$arg \max_{\bar{A}^T y \leq \bar{c}, y \geq 0} (b^T y)$$

And

$$\bar{A}^T y \leq \bar{c} \iff A^T y \leq c, -y \leq 0$$

Therefore, the dual problem is equivalent to the following problem.

$$arg \max_{A^T y \leq \bar{c}, y \geq 0} (b^T y)$$

□

Proof of (ii). We set

$$\bar{b} := \begin{pmatrix} -b & 0_n \end{pmatrix}, \bar{A} := \begin{pmatrix} -A^T & -E \end{pmatrix}$$

Then the dual of the dual problem with standard form is the following.

$$arg \max_{\bar{A}^T x \leq -\bar{b}, x \geq 0} (-c^T x)$$

And

$$\bar{A}^T x \leq -\bar{b} \iff A^T x \leq b, -x \leq 0$$

Therefore, the dual problem is equivalent to the following problem.

$$arg \min_{A^T x \leq b, y \geq 0} (c^T x)$$

□

12.2 MILP and Branch-and-Bound Method

Definition 12.2.1 (MILP:Mixed integer linear programming)**.** *Let*

(S1) $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

(S2) $S := \{(x,y) \in (\mathbb{Z}_+)^n \times (\mathbb{R}_+)^p | g(x,y) := Ax + Gy \le b\}$

We call the following problem a MILP.

$$\begin{aligned} &max\ f(x,y) := c^t x + h^t y \\ &subject\ to\ (x,y) \in S \end{aligned}$$

We succeed notations in Definition12.2.1. And we set

$$S^0 := \{(x,y) \in (\mathbb{R}_+)^n \times (\mathbb{R}_+)^p | Ax + Gy \le b\}$$

Let us assume the MILP has a opmimal solution (x^*, y^*) and the optimal optimal value z^*. So $S^0 \neq \phi$. Let us fix $(x,y) \in S^0$.

Algorithm Branch-and-Bound Method

Input: $S^0 \neq \phi$
Step 1: Take a $(x^0, y^0) \in S^0$ and $(\underline{x}, \underline{y}, \underline{z}) \leftarrow (x_0, y_0, f(x^0, y^0))$ and $\mathcal{S} \leftarrow S_0$
Step 2: Take $j \in \{1, 2, ..., n\}$. $S_{00} := \{(x,y) \in S | x_j \le \lfloor x_j^0 \rfloor\}$ and $S_{01} := \{(x,y) \in S | x_j \ge \lceil x_j^0 \rceil\}$ and
  MILP$_{00}$: $max f(S_{00})$ and MILP$_{01}$: $max f(S_{01})$.
  Delete S_0 from $\mathcal{S}$ and add S_{00} and S_{01} to $\mathcal{S}$.
Step 3: **for** $S_\alpha \in \mathcal{S}$ **do**
    Solve $LP_\alpha : max f(S_\alpha)$.
    if LP_α is not feasible **then**
      Delete S_α from $\mathcal{S}$.
    else
      We set (x^α, y^α) which is a optimal solution and z^α which is its optimal value.
      Delete S_α from $\mathcal{S}$.
      if $x^\alpha \in \mathbb{Z}_+^n$ **then**
        if $z^\alpha > \underline{z}$ **then**
          $(\underline{x}, \underline{y}, \underline{z}) \leftarrow (x^\alpha, y^\alpha, f(x^\alpha, y^\alpha))$.
        end if
      else$z^\alpha > \underline{z}$
        Take $j \in \{1, 2, ..., n\}$. $S_{\alpha 0} := \{(x,y) \in S_\alpha | x_j \le \lfloor x_j^\alpha \rfloor\}$ and $S_{\alpha 1} := \{(x,y) \in S_\alpha | x_j \ge \lceil x_j^\alpha \rceil\}$.
        Add $S_{\alpha 0}$ and $S_{\alpha 1}$ to $\mathcal{S}$.
      end if
    end if
  end for
Output: $(\underline{x}, \underline{y}, \underline{z})$.

12.3 Meyer's Fundamental Theorem

12.3.1 Main result

The propositions shown in this subsection will not be presented with proofs in this subsection, but will be presented with proofs in the subsections that follow.

Definition 12.3.1 (Polyhedron)**.** *Let* $A \in M(m,n,\mathbb{R}), b \in \mathbb{R}^m$. *We call*

$$P := \{x \in \mathbb{R}^n | Ax \le b\}$$

a Polyhedron in $\mathbb{R}^n$ *or a* $\mathcal{H}$*-polyhedron. We call the right side* $\mathcal{H}$*-representation. If* $A \in M(m,n,\mathbb{Q}), b \in \mathbb{Q}^m$ *then* P *is a rational polyhedron.*

Definition 12.3.2 (Recession cone)**.** *Let P be a nonempty polyhedron. We call*

$$rec(P) := \{r \in \mathbb{R}^n | x + \lambda r \in P, \forall x \in P, \forall \lambda \in \mathbb{R}_+\}$$

the recession cone of P.

Notation 12.3.3. *Let*

(S1) $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

We set

$$P(A,G,b) := \{(x,y) \in (\mathbb{R}_+)^n \times (\mathbb{R}_+)^p | g(x,y) := Ax + Gy \leq b\}$$

Definition 12.3.4 (Convex, Convex combination)**.** *Let $A \subset \mathbb{R}^n$. We say A is convex if $\sum_{i=1}^n \lambda_i a_i \in A$ for $a_1, ..., a_n \in A$ and $\lambda_1, ..., \lambda_n \subset [0,1]$ such that $\sum_{i=1}^n \lambda_i = 1$. We call the sum*

$$\sum_{i=1}^n \lambda_i a_i$$

convex combination of $a_1, ..., a_n$.

Proposition 12.3.5. *Let*

(S1) $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

(S2) $S := \{(x,y) \in (\mathbb{Z}_+)^n \times (\mathbb{R}_+)^p | g(x,y) := Ax + Gy \leq b\}$

Then

(i)

$$sup\{c^t x + h^t y | (x,y) \in S\} = sup\{c^t x + h^t y | (x,y) \in conv(S)\}$$

Furthermore, there is $(x,y) \in S$ such that $c^t x + h^t y = sup\{c^t x + h^t y | (x,y) \in S\}$ $\iff$ there is $(x,y) \in conv(S)$ such that $c^t x + h^t y = sup\{c^t x + h^t y | (x,y) \in S\}$

(ii) $ex(conv(S)) \subset S$

Theorem 12.3.6 (Meyer(1974)[44] Fundamental Theorem)**.** *Here are the settings and assumptions.*

(S1) $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $b \in \mathbb{Q}^m$.

(S2) $S := \{(x,y) \in P(A,G,b) | x \in (\mathbb{Z}_+)^n\}$.

Then there are $A' \in M(m,n,\mathbb{Q})$, $G' \in M(m,p,\mathbb{Q})$, $b' \in \mathbb{Q}^m$ such that

$$conv(S) = P(A', G', b')$$

By Proposition12.3.5 and Theorem12.3.6, MILP

$$\begin{aligned} &\max\ f(x,y) := c^t x + h^t y \\ &\text{subject to } (x,y) \in S \end{aligned}$$

is equal to a pure LP

$$\begin{aligned} &\max\ f(x,y) := c^t x + h^t y \\ &\text{subject to } (x,y) \in P(A', G', b') \end{aligned}$$

We set

$$\tilde{A} := \begin{pmatrix} A \\ A' \end{pmatrix}, \tilde{G} := \begin{pmatrix} G \\ G' \end{pmatrix}, \tilde{b} := \begin{pmatrix} b \\ b' \end{pmatrix},$$

Then clearly

$$S = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^p | (x,y) \in P(\tilde{A}, \tilde{G}, \tilde{b}), x \in \mathbb{Z}^n\}$$

and MILP

$$\begin{aligned} &\max\ f(x,y) := c^t x + h^t y \\ &\text{subject to } (x,y) \in S \end{aligned}$$

has a continuous ralaxation

$$\max\ f(x,y) := c^t x + h^t y$$
$$\text{subject to } (x,y) \in P(\tilde{A},\tilde{G},\tilde{b})$$

whose optimal value is equal to the one of the original MILP. And we can effectively find an optimal solution of this continuas ralaxation which is contained in S.

From the above discussion, the following can be shown.

Proposition 12.3.7. *Here are the settings and assumptions.*

(S1) $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

(S2) $S := \{(x,y) \in P(A,G,b) | x \in (\mathbb{Z}_+)^n\}$.

Then there is $M \in \mathbb{N}$ *and are* $\tilde{A} \in M(M,n,\mathbb{Q})$, $\tilde{G} \in M(M,p,\mathbb{Q})$, $\tilde{b} \in \mathbb{Q}^M$ *such that*

$$S = P(\tilde{A},\tilde{G},\tilde{b}) \cap \mathbb{Z}^n_+ \times \mathbb{R}^p_+$$

and

$$conv(S) = P(\tilde{A},\tilde{G},\tilde{b})$$

12.3.2 Fourier elimination and Farkas Lemma

Definition 12.3.8 (Conic combination)**.** *Let* $v_1,...,v_m \in \mathbb{R}^n$. *For every* $\lambda_1,...,\lambda_m \geq 0$, *we call* $\sum_{i=1}^m \lambda_i v_i$ *a conic combination of* $v_1,...,v_m$.

Theorem 12.3.9 (Fourier Elimination)**.** *Let*

(S1) $A \in M(m,n,\mathbb{R})$, $b \in \mathbb{R}^m$.

(S2) $I^+ := \{i | a_{i,n} > 0\}$, $I^- := \{i | a_{i,n} < 0\}$, $I^0 := \{i | a_{i,n} = 0\}$.

(S3) $a'_{i,k} := \dfrac{a_{i,k}}{|a_{i,n}|}$ $(i \in I^+ \cup I^-, k \in \{1,2,...,n-1\})$, $b'_i := \dfrac{b_i}{|a_{i,n}|}$ $(i \in I^+ \cup I^-)$.

(S4) $\tilde{A} := (A,b) \in M(m,n+1,\mathbb{R})$.

(S5) We set $\tilde{A}_{n-1} \in M(\#I^+ * \#I^- + \#I^0, n, \mathbb{R})$ *and* $b' \in \mathbb{R}^{(\#I^+ * \#I^- + \#I^0)}$ *by*

$$(kq\text{-th row of } \tilde{A}_{n-1}) = \frac{1}{|a_{k,n}|}(k\text{-th row of } \tilde{A}) + \frac{1}{|a_{q,n}|}(q\text{-th row of } \tilde{A})\ (\forall k \in I^+, \forall q \in I^-)$$

and

$$((\#I^+ * \#I^- + j)\text{-th row of } \tilde{A}') = (j\text{-th row of } \tilde{A})\ (j = 1,2,...,\#I^0)$$

(S6) $x^i := (x_1,...,x_i)$ $(x \in \mathbb{R}^n)$

Then

(i) $Ax \leq b, x \in \mathbb{R}^n$ *is feasible if and only if*

$$\sum_{i=1}^{n-1} (a'_{k,i} + a'_{q,i}) x_i \leq b'_k + b'_q\ (\forall k \in I^+, \forall q \in I^-),$$
$$\sum_{i=1}^{n-1} a_{p,i} x_i \leq b_p\ (\forall p \in I^0)$$

(ii) If $A \in M(m,n,\mathbb{Q})$ *and* $b \in \mathbb{Q}^m$, *then* $a'_{k,i}, a'_{q,i}, b'_k, b'_q \in \mathbb{Q}$ $(\forall k \in I^+, \forall i \in \{1,2,...,n-1\}, \forall q \in I^-)$.

(iii) $\{x \in \mathbb{R}^n | Ax \leq b\} \neq \phi \iff \{x \in \mathbb{R}^{n+1} | \tilde{A}(x^t,-1)^t \leq 0\} \neq \phi \iff \{x \in \mathbb{R}^n | \tilde{A}_{n-1}((x^{n-1})^t,-1)^t \leq 0\} \neq \phi$.

(iv) For each $i \in \{0,1,...,n-1\}$, *there is* $m_i \in \mathbb{N}$ *and* $\tilde{A}_i \in M(m_i, i+1, \mathbb{R}$ *such that every row of* $\tilde{A}_i$ *is a conic combination of rows of* $\tilde{A}$ *and*

$$\{x \in \mathbb{R}^n | Ax \leq b\} \neq \phi \iff \{x \in \mathbb{R}^i | \tilde{A}_i((x^i)^t,-1)^t \leq 0\}$$

(v) If $\tilde{A} \in M(m,n+1,\mathbb{Q})$ *then* $\tilde{A}_i \in M(m_i,i+1,\mathbb{Q})$ $i \in \{0,1,...,n-1\}$.

(vi) $\{x \in \mathbb{R}^n | Ax \leq b\} \neq \phi \iff \tilde{A}_0 \leq 0.$

Proof of the 'only if' part in (i). Let us assume $x \in \mathbb{R}^n$ such that $Ax \leq b$. Then

$$\sum_{i=1}^{n-1} a'_{k,i} x_i + x_n \leq b'_k \ (\forall k \in I^+)$$

and

$$\sum_{i=1}^{n-1} a'_{q,i} x_i - x_n \leq b'_q \ (\forall q \in I^-)$$

So, by adding the left and right sides of these two inequalities, respectively, the following holds.

$$\sum_{i=1}^{n-1} (a'_{k,i} + a'_{q,i}) x_i \leq b'_k + b'_q \ (\forall k \in I^+, \forall q \in I^-),$$
$$\sum_{i=1}^{n-1} a_{p,i} x_i \leq b_p \ (\forall p \in I^0)$$

□

Proof of the 'if' part in (i). Let us assume

$$\sum_{i=1}^{n-1} (a'_{k,i} + a'_{q,i}) x_i \leq b'_k + b'_q \ (\forall k \in I^+, \forall q \in I^-),$$
$$\sum_{i=1}^{n-1} a_{p,i} x_i \leq b_p \ (\forall p \in I^0)$$

Then

$$\sum_{i=1}^{n-1} a'_{k,i} x_i - b'_k \leq -(\sum_{i=1}^{n-1} a'_{q,i} - b'_q) \ (\forall k \in I^+, \forall q \in I^-)$$

We set

$$x_n := \min\{-(\sum_{i=1}^{n-1} a'_{k,i} - b'_k) | k \in I^+\}$$

Then

$$x_n \geq \max\{(\sum_{i=1}^{n-1} a'_{q,i} - b'_q) | q \in I^-\}$$

So, $Ax \leq b$. □

Proof of (ii)-(iv). These are followed by (i). □

Theorem 12.3.10 (Farkas Lemma I)**.** *Let*

(S1) $A \in M(m, n, \mathbb{R})$, $b \in \mathbb{R}^m$.

Then

$$\{x \in \mathbb{R}^n | Ax \leq b\} = \phi \iff \{v \in \mathbb{R}^m | A^t v = 0, b^t v < 0, v \geq 0\} \neq \phi$$

Proof of 'only if' part. By Fourier elimination method (iv), there are $m_0 \in \mathbb{N}$ and $U \in M(m_0, n, \mathbb{R})$ such that $U \geq 0$ and $U\tilde{A} = (O_{m_i,n-1}, b^0)$ and $b^0 \not\geq 0$. Then there is $u \in \mathbb{R}^{m_0}$ such that $u^t b^0 < 0$. We set

$$v := (u^t U)^t$$

Then $v \geq 0$ and $Av = 0$ and $v^t b < 0$. □

Proof of 'if' part. Let us assume $\exists v \in \mathbb{R}^m$ such that $v^t A = 0$ and $v^t b < 0$ and $v \geq 0$. For any $x \in \mathbb{R}^n$, $v^t Ax = 0$. So, $Ax \not\leq b$. □

Theorem 12.3.11 (Farkas Lemma II)**.** *Let*

(S1) $A \in M(m, n, \mathbb{R})$, $b \in \mathbb{R}^m$.

Then

$$\{x \in \mathbb{R}^n | Ax = b, x \geq 0\} \neq \phi \iff \{u \in \mathbb{R}^m | A^t u \leq 0\} \subset \{u \in \mathbb{R}^m | u^t b \leq 0\}$$

Proof of ' $\implies$ *'.* Let us fix $x \in \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$. Let us fix any $u \in \{u \in \mathbb{R}^m | A^t u \leq 0\}$. So, $b^t u \leq 0$. □

Proof of ' $\impliedby$ *'.* Let us assume

$$\{x \in \mathbb{R}^n | Ax = b, x \geq 0\} = \phi$$

Then

$$\{x \in \mathbb{R}^n | Ax \leq b, -Ax \leq -b, x \geq 0\} = \{x \in \mathbb{R}^n | Bx \leq c\} = \phi$$

Here,

$$B := \begin{pmatrix} A \\ -A \\ -I_n \end{pmatrix}, c := \begin{pmatrix} b \\ -b \\ O_{n,1} \end{pmatrix}$$

and I_n is the n-th unit matrix. By Farkas Lemma I, there are $v \in \mathbb{R}^m_+$ and $v' \in \mathbb{R}^m_+$ and $w \in \mathbb{R}^n_+$ such that

$$B^t \begin{pmatrix} v \\ v' \\ w \end{pmatrix} = 0, \begin{pmatrix} v \\ v' \\ w \end{pmatrix}^t c < 0$$

This implies

$$A(-(v - v')) = -w, -(v - v')^t b > 0$$

We set $u := -(v - v')$. Then

$$u \in \{u \in \mathbb{R}^m | A^t u \leq 0\} \setminus \{u \in \mathbb{R}^m | u^t b \leq 0\}$$

□

12.3.3 Polyhedron and Minkowski Weyl Theorem

Definition 12.3.12 (Polytope)**.** *We say* $A \subset \mathbb{R}^n$ *is a polytope if there are finite vectors* $v_1, ..., v_m \in \mathbb{R}^n$ *such that* $A = conv(v_1, ..., v_m)$. *We call* $v_1, ..., v_m$ *vertices of* A. *If* $v_1, ..., v_m \in \mathbb{Q}^n$, *we call* A *is a rational polytope.*

Definition 12.3.13 (Cone)**.** *We say* $C \subset \mathbb{R}^n$ *is a cone if* $0 \in C$ *and for every* $x \in C$ *and* $\lambda \in \mathbb{R}_+$ $\lambda x \in C$.

By the definition of cone, the following holds.

Proposition 12.3.14. *Any cone containing nonzero vector is not bounded.*

Definition 12.3.15 (Convex Cone)**.** *We say* $C \subset \mathbb{R}^n$ *is a convex cone if* C *is cone and every conic combination of finite vectors of* C *is contained in* C.

Because every intersection of convex cones is also convex cone, the following holds.

Proposition 12.3.16 (Convex Cone generated by a set)**.** *Let us assume* A *is any subset of* $\mathbb{R}^n$. *Then there is the minimum convex cone containing* A. *We denote this convex cone by* $cone(A)$.

Definition 12.3.17 (Polyhedral cone)**.** *Let*

(S1) $A \in M(m, n, \mathbb{Q})$.

We call

$$P := \{x \in \mathbb{R}^n | Ax \leq 0\}$$

a Polyhedral cone.

Theorem 12.3.18 (Minkowski Weyl Theorem for cones)**.** *Let*

(S1) $C \subset \mathbb{R}^n$.

Then C is a Polyhedral cone if and only if C is finite generated cone.

STEP1. Proof of 'if' part. Let us assume C is finite generated cone. Then there is $r_1, ..., r_k \in \mathbb{R}$ such that $C = cone(r_1, ..., r_k)$. We set $R = (r_1, ..., r_k)$.

By applying Fourier elimination method k times to the the following inequality

$$-\mu \leq 0, R\mu \leq x, -R\mu \leq -x$$

and Fourier elimination method (vi), there is $A \in M(m, n, \mathbb{R})$ such that the above inequality is equivalent to

$$Ax \leq 0$$

So, $C = \{x \in \mathbb{R}^n | Ax \leq 0\}$.

□

STEP2. Proof of 'only if' part. Let us assume C is a Polyhedral cone. So, there is $A \in M(m, n, \mathbb{R})$ such that $C = \{x \in \mathbb{R}^n | Ax \leq 0\}$. We set $C^* := \{y \in \mathbb{R}^n | \exists \nu \in \mathbb{R}^m_+ \text{ such that } A^t v = y\}$. Then

$$C^* = cone(a^1, ..., a^m)$$

Here, $a^i \in \mathbb{R}^n$ is the i-th row vector of A $(i = 1, 2, ..., m)$. By STEP1, there is $R \in M(n, k, \mathbb{R})$ such that

$$C^* = \{y \in \mathbb{R}^n | R^t y \leq 0\}$$

We denote the i-th column vector of R by r^i $(i = 1, 2, ..., k)$. We will show

$$C = cone(r_1, ..., r_k)$$

Let us fix any $x \in cone(r_1, ..., r_k)$. Then there are $\nu_1, ..., \nu_k \in \mathbb{R}_+$ such that $x = R\nu$. Because $a_i = A^t e_i$ $(i = 1, 2, ..., m)$, $a_i \in C^*$ $(i = 1, 2, ..., m)$. So, $AR \leq 0$. This implies $Ax = AR\nu \leq 0$. This means $x \in C$. We have shown $cone(r_1, ..., r_k) \subset C$.

Let us fix any $\bar{x} \in cone(r_1, ..., r_k)^c$. So, $\{\nu \in \mathbb{R}^k | R\nu = \bar{x}, \nu \geq 0\} = \phi$. By Farkas Lemma II, there is $y \in \mathbb{R}^n$ such that $R^t y \leq 0$ and $y^t \bar{x} > 0$. So, $y \in C^*$. Then there are $\nu \in \mathbb{R}^m_+$ such that $y = A^t \nu$. So, $\nu^t A\bar{x} > 0$. Because $\nu \in \mathbb{R}^m_+$, this implies $A\bar{x} \nleq 0$. This means $\bar{x} \in C^c$. Consequently $C \subset cone(r_1, ..., r_k)$. □

Definition 12.3.19 (Minkowski sum)**.** *Let $A, B \subset \mathbb{R}^n$. We call*

$$A + B$$

the Minkowski sum of A and B.

Proposition 12.3.20. *Let*

(i) Minkowski sum of any two convex set is convex.

(ii) For any two subset $A, B \subset \mathbb{R}^n$,

$$conv(A + B) = conv(A) + conv(B)$$

Proof of (i). Let $A, B \subset \mathbb{R}^n$ be convex. For any $a_1, ..., a_m \in A$ and $b_1, ..., b_m \in B$ and $\lambda_1, ..., \lambda_m \subset [0, 1]$ such that $\sum_{i=1}^m \lambda_i = 1$,

$$\sum_{i=1}^m \lambda_i (a_i + b_i) = \sum_{i=1}^m \lambda_i a_i + \sum_{i=1}^m \lambda_i b_i \in A + B$$

So, $A + B$ is convex. □

Proof of (ii). By (i), $conv(A) + conv(B)$ is convex. And $A + B \subset conv(A) + conv(B)$. So, $conv(A + B) \subset conv(A) + conv(B)$. Let us fix any $a_1, .., a_k \in A$ and $b_a, ..., b_l \in B$ and $\lambda_1, ..., \lambda_k, \mu_1, ..., \mu_l \in [0, 1]$ such that $\sum_{i=1}^k \lambda_i = 1$ and $\sum_{i=1}^l \mu_i = 1$. Then

$$\sum_{i=1}^k \lambda_i a_i + \sum_{j=1}^l \mu_j b_j = \sum_{j=1}^l \mu_j (\sum_{i=1}^k \lambda_i a_i + b_j) = \sum_{j=1}^l \mu_j (\sum_{i=1}^k \lambda_i (a_i + b_j)) = \sum_{i,j} \lambda_i \mu_j (a_i + b_j) \in conv(A + B)$$

□

Theorem 12.3.21 (Minkowski-Weyl Theorem). *A subset $P \subset \mathbb{R}^n$ is a Polyhedron if and only if there is a polytope Q a finite generated cone C such that*

$$P = Q + C$$

We call the right side $\mathcal{V}$-representation and call P a $\mathcal{V}$-polyhedron.

Proof of 'only if' part. Let us fix $A \in M(m, n, \mathbb{R})$ and $b \in \mathbb{R}^m$ such that $P = \{x \in \mathbb{R}^n | Ax \leq b\}$. We set

$$C_P := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} | Ax - yb \leq 0, y \leq 0\}$$

Then clearly

$$P = \{x \in \mathbb{R}^n | (x, 1) \in C_P\}$$

By Minkowski Weyl Theorem for cones, there are $r^1, r^2, ..., r^K \mathbb{R}^{n+1}$ such that

$$C_P := cone(r^1, r^2, ..., r^K)$$

Because C_P is a cone, we can assume $r^i_{n+1} = 0$ or 1 $(\forall i)$. So, there are $u_1, ..., u_k \in \mathbb{R}^n$ and $v_1, ..., v_l \in \mathbb{R}^n$ such that

$$C_P = cone(\begin{pmatrix} u_1 \\ 1 \end{pmatrix}, ... \begin{pmatrix} u_k \\ 1 \end{pmatrix}, \begin{pmatrix} v_1 \\ 0 \end{pmatrix}, ..., \begin{pmatrix} v_l \\ 0 \end{pmatrix})$$

So,

$$P = conv(u^1, ..., u^k) + cone(v^1, ..., v^l)$$

□

Proof of 'if' part. We assume we can get

$$P = conv(u^1, ..., u^k) + cone(v^1, ..., v^l)$$

Then

$$P = cone(\begin{pmatrix} u^1 \\ 1 \end{pmatrix}, ..., \begin{pmatrix} u^k \\ 1 \end{pmatrix}, \begin{pmatrix} v^1 \\ 0 \end{pmatrix}, ..., \begin{pmatrix} v^l \\ 0 \end{pmatrix}) \cap \mathbb{R}^n \times \{1\}$$

Because $cone(\begin{pmatrix} u^1 \\ 1 \end{pmatrix}, ..., \begin{pmatrix} u^k \\ 1 \end{pmatrix}, \begin{pmatrix} v^1 \\ 0 \end{pmatrix}, ..., \begin{pmatrix} v^l \\ 0 \end{pmatrix})$ is a Polyhedral cone, P is a Polyhedron. □

Proof of the last part. □

Proposition 12.3.22. *Let*

(i) *Bounden Polyhedron is polytone.*

(ii) *If $A \in M(m, n, \mathbb{Q})$ and $b \in \mathbb{Q}^m$, then there are $v_1, ..., v_k \in \mathbb{Q}^n$ and $r_1, ..., r_l \in \mathbb{Z}^n$ such that*

$$P := \{x \in \mathbb{R}^n | Ax \leq b\} = conv(v_1, ..., v_k) + cone(r_1, ..., r_l)$$

If P is bounded, P is a rational polytope.

(iii) *$P \subset \mathbb{R}^n$ is a rational polyhedron if and only if P is a minkowski sum of a rational polytope and a convex cone generated by finite rational vectors.*

Proof of (i). By Proposition12.3.14, (i) holds. □

Proof of (ii). By the proof of Theorem12.3.18, (ii) holds. □

Proof of (iii). By the proof of Theorem12.3.18, (iii) holds. □

12.3.4 Perfect formulation and Meyer's Foundamental theorem

Proposition 12.3.23. *Here are the settings and assumptions.*

(S1) $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

(S2) $S := \{(x,y) \in (\mathbb{Z}_+)^n \times (\mathbb{R}_+)^p | g(x,y) := Ax + Gy \le b\}$

Then

(i)

$$sup\{c^t x + h^t y | (x,y) \in S\} = sup\{c^t x + h^t y | (x,y) \in conv(S)\}$$

Furthermore, there is $(x,y) \in S$ *such that* $c^t x + h^t y = sup\{c^t x + h^t y | (x,y) \in S\} \iff$ *there is* $(x,y) \in conv(S)$ *such that* $c^t x + h^t y = sup\{c^t x + h^t y | (x,y) \in S\}$

(ii) $ex(conv(S)) \subset S$

Proof of the first part of (i). Because $S \subset conv(S)$,

$$sup\{c^t x + h^t y | (x,y) \in S\} \le sup\{c^t x + h^t y | (x,y) \in conv(S)\}$$

We can assume $z^* = sup\{c^t x + h^t y | (x,y) \in S\} < \infty$. Let us set $H := \{(x,y) \in \mathbb{R}^{n+p} | c^t x + h^t y \le z^*\}$. Because H is convex and $S \subset H$, $conv(S) \subset H$. So,

$$sup\{c^t x + h^t y | (x,y) \in S\} \ge sup\{c^t x + h^t y | (x,y) \in conv(S)\}$$

□

Proof of the last part of (i). The part of $\Longrightarrow$ is clear. We set $d := (c,h)$. Let us assume there is $\bar{z} = (\bar{x}, \bar{y})$ such that $d^t \bar{z} = sup\{c^t x + h^t y | (x,y) \in conv(S)\}$. Then there are $\lambda_1, ..., \lambda_k > 0$ and $z_1, ..., z_k \in S$ such that $\bar{z} = \sum_{i=1}^k \lambda_i z_i$. Clearly $d^t z_i \le d^t \bar{z}$ $(\forall i)$. Because $d^t \bar{z} = \sum_{i=1}^k d^t \lambda_i z_i$, there is i such that $d^t z_i \ge d^t \bar{z}$. So, $d^t z_i = sup\{c^t x + h^t y | (x,y) \in conv(S)\}$. □

Proof of (ii). Let us fix any $v \in ex(conv(S))$. Because $ex(conv(S)) \subset conv(S)$, there are $\lambda_1, ..., \lambda_m \in (0,1]$ and $v_1, ..., v_m \in S$ such that $v = \sum_{i=1} \lambda_i v^i$. We can assume $m > 1$. We set $v' := \sum_{i=2}^m \frac{\lambda_i}{1-\lambda_1} v^i$. Then $v' \in conv(S)$. Because $v = \lambda_1 v_1 + (1-\lambda_1) v'$ and $v \in ex(conv(S))$, $v = v_1 \in S$. □

Proposition 12.3.24. *Let* $r^1, ..., r^K \in \mathbb{R}^n$. *Then*

$$conv(\sum_{i=1}^K \mathbb{Z}_+ r^i) = cone(r_1, ..., r^K)$$

Proof. We will show this by Mathematical induction. If $K = 1$, then this proposition holds. Let us fix any $k \in \mathbb{N}$ and assume this proposition holds for every $K \le k$.

We set $C := conv(\sum_{i=1}^{k+1} \mathbb{Z}_+ r^i)$. Clearly $C \subset cone(r^1, ..., r^{k+1})$. Let us fix $x \in cone(r^1, ..., r^{k+1})$. Then there are $\mu_1, ..., \mu_{k+1} \in \mathbb{R}_+$ such that $x = \sum_{i=1}^{k+1} \mu_i r^i$. We can assume $\mu_{k+1} > 0$. We set $\lambda := \frac{2\mu_{k+1}}{\lceil 2\mu_{k+1} \rceil}$. Because $0 \in C$, $2\mu_{k+1} r^{k+1} = (1-\lambda)0 + \lambda \lceil 2\mu_{k+1} \rceil r^{k+1} \in C$. By Mathematical induction assumption, $\sum_{i=1}^k 2\mu_i r^i \in C$. So,

$$\sum_{i=1}^{k+1} \mu_i r^i = \frac{1}{2}(2\mu_{k+1} r^{k+1} + \sum_{i=1}^k 2\mu_i r^i) \in C$$

So, $cone(r^1, ..., r^{k+1}) \subset C$. □

Theorem 12.3.25 (Meyer(1974)[44] Fundamental Theorem)**.** *Here are the settings and assumptions.*

(S1) $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

(S2) $S := \{(x,y) \in P(A,G,b) | x \in \mathbb{Z}^n\}$.

Then there are $A' \in M(m,n,\mathbb{Q})$, $G' \in M(m,p,\mathbb{Q})$, $b' \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$ *such that*

$$conv(S) = P(A', G', b')$$

STEP1. Decomposition of S. We can assume $S \neq \phi$. Then by Proposition12.3.22, there are $v^1, ..., v^t \subset \mathbb{Q}^{n+p}$ and $r^1, ..., r^q \subset \mathbb{Z}^{n+p}$ such that

$$P := P(A, G, b) = conv(v^1, ..., v^t) + cone(r^1, ..., r^q)$$

We set

$$T := \{\sum_{i=1}^{s} \lambda_i v^i + \sum_{j=1}^{q} \mu_j r^j | 0 \leq \lambda_i, \mu_j \leq 1 \ (\forall i, j), \sum_{i=1}^{s} \lambda_i = 1\} = conv(v^1, ..., v^t) + \sum_{j=1}^{q} [0, 1] r_j$$

Then T is bounded. There is $M \in \mathbb{N}$ and $D \in M(M, n+p, \mathbb{Q})$ such that

$$T = \{z \in \mathbb{R}^{n+p} | \exists \lambda \in \mathbb{R}^n_+, \exists \mu \in \mathbb{R}^p_+ \text{ s.t. } D \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \leq z, \sum_{i=1}^{s} \lambda_i \leq 1, -\sum_{i=1}^{s} \lambda_i \leq -1, \mu \leq 1\}$$

By Fourier elimination method, there are $C \in M(M, n, \mathbb{R})$ and $d \in \mathbb{Q}^n$ such that $T = \{x \in \mathbb{R}^n | Cx \leq d\}$. So, by Proposition12.3.22, T is a rational polytope.

Let

$$T_I := \{(x, y) \in \mathbb{Z}^n \times \mathbb{R}^p | (x, y) \in T\}, R_I := \{\sum_{j=1}^{q} \mu_j r^j | \mu_j \in \mathbb{Z}_+ \ (\forall j)\}$$

We will show

$$S = T_I + R_I$$

Because $T_I + R_I \subset T$ and i-th component of $T_I + R_I$ is integer for every $i \in \{1, 2, ..., s\}$, $T_I + R_I \subset S$.

Let us fix any $(x, y) \in \mathbb{Z}^n \times \mathbb{R}^p$ such that $(x, y) \in S$. Then there are $\lambda_1, ..., \lambda_s, \mu_1, ..., \mu_q \in [0, 1]$ such that $\sum_{i=1}^{s} \lambda_i = 1$ and

$$(x, y) = \sum_{i=1}^{s} \lambda_i v^i + \sum_{j=1}^{q} \mu_j r^j$$

We set

$$(x', y') := \sum_{i=1}^{s} \lambda_i v^i + \sum_{j=1}^{q} (\mu_j - \lfloor \mu_j \rfloor) r^j, r := \sum_{j=1}^{q} \lfloor \mu_j \rfloor r^j$$

Then $(x', y') \in T_I$ and $r \in R_I$. So, $(x, y) \in T_I + R_I$. Consequently, $S = T_I + R_I$. □

STEP2. Proof that $conv(S)$ is a rational polyhedron. By Proposition12.3.20 and STEP1,

$$conv(S) = conv(T_I) + conv(R_I)$$

Because $conv(R_I) = conv(r^1, ..., r^q)$, by Proposition12.3.24, $conv(R_I)$ is a rational polyhedral cone. So, it is enough to show

$$conv(T_I) \text{ is a rational polytope}$$

Since T is bounded, $X := \{x \in \mathbb{Z}^n | \exists y \in \mathbb{R}^p \text{ such that } (x, y) \in T_I\}$ is bounded and so is a finite set.

For each $x \in X$, we set $T_x := \{(x, y) | \exists y \in \mathbb{R}^p \text{ such that } (x, y) \in T_I\}$. For any $\bar{x} \in X$,

$$T_{\bar{x}} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p | x = \bar{x} \text{ and } (x, y) \in T\}$$

Because T is a rational polytope, $T_{\bar{x}}$ is a rational polytope. We denote th set of all vertices of $T_{\bar{x}}$ by $V_{\bar{x}}$ for any $\bar{x} \in X$. We set $V := \cup_{x \in X} V_x$. V is a finite set. We will show

$$conv(T_I) = conv(V)$$

Because $T_I = \cup_{x \in X} T_x = \cup_{x \in X} conv(V_x) \subset conv(V)$, $conv(T_I) \subset conv(V)$. Because $V = \cup_{x \in X} V_x \subset \cup_{x \in X} conv(V_x) = \cup_{x \in X} T_x = conv(T_I)$, $conv(V) \subset conv(T_I)$. So, $conv(T_I) = conv(V)$. Consequently, $conv(T_I)$ is a rational polytope. □

By the proof of Theorem12.3.23, the following holds.

Theorem 12.3.26. *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $G \in M(m, p, \mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

(S2) $S := \{(x, y) \in P(A, G, b) | x \in \mathbb{Z}^n\}$.

Then there are

$$a_1, ..., a_k \in P(A, G, b) \cap \mathbb{Z}^n \times \mathbb{Q}^p = S$$

and

$$r_1, ..., r_l \in \mathbb{Z}^{n+p}$$

such that

$$conv(S) = conv(a_1, ..., a_k) + cone(r_1, ..., r_l)$$

12.3.5 Sharp MILP Formulation

Definition 12.3.27 (MILP Formulation)**.** *Here are the settings and assumptions.*

(S1) $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $B \in M(m,t,\mathbb{Q})$, $b \in \mathbb{Q}^m$.

(S2) $S \subset \mathbb{Q}^n$.

(S3) $T(A,G,B,b) := \{(x,y,z) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Z}^t | Ax + Gy + Bz \leq b\}$.

We say (A,G,B,b) *is a MILP formulation for* S *if and only if* S *is equal to the image of*

$$p_n : T(A,G,B,b) \ni (x,y,z) \mapsto x \in \mathbb{Q}^n$$

Clearly the following holds.

Proposition 12.3.28. *Here are the settings and assumptions.*

(S1) $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

(S2) $S := \{(x,y) \in P(A,G,b) | x \in (\mathbb{Z}_+)^n\}$.

(S3) *We set*

$$\tilde{A} := \begin{pmatrix} A \\ E_n \\ O_{p,n} \\ O_{n,n} \end{pmatrix}, \tilde{G} := \begin{pmatrix} G \\ O_{n,p} \\ -E_p \\ O_{n,p} \end{pmatrix}, \tilde{B} := \begin{pmatrix} B \\ -E_n \\ O_{p,n} \\ -E_n \end{pmatrix}, \tilde{b} := \begin{pmatrix} b \\ 0_n \\ 0_p \\ 0_n \end{pmatrix}$$

Then $(\tilde{A}, \tilde{G}, \tilde{B}, \tilde{b})$ *is a MILP formultation for* S.

Definition 12.3.29 (Sharp MILP Formulation)**.** *Here are the settings and assumptions.*

(S1) $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $B \in M(m,t,\mathbb{Q})$, $b \in \mathbb{Q}^m$.

(S2) $S \subset \mathbb{Q}^n$.

(Aq) (A,G,B,b) *is a MILP formulation for* S.

We say (A,G,B,b) *is sharp MILP formulation for* S *if and only if* $conv(S)$ *is equal to the image of*

$$p_n : \tilde{T}(A,G,B,b) \ni (x,y,z) \mapsto x \in \mathbb{Q}^n$$

Here, $\tilde{T}(A,G,B,b)$ *is a LP relaxation of* $T(A,G,B,b)$.

Theorem 12.3.30. *Here are the settings and assumptions.*

(S1) $S \subset \mathbb{Q}^n$.

(A1) *There are* $A \in M(m,n,\mathbb{Q})$, $G \in M(m,p,\mathbb{Q})$, $B \in M(m,t,\mathbb{Q})$, $b \in \mathbb{Q}^m$ *such that* (A,G,B,b) *is a MILP formulation for* S.

Then there there are $M \in \mathbb{N}$ *and* $\tilde{A} \in M(M,n,\mathbb{Q})$, $\tilde{G} \in M(M,p,\mathbb{Q})$, $\tilde{B} \in M(M,t,\mathbb{Q})$, $\tilde{b} \in \mathbb{Q}^M$ *such that* $(\tilde{A}, \tilde{G}, \tilde{B}, \tilde{b})$ *is a sharp MILP formulation for* S.

Proof. We set

$$T_I := \{(x,y,z) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Z}^t | Ax + Gy + Bz \leq b\}$$

and $p_1 : T_I \ni (x,y,z) \mapsto x \in \mathbb{Q}^n$. Because (A,G,B,b) is a MILP formulation for S,

$$p_1(T_I) = S$$

By Theorem12.3.4, there are $M \in \mathbb{N}$ and $\tilde{A} \in M(M,n,\mathbb{Q})$, $\tilde{G} \in M(M,p,\mathbb{Q})$, $\tilde{B} \in M(M,t,\mathbb{Q})$, $\tilde{b} \in \mathbb{Q}^M$ such that

$$T_I = \{(x,y,z) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Z}^t | \tilde{A}x + \tilde{G}y + \tilde{B}z \leq \tilde{b}\}$$

$$conv(T_I) = \{(x,y,z) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Q}^t | \tilde{A}x + \tilde{G}y + \tilde{B}z \leq \tilde{b}\}$$

Because $conv(S) = conv(p_1(T_I)) = p_1(conv(T_I))$,

$$conv(S) = p_1(conv(T_I))$$

So, $(\tilde{A}, \tilde{G}, \tilde{B}, \tilde{b})$ is a sharp MILP formulation for S. □

12.3.6 Review

Meyer theorem states that the convex hull of the feasible region of MILP is a rational polyhedron. So, the feasibility and the optimal value of MILP are equivalent to the feasibility and the optimal value of some LP, respectively. By methods such as simplex method, we can find this LP solution in extreme points of feasible reasion. By Proposition12.3.23, this extreme point is a solution of original MILP problem.

I think there are the following three ideas that are important in the proof of Meyer theorem.

1. Fourier elimination method
2. Expressing the feasible region of MILP or LP in terms of the Minkowski sum of bounded and unbounded parts
3. Going back and forth between integer and continuous parts of a polyhedron

Fourier elimination method plays an important role throughout this section. Fourier elimination method is a method of solving linear inequalities

$$Ax \leq b \tag{12.3.1}$$

focusing on the sign of the coefficients of a certain variable and using only non-negative multipliers to eliminate the variable. (12.3.1) corresponds to another two linear inequalities. If there is a solution of (12.3.1), then there is $U \in M(m_0, n, \mathbb{R})$ such that $U \geq 0$ and $UA = 0$ and

$$0 \leq Ub \tag{12.3.2}$$

By focusing on row vectors of U, if there is no solutions of (12.3.1), then there is $u \in \mathbb{R}^n_+$ such that

$$A^t u = 0, u^t b < 0, u \geq 0 \tag{12.3.3}$$

Correspondance between (12.3.1) and (12.3.3) is stated by Farkas Lemma.

For idea2 on LP feasible reasion P, we state this idea as Minkowski Weyl Theorem.

$$P = conv(v^1, ..., v^s) + cone(r^1, ..., r^q) \tag{12.3.4}$$

By increasing the dimension of the solution space of the simultaneous inequalities by one as follows, Minkowski Weyl Theorem is boil down to the case in P is a polyhedral cone.

$$P = \tilde{P} \cap \mathbb{R}^n \times \{1\}, \tilde{P} := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} | (A, -b) \begin{pmatrix} x \\ y \end{pmatrix} \leq 0\} \tag{12.3.5}$$

By Fourier elimination method and Farkas Lemma, any polyhedral cone is equivalent to finite generated convex cone.

Meyer theorem is the following.

Theorem 12.3.31. *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $G \in M(m, p, \mathbb{Q})$, $b \in \mathbb{Q}^m$, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^p$.

(S2) $S := \{(x, y) \in P(A, G, b) | x \in \mathbb{Z}^n\}$.

Then $conv(S)$ *is a rational polyhedron.*

In the proof of Meyer theorem, we focus on Polyhedron $P := P(A, G, b)$ which is containing S. By Minkowski Weyl Theorem, we get

$$P = conv(v^1, ..., v^s) + cone(r^1, ..., r^q)$$

We focus a bounded part of P

$$T = conv(v^1, ..., v^s) + \sum_{j=1}^{q} [0, 1] r_j$$

We denote a integer part of T by T_I and denote a integer part of $cone(r^1, ..., r^q)$ by R_I. Then we get

$$S = T_I + R_I$$

So,

$$conv(S) = conv(T_I) + conv(R_I)$$

Because $conv(T_I)$ is a rational polytope and $conv(R_I)$ is a rational polyhedral cone, $conv(S)$ is a rational polyhedron.

12.4 MILP formulation

12.4.1 Minimal formulation

Definition 12.4.1 (Implied equations, redundant inequalities, and facet)**.** *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $b \in \mathbb{Q}^m$, $P := \{x \in \mathbb{Q}^n | Ax \leq b\}$. a_i *is the i-th row vector of A..*

Then

(i) *We say* $F \subset P$ *is a face of* P *if and only if* $F = \{x | a_i^T x = b_i \ (\forall i \in L)\}$ *for some* $L \subset \{1, 2, ..., m\}$.

(ii) *We say* $F \subset P$ *is a proper face of* P *if and only if* F *is a face and* $F \neq \phi$ *and* $F \neq P$.

(iii) *We say* $F \subset P$ *is a facet of* P *if and only if* F *is a proper face and maximum with respect to inclusion.*

(iv) *We say* $a_i x \leq b_i \ (i \in L)$ *is implied equations of* P *if and only if* $a_i x \leq b_i \ (\forall i \in L)$ *for any* $x \in P$.

(v) *We say* $a_i x \leq b_i \ (i \in L)$ *is facet defining inequalities of* P *if and only if* $F := \{x | a_i x \leq b_i \ (\forall i \in L)\}$ *is a facet of* P.

(vi) *We say* $a_i x \leq b_i \ (i \in L)$ *is redundant inequalities of* P *if and only if there is a subset* $I \subset \{1, 2, ..., m\}$ *such that* $P = \{x | a_i x \leq b_i \ (\forall i \in I \setminus L)\}$.

(vii) *We say* $L \subset \{1, 2, ..., m\}$ *is a minimal formulation of* P *if and only if* $P = \{x | a_i x \leq b_i \ (\forall i \in L)\}$ *and there is no* iL *such that* $a_i \leq b$ *is a redundant inequality of* P.

12.4.2 Locally ideal formulation

Proposition 12.4.2 (Standard equity form for LP)**.** *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $b \in \mathbb{Q}^m$.

(S2) $S := \{x \in \mathbb{Q}^n | Ax \leq b\}$.

(S3) *We set for* $x \in S$,

$$\Phi(x) := (y^+, y^-, z)$$

Here,

$$y_i^+ := \max\{x_i, 0\} \ (i = 1, 2, ..., n)$$
$$y_i^- := \max\{-x_i, 0\} \ (i = 1, 2, ..., n)$$
$$z_j := (a_j, x) - b_j \ (j = 1, 2, ..., m)$$

(S4) $\tilde{S} := \{(y^+, y^-, z) \in \mathbb{Q}_+^n | A(y^+ - y^-) + z \leq b\}$.

Then Φ *is a bijective from* S *to* $\tilde{S}$. *We call* $\tilde{S}$ *the standard equity form of* S. *We call each* z_j *a slack variable.*

Definition 12.4.3 (Basic feasible solution for LP.)**.** *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $b \in \mathbb{Q}^m$.

Then

(i) *For* $x \in \mathbb{Q}^n$, *we say* $\bar{x}$ *is a basic solution of* $Ax = b$ *if and only if* $\{a_i | a_i$ *is the i-th column of* A *and* $\bar{x}_i > 0\}$ *are linear independent.*

(ii) *For* $x \in \mathbb{Q}_+^n$, *we say* $\bar{x}$ *is a basic feasible solution of*

$$Ax = b, x \geq 0$$

if and only if x *is a basic solution of* $Ax = b$.

Proposition 12.4.4. *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $b \in \mathbb{Q}^m$.

(S2) x *is a solution of* $Ax \leq b, x \geq 0$.

(S3) $z = (z_1, .., z_m)$ *are nonzero slack variables for* $Ax + z = b, x, z \geq 0$.

(S4) $I := \{i \in \{1, 2, ..., m\} | a_i^T x = b_i\}$. *Here* a_i *is the i-th row vector of* A.

(S5) $J := \{j \in \{1, 2, ..., n\} | x_j \neq 0\}$.

Then (x, z) *is a basic feasible solution iff* $\{\{a_{i,j}\}_{i\in I}\}_{j\in J}$ *are linear independent.*

Proof. We set $I' := \{i \in \{1, 2, ..., m\}|a_i^T x < b_i\}$. (x, z) is a basic feasible solution iff $\{a^j\}_{j\in J} \cup \{e_i\}_{i\in I'}$ are linear independent. Here a^j is the j-th column of A. This is equivalent to $\{a^j - \sum_{i\in I'} a_{i,j}e_i\}_{j\in J} \cup \{e_i\}_{i\in I'}$ are linear independent. So, (x, z) is a basic feasible solution iff $\{\{a_{i,j}\}_{i\in I}\}_{j\in J}$ are linear independent. □

Definition 12.4.5 (Locally ideal)**.** *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $G \in M(m, p, \mathbb{Q})$, $B \in M(m, t, \mathbb{Q})$, $b \in \mathbb{Q}^m$.

(S2) $S \subset \mathbb{Q}^n$.

(S3) $T(A, G, B, b) := \{(x, y, z) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Z}^t|Ax + Gy + Bz \le b\}$.

(S4) $\tilde{S} := \{w \in \mathbb{Q}^M|Cw = c, w \ge 0\}$ *is a standard equity form of* S *and* Φ *is the bijection from* S *to* $\tilde{S}$ *in Proposition12.4.2.*

We say (A, G, B, b) *is a locally ideal MILP formulation for* S *if and only if* $\tilde{S}$ *has at most one basic feasible solution and for any basic feasible solution of* $\tilde{S}$ w, $\Phi^{-1}(w) \in \mathbb{Q}^{n+p} \times \mathbb{Z}^t$.

We will show an example of MILP formulation which is not locally ideal but sharp.

Example 12.4.6. *Here are the settings and assumptions.*

(S1) $S = \cup_{i=1}^n P_i$. $P_i := \{x \in \mathbb{Q}^n||x_i| \le 1, x_j = 0\ (j \ne i)\}\ (i = 1, 2, ..., n)$.

Then

(i) The following is a MILP formulation for S.

$$y_j - 1 \le x_i \le 1 - y_j\ (i = 1, 2, .., n, j \ne i), \tag{12.4.1}$$
$$y_i \ge 0,\ (i = 1, 2, .., n), \tag{12.4.2}$$
$$\sum_{i=1}^n y_i = 1 \tag{12.4.3}$$
$$y \in \mathbb{Z}^n$$

(ii) $conv(S) = \{x \in \mathbb{Q}^n| \sum_{i=1}^n |x_i| \le 1\}$

(iii) Equalities and Inequalities in (i) and the following is a sharp MILP formulation for S.

$$\sum_{i=1}^n r_i x_i \le 1\ (r \in \{-1, 1\}^n) \tag{12.4.4}$$

(iv) If $n = 3$*, the formulation in (iii) is not locally ideal.*

(v) The following is a sharp and locally ideal MILP formulation for S.

$$-y_i \le x_i \le y_i\ (i = 1, 2, .., n), \tag{12.4.5}$$
$$y_i \ge 0,\ (i = 1, 2, .., n), \tag{12.4.6}$$
$$\sum_{i=1}^n y_i = 1 \tag{12.4.7}$$
$$y \in \mathbb{Z}^n$$

Proof of (i). It is clear. □

Proof of (ii). The part of $\subset$ is clear. Let us fix any x in the right side. We take $s \ge 1$ such that $\sum_{i=1}^n s|x_i| = 1$. Then

$$x = \sum_{i=1}^n r|x_i|\frac{sign(x_i)}{r}e_i$$

So, $x \in conv(S)$. □

Proof of (iii). We set $T := \{(x, y) \in \mathbb{Q}^n \times \mathbb{Q}^n|(x, y)$ satisfies equalities and inequalities of (i)$\}$. Clearly $p_1(T) \subset conv(S)$. Clearly T is convex. Because $P_i \times \{e_i\} \subset T\ (\forall i)$, $S \subset p_1(T)$. So, $conv(S) \subset T$. □

Proof of (iv). Clearly $x_1 = x_2 = y_1 = y_2 = \frac{1}{2}, x_3 = y_3 = 0$ is a feasible solution. We will show this is a basic feasible solution. By Proposition12.4.4, it is enough to show the column vectors of

	x_1	x_2	y_1	y_2
$x_1 \leq 1 - y_1$	1	0	0	1
$x_2 \leq 1 - y_2$	0	1	1	0
$y_1 + y_2 = 1$	0	0	1	1
$x_1 + x_2 = 1$	1	1	0	0

are linear independent. Because this matrix is nonsingular, the column vectors of this matrix are linear independent. □

Proof of (v). By the same argument as the proof of (iii), we can show this formulation is sharp. For locally ideal property, it is enough to show for any basic feasible solution (x^+, x^-, y, z) there is $\#\{i|y_i \neq 0\} = 1$. Because $\sum_{i=1}^n y_i = 1$, $\#\{i|y_i \neq 0\} \geq 1$. For aiming contradiction, let us assume $\#\{i|y_i \neq 0\} > 1$. So, there are $i_1 \neq i_2$ such that $y_{i_1}, y_{i_2} > 0$. We can assume $i_1 =, i_2 = 2$. We will show in each case of the followings.

case1 $|x_1| < y_1$ or $|x_2| < y_2$.

case2 $|x_1| = y_1$ and $|x_2| = y_2$.

In case1, we can assume $|x_1| < y_1$. If $|x_2| < y_2$, then By Proposition12.4.4, the clumns vectors of the following matrix are linear independent.

	y_1	y_2
$*$	0	0
...	...	....
$*$	0	0
$\sum_i y_i = 1$	1	1

This is contradiction. So, $|x_{i_2}| = y_{i_2}$. By Proposition12.4.4, the clumns vectors of the following matrix are linear independent.

	y_1	y_2	x_2^*
$*$	0	0	0
...	...	...	0
$*$	0	0	0
$q_2y_2 + r_2x_2 \leq 0$	0	q_2	r_2
$\sum_i y_i = 1$	1	1	0

Here, $q_2r_2 \neq 0$. So, the clumns vectors of the following matrix are linear independent.

	y_1	y_2	x_2^*
$*$	0	0	0
...	...	...	0
$*$	0	0	0
$q_2y_2 + r_2x_2 \leq 0$	0	0	r_2
$\sum_i y_i = 1$	1	0	0

This is contradiction.

In case2, By Proposition12.4.4, the clumns vectors of the following matrix are linear independent.

	y_1	y_2	x_1^*	x_2^*
$*$	0	0	0	0
...	...	...	...	...
$*$	0	0	0	0
$q_1y_1 + r_1x_1 \leq 0$	q_1	0	r_1	0
$q_2y_2 + r_2x_2 \leq 0$	0	q_2	0	r_2
$\sum_i y_i = 1$	1	1	0	0

Here, $q_1r_1q_2r_2 \neq 0$. So, the clumns vectors of the following matrix are linear independent.

	y_1	y_2	x_1^*	x_2^*
$*$	0	0	0	0
...	...	...	...	...
$*$	0	0	0	0
$q_1y_1 + r_1x_1 \leq 0$	0	0	r_1	0
$q_2y_2 + r_2x_2 \leq 0$	0	0	0	r_2
$\sum_i y_i = 1$	1	1	0	0

This is contradiction.

Consequently, $\#\{i|y_i \neq 0\} \leq 1$. □

Memo 12.4.7. *We measured execution times in three formulations in Example12.4.6. Here are the settings.*

Version of SCIP: SCIP9.0.0.0

Target Machine: Ubuntu Desktop 22.04

Host Machine: Windows10

CPU:Inter Core i7-6700T@2.8GHz

DRAM: 32GB

Here are the problem.

$$maximize \sum_{i=1} c_i x_i, x \in S$$

For $n = 1000$*, the execution times are below.*

$$\text{Simple formulation(i)} : 119sec$$
$$\text{Locally ideal formulation(v)} : 0.028sec$$

For sharp formulation(iii) and $n = 17$*, the execution time is 229sec. See [45] for sample code.*

Proposition 12.4.8. *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $b \in \mathbb{Q}^m$.

(S2) $S := \{x \in \mathbb{Q}^{n_1} \times \mathbb{Q}^{n_2} | Ax \leq b\}$.

(S3) $T(A, G, B, b) := \{(x, y, z) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Z}^t | Ax + Gy + Bz \leq b\}$.

(S4) $\tilde{S} := \{w \in \mathbb{Q}^M | Cw = c, w \geq 0\}$ *is a standard equity form of* S *and* Φ *is the bijection from* S *to* $\tilde{S}$ *in Proposition12.4.2.*

Then

(i) For any $x \in ex(S)$, $\Phi(x)$ *is a basic feasible solution.*

(ii) For any $x \in S \setminus \{0\}$ *such that* $\Phi(x)$ *is a basic feasible solution,* $\Phi(x) \in ex(\tilde{S})$.

(iii) Let us assume $S \subset [0, \infty)^{n_1+n_2}$. *Then for any* $x \in S$ *such that* $\Phi(x)$ *is a basic feasible solution,* $x \in ex(S)$.

Proof of (i). Let z is a slack variable for $Ax \leq b$. $I_n := \{i|x_i \neq 0\}$. $J_0 := \{j|z_j = 0\}$. If $J_0 = \phi$, then $Ax < b$. So, there is $x^1, x^2 \in \mathbb{Q}^n$ $t \in (0, 1)$ such that $Ax^1 Ax^2 < b$ and $x = tx^1 + (1-t)x^2$. This is contradiction. So, $J_0 \neq \phi$. If $x = 0$, $\Phi(x)$ is clearly basic feasible solution. So, we can assume $x \neq 0$. It is enough to show $\{\{a_{i,j}\}_{i \in I_n} | j \in J_0\}$ are linear independent. For aiming contradiction, let us assume $\{\{a_{i,j}\}_{i \in I_n} | j \in J_0\}$ are linear dependent. We set $k := \#I_n$ and

$$A' := \{a_{i,j}\}_{i \in I_n, j \in \{1,2,...,n\}}, A'' := \{a_{i,j}\}_{i \notin I_n, j \in \{1,2,...,n\}}, b' := \{b_i\}_{i \in I_n}, b'' := \{b_i\}_{i \notin I_n}$$

Then there is a $c \in \mathbb{Q}^n \setminus \{0\}$ such that

$$c_i = 0 \ (\forall i \notin I_n), x + sc \text{ is a solution of } A'x = b' \ (\forall s \in \mathbb{R})$$

Because $A''x < b''$, there is $s > 0$ such that $A''(x + sc) < b''$ and $A''(x - sc) < b''$. This means that $x \notin ex(S)$. This is contradiction. □

Proof of (ii). Let us fix any $x \in S \setminus \{0\}$ such that $\Phi(x)$ is a basic feasible solution. We can assume $|x_1|, .., |x_k| > 0, x_{k+1} = ... = x_n = 0$ and

$$a_{1,1}x_1 + ... + a_{1,k}x_k = b_1$$
$$...$$
$$a_{l,1}x_1 + ... + a_{l,k}x_k = b_l$$

and

$$rank \begin{pmatrix} a_{1,1} & ... & a_{1,k} \\ ... & ... & ... \\ a_{l,1} & ... & a_{l,k} \end{pmatrix} = k$$

So, the equation

$$a_{1,1}\bar{x}_1 + ... + a_{1,k}\bar{x}_k = b_1$$
$$...$$
$$a_{l,1}\bar{x}_1 + ... + a_{l,k}\bar{x}_k = b_l$$

has the unique solution.

For aiming contradiction, let us assume that $\Phi(x) \notin ex(S)$. Then there are $x^1 := (x^{1,+}, x^{1,-}, z^1)$ and $x^2 := (x^{2,+}, x^{2,-}, z^2)$ and $t \in (0,1)$ such that $x = tx^1 + (1-t)x^2$. So, $x_i^{j,+} = x_i^{j,-} = 0$ ($\forall i > k, j = 1,2$) and $x_i^{j,+} = \delta_{+,sign(x_i)} x_i^{j,+}$ ($\forall i \le k, j = 1,2$) and $x_i^{j,-} = \delta_{+,sign(x_i)} x_i^{j,-}$ ($\forall i \le k, j = 1,2$). This implies $(x_1^{1,sign(x_1)}, ..., x_k^{1,sign(x_1)})$ and $(x_1^{2,sign(x_2)}, ..., x_k^{2,sign(x_2)})$ satisfies the equation

$$a_{1,1}\bar{x}_1 + ... + a_{1,k}\bar{x}_k = b_1$$
$$...$$
$$a_{l,1}\bar{x}_1 + ... + a_{l,k}\bar{x}_k = b_l$$

This is contradiction. □

Proof of (iii). (iii) is followed by the same argument of the proof of (ii). □

Definition 12.4.9 (Affine combination, Affine independent)**.**

(i) For $x_1, ..., x_m \in \mathbb{Q}^n$,

$$\sum_{i=1}^{m} \lambda_i x_i, \lambda_1, ..., \lambda_m \in \mathbb{Q}, \sum_{i=1}^{m} \lambda_i = 1$$

is called an affine combination of $x_1, ..., x_m$.

(ii) We say $x_1, ..., x_m \in \mathbb{Q}^n$ are affinely independent if for any $\lambda_1, ..., \lambda_m \in \mathbb{Q}$ such that $\sum_{i=1}^m \lambda_i = 0$ and $\sum_{i=1}^m \lambda_i x_i = 0$, $\lambda = 0$.

Definition 12.4.10 (Dimension)**.** *For $S \subset \mathbb{R}^n$,*

$$dim(S) := \max\{\#A | A \textit{ is a finite subset of } S \textit{ and } A \textit{ is affinely independent}\} - 1$$

Definition 12.4.11 (Pointed set)**.** *We say convex subset $S \subset \mathbb{R}^n$ is pointed if and only if $ex(S) \neq \phi$.*

Proposition 12.4.12. *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $b \in \mathbb{Q}^n$.

(S2) $P := \{x \in \mathbb{Q}_+^n | Ax \le b\}$.

(A1) $P \neq \phi$.

Then P is pointed.

Proof. For $y \in P$, We set

$$N(y) := \#\{i | y_i \neq 0\}, M(y) := \#\{j | a_j^T y = b_j\}$$

Here, a_j is the j-th row vector of A. We set

$$K := \max\{N(y) | y \in P\}$$

If $K = n$, clearly $0 \in ex(P)$. So, we can assume $K < n$. We set

$$L := \max\{M(y) | y \in P, N(y) = K\}$$

Because $K < n$, $L > 0$. There is $x \in P$ such that $N(x) = K, M(x) = L$. We set $k := n - K$. We can assume $x_1, ..., x_k > 0, x_{k+1} = ... = x_n = 0$ and

$$A'x' = b'$$

Here, $A' := \{a_{i,j}\}_{i=1,...,L,j=1,2,...,n}$, $x' := (x_1, ..., x_k)$, $b' := (b_1, ..., b_L)^T$. For aiming contradiction, let us assume $rank(A') < n$. there is $r' \in \mathbb{Q}^k$ such that

$$A'(x' + tr) = b' \ (\forall t \in \mathbb{R})$$

So, there is $y \in P$ such that $N(y) > N(x)$ or $M(x) < M(y)$. This is contradiction. So, $rank(A') = n$. From this, $x \in ex(P)$. □

Definition 12.4.13 (Edge)**.** *Let P be a nonempty polyhedron in $\mathbb{R}^n$. We call a face of P whose dimension is 1 an edge of P.*

Proposition 12.4.14. *Here are the settings and assumptions.*

(S1) $a_1, ..., a_k \mathbb{Q}^n$.

(A1) For any i, $a_i \notin conv(a_1, ..., a_{i-1}, a_{i+1}, ..., a_k)$.

Then

$$ex(conv(a_1, ..., a_k)) = \{a_1, ..., a_k\}$$

Proof. By Proposition12.3.23, it is enogh to show *supset* part. Let us assume there is i such that $a_i \notin ex(conv(a_1, ..., a_k))$. We can assume $i = k$. So there are $\lambda_1, ..., \lambda_k, \eta_1, ..., \eta_k, t \in (0, 1)$ such that $\sum_{i=1}^k \lambda_i = 1$ and $\sum_{i=1}^k \eta_i = 1$ and $a_k = t \sum_{i=1}^k \lambda_i a_i + (1 - t) \sum_{i=1}^k \eta_i a_i$ and $\sum_{i=1}^k \lambda_i a_i \neq a_k$ and $\sum_{i=1}^k \eta_i a_i \neq a_k$. So, $t\lambda_k + (1 - t)\eta_k < 1$. This implies

$$a_k = \sum_{i=1}^{k-1} \frac{t\lambda_i + (1 - t)\eta_i}{1 - t\lambda_k - (1 - t)\eta_k} a_i$$

So, $a_k \in conv(a_1, ..., a_{k-1})$. This is contradiction. □

Proposition 12.4.15. *Here are the settings and assumptions.*

(S1) $P \subset \mathbb{R}^n$ is a Poryhedron.

(S2) $a_1, ..., a_k, r_1, ..., r_l \mathbb{Q}^n$ such that $P = conv(a_1, ..., a_k) + cone(r_1, ..., r_l)$.

(A1) For any i, $a_i \notin conv(a_1, ..., a_{i-1}, a_{i+1}, ..., a_k)$.

Then

$$ex(P) \subset \{a_1, ..., a_k\}$$

Proof. By Proposition12.3.23, it is enough to show $ex(P) \subset conv(a_1, ..., a_k)$. Let us fix any $x \in P \setminus conv(a_1, ..., a_k)$. There are $y \in conv(a_1, ..., a_k)$ and $z \in cone(r_1, ..., r_l) \setminus \{0\}$ such that $x = y + z$. Because $2z, 0 \in cone(r_1, ..., r_l)$, $y, y + 2z \in P$. So, $z = \frac{1}{2}(y + y + 2z)$. This means $x \notin ex(P)$. Consequently, $ex(P) \subset conv(a_1, ..., a_k)$. □

Proposition 12.4.16. *Here are the settings and assumptions.*

(S1) $P \subset \mathbb{R}^n$ is a Poryhedron.

(A1) P is pointed.

Then there are $a_1, ..., a_k, r_1, ...r_l \in \mathbb{Q}^n$

$$P = conv(a_1, ..., a_k) + cone(r_1, ..., r_l), ex(P) = \{a_1, ..., a_k\}, 0 \notin cone(r_1, ..., r_l)$$

Proof. By Minkowski-Weyl Theorem, Then there are $a_1, ..., a_k, r_1, ...r_l \in \mathbb{Q}^n$

$$P = conv(a_1, ..., a_k) + cone(r_1, ..., r_l)$$

For aiming contradiction, let us assume $0 \notin ex(cone(r_1, ..., r_l))$. The there are $z_1 \neq z_2 \in cone(r_1, ..., r_l)$ such that $0 = \frac{1}{2}(z_1 + z_2)$. For any i,

$$a_i = \frac{1}{2}((a_i + z_1) + (a_i + z_2))$$

This implies $a_i \notin ex(P)$. By Proposition12.4.15, $ex(P) = \phi$. This is contradiction. So, $0 \in ex(cone(r_1, ..., r_l))$.

By dropping some elements if necessary, we can assume that for each i

$$a_i \notin conv(a_1, ..., a_{i-1}, a_{i+1}, ..., a_k) + cone(r_1, ..., r_l)$$

For aiming contradiction, let us assume $a_i \notin ex(P)$. We can assume $i = k$. Then there are $y_1, y_2 \in conv(a_1, ..., a_k)$ and $z_1, z_2 \in cone(r_1, ..., r_l)$, $t \in (0, 1)$ such that

$$a_k = t(y_1 + z_1) + (1 - t)(y_2 + z_2), y_1 + z_1 \neq a_k, y_2 + z_2 \neq a_k$$

There are $\lambda_1, ..., \lambda_k, \eta_1, ..., \eta_k, t \in (0, 1)$ such that $\sum_{i=1}^k \lambda_i = 1$ and $\sum_{i=1}^k \eta_i = 1$ and $y_1 = \sum_{i=1}^k \lambda_i a_i$ and $y_2 = \sum_{i=1}^k \eta_i a_i$. If $y_1 = y_2 = a_k$, $0 = tz_1 + (1-t)z_2$. This contradicts with $0 \notin ex(P)$. So, $y_1 \neq a_k$ or $y_2 \neq a_k$. This implies $t\lambda_k + (1-t)\eta_k < 1$. So,

$$a_k = \frac{1}{1 - t\lambda_k - (1 - t)\eta_k}(\sum_{i=1}^k (t\lambda_i + (1 - t)\eta_i)a_i + tz_1 + (1 - t)z_2)$$

This means $a_k \in conv(a_1, ..., a_{k-1}) + cone(r_1, ..., r_l)$. This is contradiction. □

Proposition 12.4.17. *Here are the settings and assumptions.*

(S1) $S \subset \mathbb{Q}^n$.

(A1) (A, B, D, b) *is a locally ideal MILP formulation for* S.

(S2) We set

$$P := \{(x, u, y) \in \mathbb{Q}^n \times \mathbb{Q}^p \times \mathbb{Q}^t | Ax + Bu + Dy \leq b\}, P_I := \{(x, u, y) \in P | y \in \mathbb{Z}^t\}$$
$$p : P \ni (x, u, y) \mapsto x \in \mathbb{Q}^n$$

(A2) P *is pointed.*

then (A, B, D, b) *is a sharp formulation for* S.

Proof. By Proposition12.4.16, there are $a_1, ..., a_k, r_1, ...r_l \in \mathbb{Q}^n$

$$P = conv(a_1, ..., a_k) + cone(r_1, ..., r_l), ex(P) = \{a_1, ..., a_k\}, 0 \notin cone(r_1, ..., r_l),$$
$$a_i = (\hat{x}^i, \hat{u}^i, \hat{y}^i)\ (i = 1, 2, ..., k), r_j = (\tilde{x}^j, \tilde{u}^j, \tilde{y}^j) \in \mathbb{Z}^n \times \mathbb{Z}^s \times \mathbb{Z}^t\ (j = 1, 2, ..., l)$$

By the assumption of locally idealness and Proposition12.4.8, $\hat{y}^i \in \mathbb{Z}^t$ $(\forall i)$.

Let us fix any $(x, u, y) \in P$. There are $\lambda_1, ..., \lambda_k, \eta_1, ..., \eta_k \in [0, 1) \in \mathbb{Q}$ such that $\sum_{i=1}^k \lambda_i = 1$ and $\sum_{i=1}^k \eta_i = 1$ and

$$(x, u, y) = \sum_{i=1}^k \lambda_i(\hat{x}^i, \hat{u}^i, \hat{y}^i) + \sum_{j=1}^l \eta_j(\tilde{x}^j, \tilde{u}^j, \tilde{y}^j)$$

We set

$$(x^1, u^1, y^1) := \sum_{i=1}^k \lambda_i(\hat{x}^i, \hat{u}^i, \hat{y}^i)$$

Because $y^1 \in \mathbb{Z}^t$ and (A, B, D, b) is a MILP formulation for S, $x^1 \in S$. Without loss of generality, we can assume $\lambda_1 > 0$. There is $\alpha \in \mathbb{Z} \cap (1, \infty)$ such that $\frac{\alpha}{\lambda_1} \sum_{j=1}^l \eta_j \in \mathbb{Z}^t$. We set

$$(x^2, u^2, y^2) := \sum_{i=1}^k \lambda_i(\hat{x}^i, \hat{u}^i, \hat{y}^i) + \alpha \sum_{j=1}^l \eta_j(\tilde{x}^j, \tilde{u}^j, \tilde{y}^j)$$

Then

$$(x^2, u^2, y^2) = \lambda_1((\hat{x}^1, \hat{u}^1, \hat{y}^1) + \frac{\alpha}{\lambda_1} \sum_{j=1}^l \eta_j(\tilde{x}^j, \tilde{u}^j, \tilde{y}^j)) + \sum_{i=2}^k \lambda_i(\hat{x}^i, \hat{u}^i, \hat{y}^i) \in conv(a_1, ..., a_k)$$

So, $x^2 \in conv(p(P))$. So, $x = (1 - \frac{1}{\alpha})x^1 + \frac{1}{\alpha}x^2 \in conv(p(P))$. Consequently, (A, B, D, b) is a sharp formulation. □

Proposition 12.4.18. *Here are the settings and assumptions.*

(S1) $A \in M(m, n, \mathbb{Q})$, $b \in \mathbb{Q}^m$.

(S2) $S := \{x \in \mathbb{Q}_+^{n_1} \times \mathbb{Z}_+^{n_2} | Ax \leq b\}$.

(S3) $\tilde{A} := \begin{pmatrix} A_1 & A_2 \\ O_{n_2,n_1} & E_{n_2} \end{pmatrix}$. *Here* $A_1 := (a^1, ..., a^{n_1})$ *and* $A_2 := (a^{n_1+1}, ..., a^n)$ *and each* a^i *is the i-th column vector of A.* $B := \begin{pmatrix} O_{m,n_2} \\ -E_{n_2} \end{pmatrix}$, $\tilde{b} := \begin{pmatrix} b \\ 0_{n_2} \end{pmatrix}$

(A1) $Q := \{x \in \mathbb{Q}_+^{n_1} \times \mathbb{Q}_+^{n_2} | Ax \leq b\}$ *has at least one basic feasible solution.*

Then $(\tilde{A}, B, \tilde{b})$ *is a locally ideal formulation for* S *iff* $(\tilde{A}, B, \tilde{b})$ *is a sharp formulation for* S*.*

Proof. Proposition12.4.12 and Propositon12.4.17, it is enough to show 'if' part. Let us assume $(\tilde{A}, B, \tilde{b})$ is a sharp formulation for S. Then

$$conv(S) = p(\{(x', x'', y) \in \mathbb{Q}^{n_1} \times \mathbb{Q}^{n_2} \times \mathbb{Q}^n | (x', x'') \in Q, x'' = y\}) = Q$$

By Theorem12.3.26, there are $\boldsymbol{a}_1, ..., \boldsymbol{a}_k \in S$ and $\boldsymbol{r}_1, ..., \boldsymbol{r}_l \subset \mathbb{Z}^n$ such that

$$Q = conv(\boldsymbol{a}_1, ..., \boldsymbol{a}_k) + cone(\boldsymbol{r}_1, ..., \boldsymbol{r}_l)$$

We can assume $\boldsymbol{a}_1, ..., \boldsymbol{a}_k$ are distinct. Let us fix any x which is a basic feasible solution of Q. By Proposition12.4.8, $x \in ex(Q)$. We will show $x \in S$. There are $\lambda_1, ..., \lambda_k, \eta_1, ..., \eta_l \in [0, 1]$ such that $\sum_{i=1}^k \lambda_i = \sum_{i=1}^l \eta_i = 1$ and

$$x = \sum_{i=1}^k \lambda_i \boldsymbol{a}_i + \sum_{j=1}^l \eta_j \boldsymbol{r}_j$$

For aiming contradiction, let us assume there is j such that $\eta_j > 0$. We can assume $j = 1$. Then We set

$$x^1 = \sum_{i=1}^k \lambda_i \boldsymbol{a}_i + \frac{1}{2}\eta_1 \boldsymbol{r}_1 + \sum_{j=2}^l \eta_j \boldsymbol{r}_j, x^2 = \sum_{i=1}^k \lambda_i \boldsymbol{a}_i + \frac{3}{2}\eta_1 \boldsymbol{r}_1 + \sum_{j=2}^l \eta_j \boldsymbol{r}_j$$

Then $x^1 \neq x^2$ and $x = \frac{1}{2}(x^1 + x^2)$. This contradicts with $x \in ex(Q)$. So,

$$x = \sum_{i=1}^k \lambda_i \boldsymbol{a}_i$$

For aiming contradiction, let us assume there is $i_1 \neq i_2$ such that $\lambda_{i_1}, \lambda_{i_2} > 0$. We can assume $i_1 = 1, i_2 = 2$. We set

$$y^1 = (\lambda_1 + \lambda_2)\boldsymbol{a}_1 + \sum_{i=3}^k \lambda_i \boldsymbol{a}_i, y^2 = (\lambda_1 + \lambda_2)\boldsymbol{a}_2 + \sum_{i=3}^k \lambda_i \boldsymbol{a}_i$$

Because $\boldsymbol{a}_1 \neq \boldsymbol{a}_2$, $y^1 \neq y^2$. And $x = \frac{\lambda_1}{\lambda_1 + \lambda_2} y^1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} y^2$. This contradicts with $x \in ex(Q)$. So, $x \in S$. □

12.5 Cutting Plane

It is known that Gomory's mixed integer inequalities are useful, which is proposed in [47](1976). Although the method that simply uses those inequalities has a problem of convergence speed, many MILP solvers are attracted the revised method based on Gomory's mixed integer inequalities, which is proposed in [48].

Definition 12.5.1 (Valid Inequality)**.** *Let* $P \subset \mathbb{R}^n, c \in \mathbb{R}^n, \delta \in \mathbb{R}$*. We say the inequality* $c^T x \leq \delta$ *is valid if* $c^T x \leq \delta$ *for any* $x \in P$*.*

Definition 12.5.2 (Split)**.** *Here are the settings and assumptions.*

(S1) $b \in \mathbb{Q}^b$.

(S2) $A \in M(m, n, \mathbb{Q})$.

(S3) $P := \{x \in \mathbb{R}^n | Ax \leq b\}$.

(S5) $I \subset \{1, 2, ..., n\}$.

(S6) $S := \{z \in P | z_i \in \mathbb{Z}\ (\forall i \in I)\}$.

(S7) $C := \{1, 2, ..., n\} \setminus I$.

(S8) $\pi \in \mathbb{Z}^n$ *such that* $\pi_j = 0$ $(\forall j \in C)$.

(S9) $\pi_0 \in \mathbb{Z}$.

Then

(i) *We say* (π, π_0) *is a split if for any* $x \in S$, $(\pi, x) \leq \pi_0$ *or* $(\pi, x) \geq \pi_0 + 1$.

(ii) *Let us assume* (π, π_0) *is a split. Then we set*

$$\Pi_1 := P \cap \{x | \pi x \leq \pi_0\}$$
$$\Pi_2 := P \cap \{x | \pi x \geq \pi_0 + 1\}$$
$$P^{(\pi,\pi_0)} := conv(\Pi_1 \cup \Pi_2)$$

(iii) *We set*

$$P^{split} := \cap_{(\pi,\pi_0):split} P^{(\pi,\pi_0)}$$

Theorem 12.5.3. *Here are the settings and assumptions.*

(S1) $b \in \mathbb{Q}^b$.

(S2) $A \in M(m, n, \mathbb{Q})$.

(S3) $P := \{x \in \mathbb{R}^n | Ax \leq b\}$.

(S5) $I \subset \{1, 2, ..., n\}$.

(S6) $S := \{z \in P | z_i \in \mathbb{Z}\ (\forall i \in I)\}$.

(S7) $C := \{1, 2, ..., n\} \setminus I$.

(S8) $\pi \in \mathbb{Z}^n$ *such that* $\pi_j = 0$ $(\forall j \in C)$.

(S9) $\pi_0 \in \mathbb{Z}$.

(S10) $B_\pi := \{u | u$ *is a basic solution of* $uA = \pi$

(A1) (π, π_0) *is a split.*

Then $P^{(\pi,\pi_0)}$ *is the set of all points in* P *satisfying the inequalities*

$$\frac{u^+(b - Ax)}{ub - \pi_0} + \frac{u^-(b - Ax)}{\pi_0 + 1 - ub} \geq 1\ (\forall u \in \{u \in B_\pi | \pi_0 < ub < \pi_0 + 1\})$$

Here,

$$u_i^+ := \max(u_i, 0), u_i^- := -\min(u_i, 0)\ (i = 1, 2, ..., n)$$

Corollary 12.5.4. *We take over the settings and assumptions in 12.5.3. And the followings are additionals.*

(S11) $A_I := \{a_{i,j}\}_{1 \leq i \leq n, j \in I}, A_C := \{a_{i,j}\}_{1 \leq i \leq n, j \in C}$

Then P^{split} *is the set of all points in* P *satisfying the inequalities*

$$\frac{u^+(b - Ax)}{f} + \frac{u^-(b - Ax)}{1 - f} \geq 1$$

for any $u \in \mathbb{R}^n$ *such that* uA_I *is integral*

(i) uA_I *is integral.*

(ii) $uA_C = 0$.

(iii) $ub \notin \mathbb{Z}$.

(iv) *Rows of* A *corresponding to nonzero entries of* u *are linearly independent.*

Here,

$$f := ub - \lfloor ub \rfloor$$

We will show Gemory's Mixed Integer Inequalities.

Proposition 12.5.5 (Gemory's Mixed Integer Inequalities)**.** *The followings are settings and assumptions.*

(S1) $b \in \mathbb{Q}^b$.
(S2) $A \in M(m, n, \mathbb{Q})$.
(S3) $P := \{x \in \mathbb{R}^n | Ax = b,\ x \geq 0\}$.
(S5) $I \subset \{1, 2, ..., n\}$.
(S6) $S := \{z \in P | z_i \in \mathbb{Z}\ (\forall i \in I)\}$.
(S7) $C := \{1, 2, ..., n\} \setminus I$.
(S8) $\pi \in \mathbb{Z}^n$ *such that* $\pi_j = 0\ (\forall j \in C)$.
(S9) $\pi_0 \in \mathbb{Z}$.
(S10) $B_\pi := \{u | u$ *is a basic solution of* $uA = \pi$
(S11) $A_I := \{a_{i,j}\}_{1 \leq i \leq n, j \in I}, A_C := \{a_{i,j}\}_{1 \leq i \leq n, j \in C}$
(S12) $(u, v) \in \mathbb{R}^m \times \mathbb{R}^n$ *such that*

$$uA_I - v_I \in \mathbb{Z}^I$$

and

$$uA_C - v_C = 0$$

and

$$ub \notin \mathbb{Z}$$

(S13) $f := ub - \lfloor ub \rfloor > 0$.
(S14)

$$\alpha := uA, \beta := ub, f_j := \alpha_j - \lfloor \alpha_j \rfloor\ (j \in I)$$

Then

$$\sum_{j \in I, f_j \leq f} \frac{f_j}{f} x_j + \sum_{j \in I, f_j > f} \frac{1 - f_j}{1 - f} x_j + \sum_{j \in C, \alpha_j \leq 0} \frac{\alpha_j}{f} x_j + \sum_{j \in C, \alpha_j < f} \frac{\alpha_j}{1 - f} x_j \geq 1$$

We call this Gomory's mixed integer inequality*.*

Proof. Remark that

$$P = \{x \in \mathbb{R}^n | Ax \leq b,\ -Ax \leq -b, -x \leq 0\}$$

And

$$\frac{v^+ x}{f} + \frac{v^- x}{1 - f} \geq 1$$

□

Proposition 12.5.6 (Gomory's Mixed Integer Cuts)**.** *The followings are settings and assumptions.*

(S1) $b \in \mathbb{Q}^b$.
(S2) $A \in M(m, n, \mathbb{Q})$.
(S3) $P := \{x \in \mathbb{R}^n | Ax = b,\ x \geq 0\}$.
(S4) $I \subset \{1, 2, ..., n\}$.
(S5) $S := \{z \in P | z_i \in \mathbb{Z}\ (\forall i \in I)\}$.
(S6) $C := \{1, 2, ..., n\} \setminus I$.
(S7) $B \subset \{1, 2, ..., n\}$ *such that* B *defines a feasible basis of the equation* $Ax = b, x \geq 0$*. I mean, there are* $\{\bar{b}_i\}_{i \in B}$ *and* $\{\bar{a}_{i,j}\}_{i \in B, j \in N}$ *such that the equation*

$$x_i + \sum_{j \in N} \bar{a}_{i,j} x_j = \bar{b}_i\ (\forall i \in B)$$

is equal to $Ax = b, x \geq 0$*. Here,* $N := \{1, 2, ..., n\} \setminus B$*.*
(S8) $x_i^* := \bar{b}_i\ (i \in B)$, $x_j^* := 0\ j \in N$*. We call* x^* *a tableau associated with* B*.*
(S9) $f_{0,i} := \bar{b}_i - \lfloor \bar{b}_i \rfloor\ (i \in I)$*.* $f_j := a_{i,j} - \lfloor a_{i,j} \rfloor\ (j \in N)$*.*
(A1) $x^* \notin S$*.*

Then

(i) $f_{0,i} > 0\ (\exists i \in B \cap I)$*.*
(ii) *For any* $i \in B \cap I$ *such that* $f_{0,i} > 0$*,*

$$\sum_{j \in N \cap I, f_j \leq f_0} \frac{f_j}{f_0} x_j + \sum_{j \in N \cap I, f_j > f_0} \frac{1 - f_j}{1 - f_0} x_j + \sum_{j \in N \cap C, \bar{a}_{i,j} \geq 0} \frac{\bar{a}_{i,j}}{f_0} x_j + \sum_{j \in N \cap C, \bar{a}_{i,j} < 0} \frac{\bar{a}_{i,j}}{1 - f_0} x_j \geq 1$$

12.6 Reformulation and Relaxation

12.6.1 Lagrangian Relaxation

Lagragian Relaxation is a relaxation method for the MILP problem which has complicated constraints. In Lagragian Relaxation, a subset of all constraints is fixed. A Lagragian Relaxation is a problem derived by moving the subset to the objective with a penalty coefficients. Lagragian Dual is the minimum of solutions of all Lagragian Relaxation. It is known that lagragian dual gives stronger bound than one given by linear relaxation.

Definition 12.6.1 (Lagragian Relaxation). *Here are the settings and assumptions.*

(S1) $c \in \mathbb{Q}^n$.

(S2) $b \in \mathbb{Q}^m$.

(S3) $A \in M(m, n, \mathbb{Q})$.

(S4) $p \in \mathbb{N}_{\leq n}$.

(S5) $S := \{x \in \mathbb{Z}^p \times \mathbb{Q}^{n-p}_{\geq 0} | Ax \leq b\}$.

(S6) $m_1 \in \mathbb{N}_{\leq n}, m_2 := m - m_1$.

(S7) $A_1 := \begin{pmatrix} a_1 \\ ... \\ a_{m_1} \end{pmatrix}, A_2 := \begin{pmatrix} a_{m_1+1} \\ ... \\ a_m \end{pmatrix}$. *Here, a_i is the i-th row vector of A $(i = 1, 2, ..., n + p)$.*

(S8) $Q := \{x \in \mathbb{R}^n_+ | A_2 x \leq b^2, x_j \in \mathbb{Z}\ (j = 1, 2, ..., p)\}$.

We call

$$z_I := \max_{x \in S} cx$$
$$\Longleftrightarrow$$
$$\begin{gathered} z_I := \max cx \\ Ax \leq b \\ x_j \in \mathbb{Z}\ (j = 1, 2, ..., p) \\ x_j \in \mathbb{Q}_{\geq 0}\ (j = 1, 2, ..., n) \end{gathered}$$

the original problem. And, for $\lambda \in \mathbb{R}^{m_1}_{>0}$, we call

$$z_{LR}(\lambda) := \max_{x \in Q}(cx + \lambda(b^1 - A_1 x))$$
$$\Longleftrightarrow$$
$$\begin{gathered} z_{LR} := \max(cx + \lambda(b^1 - A_1 x)) \\ A_2 x \leq b^2 \\ x_j \in \mathbb{Z}\ (j = 1, 2, ..., p) \\ x_j \in \mathbb{Q}_{\geq 0}\ (j = 1, 2, ..., n) \end{gathered}$$

$LR(\lambda)$, lagrangian relaxation.

Proposition 12.6.2. *We take over notations and settings in Definition12.6.1. Then*

$$z_{LR}(\lambda) \geq z_I\ (\forall \lambda \in \mathbb{R}^{m_1}_{>0})$$

Proof. If $z_I = -\infty$ the claim of the proposition clearly holds. Let us assume that $z_I > -\infty$. Let us fix any $x \in Q$. Since any constraint of $LR(\lambda)$ is contained in the constraints of the original problem, x satisfies the constraints of $LR(\lambda)$. And, since $\lambda \geq 0$,

$$cx \leq (cx + \lambda(b^1 - A_1 x)) \leq z_{LR}$$

That implies that

$$z_I \leq z_{LR}$$

□

Definition 12.6.3 (Unimodular Matrix). *Let $A \in M(m, n, \mathbb{Z})$. We say A is unimodular if $rank(A) = m$ and for every m-th squared submatrix B*

$$\det B = 0, \pm 1$$

Definition 12.6.4 (Totally Unimodular Matrix). *Let $A \in M(m, n, \mathbb{R})$. We say A is totally unimodular if for every squared submatrix B*

$$\det B = 0, \pm 1$$

Clearly the following holds.

Proposition 12.6.5. *For any totally unimodular matrix $A \in M(m, n, \mathbb{R})$, each $a_{i,j} = 0$ or 1.*

Theorem 12.6.6 (Lagrangian Dual). *We take over notations and settings in Definition12.6.1. And*

- *(A1)* $\{x | A_1 x \leq b^1, x \in conv(Q)\} \neq \phi$.
- *(S1)* $z_{LD} := \min_{\lambda > 0} z_{LR}(\lambda)$. *We call the problem to find z_{LD} the Lagragian Dual of the original problem.*
- *(A2) The solution of* $\max\{cx | A_1 x \leq b^1, x \in conv(Q)\}$ *exists.*
- *(A3)* $-\infty < z_{LD}$ *exists.*
- *(A4) For any $\lambda \geq 0$, $z_{LR}(\lambda) \in (-\infty, \infty)$ exists.*

Then

$$z_{LD} = \max\{cx | A_1 x \leq b^1, x \in conv(Q)\}$$

Proof. From Meyer Theorem, there is $C \in M(m', n; \mathbb{Q})$ and $d \in \mathbb{R}^{m'}$ such that

$$conv(Q) = \{x \in \mathbb{R}^n_{\geq 0} | Cx \leq d\}$$

Let us fix any $\lambda \geq 0$. From Proposition12.1.6 and Proposition12.1.7,

$$\begin{aligned} & z_{LR}(\lambda) = \max\{cx + \lambda^T(b^1 - A_1 x) | Cx \leq d\} = -\min\{-cx - \lambda^T b^1 + \lambda^T A_1 x | -Cx \geq -d\} \\ = \; & -\max\{-\lambda^T b^1 - d^T \mu | -C^T \mu \leq -c + A_1^T \lambda, \mu \geq 0\} = \min\{\lambda^T b^1 + d^T \mu | C^T \mu \geq c - A_1^T \lambda, \mu \geq 0\} \end{aligned}$$

Then

$$z_{LD} = \min\{\lambda^T b^1 + d^T \mu | C^T \mu \geq c - A_1^T \lambda, \lambda \geq 0, \mu \geq 0\}$$

From Proposition12.1.6 and Proposition12.1.7,

$$\begin{aligned} & \min\{\lambda^T b^1 + d^T \mu | C^T \mu \geq c - A_1^T \lambda, \lambda \geq 0, \mu \geq 0\} \\ = \; & \min\{d^T \mu + \lambda^T b^1 | C^T \mu + A_1^T \lambda \geq c, \lambda \geq 0, \mu \geq 0\} = \max\{c^T x | Cx \leq d, A_1 x \leq b^1\} = \max\{c^T x | x \in conv(Q), A_1 x \leq b^1\} \end{aligned}$$

Therefore,

$$z_{LD} = \max\{c^T x | x \in conv(Q), A_1 x \leq b^1\}$$

□

Corollary 12.6.7. *We take over notations and settings and assumptions in Theorem12.6.6. Then*

$$z_I \leq z_{LD} \leq z_{LP}$$

Here, z_{LP} is an optimal solution of the continuous relaxation of the original problem.

Proof of $z_I \leq z_{LD}$. Let us fix any $x \in S$ and $\lambda \in \mathbb{R}^m_{\geq 0}$. Then

$$z_I \leq c^T x \leq c^T x + \lambda^T(b^1 - A_1 x) \leq z_{LR}(\lambda)$$

Therefore, $z_I \leq z_{LR}$. □

Proof of $z_{LD} \leq z_{LP}$. From Theorem12.6.6, $z_{LD} = \max\{c^T x | x \in conv(Q), A_1 x \leq b^1\}$ Since

$$conv(Q) \cap \{x \in \mathbb{R}^n_{\geq 0} | A_1 x \leq b^1\} \subset conv(S)$$

$\max\{c^T x | x \in conv(Q), A_1 x \leq b^1\} \leq z_{LP}$. □

Corollary 12.6.8. *We take over notations and settings and assumptions in Theorem12.6.6. And*

(A2) $conv(Q) = \{x \in \mathbb{R}^n_{\geq 0} | A_2 x \leq b^2\}$.

Then

$$z_{LD} = z_{LP}$$

Corollary 12.6.9. *We take over notations and settings and assumptions in Theorem12.6.6. And*

(A2) A_2 *is totally unimodular.*

(A3) b^2 *is integer.*

Then

$$z_{LD} = z_{LP}$$

Proposition 12.6.10 (A formulation of Lagragian Relaxation with extreme points and extreme rays)**.** *We take over notations and settings and assumptions in Theorem12.6.1. And*

(S9) $\{v^k\}_{k\in K}$ *is a finite subset of* $conv(Q)$.

(S10) We pick $\{v^k\}_{k\in K} \subset \mathbb{Q}^n$ *and* $\{u^h\}_{h\in H} \subset \mathbb{Q}^n$ *such that* $conv(Q) = conv(\{v^k\}_{k\in K}) + cone(\{u^h\}_{h\in H})$. *Remark that such* $\{v^k\}_{k\in K}$ *and* $\{u^h\}_{h\in H}$ *exist by Meyer's theorem.*

Then

(i)

$$z_{LR}(\lambda) = \lambda^T b^1 + \max_{k\in K}(c - \lambda^T A_1)v^k$$

(ii)

$$z_{LD} = \min_{\lambda \geq 0, (c-\lambda^T A_1)r^h \leq 0 \ (\forall h \in H)} (\lambda^T b^1 + \max_{k\in K}(c - \lambda^T A_1)v^k)$$

Proof of (i). Since $conv(Q) = conv(\{v^k\}_{k\in K}) + conv(\{r^h\}_{h\in H})$, for each $x \in conv(Q)$, there are $\alpha \geq 0$ and $\beta \geq 0$ such that $\sum_k \alpha_k = 1$ and $x = \sum_k \alpha_k v^k + \sum_h \beta_h r^h$. Then

$$\begin{aligned} z_{LR} &= \max_{\alpha\geq 0, \beta \geq 0, \sum_k \alpha_k = 1} (c^T(\sum_k \alpha_k v^k + \sum_h \beta_h r^h) + \lambda^T(b^1 - A_1(\sum_k \alpha_k v^k + \sum_h \beta_h r^h))) \\ &= \max_{x\in conv(Q), \beta\geq 0} (\lambda^T b^1 + (c^T - \lambda^T A_1)x + \sum_h \beta_h (c^T - \lambda^T A_1) r^h) \end{aligned}$$

If there is $c^T - \lambda^T A_1 \not\leq 0$, by reaching some $\beta_h \to \infty$, we get $z_{LR} = \infty$. That is a contradiction. Therefore, $c^T - \lambda^T A_1 \leq 0$. So,

$$z_{LR} = \max_{x\in conv(Q)} (\lambda^T b^1 + (c^T - \lambda^T A_1)x) = \max_{x \in ex(conv(Q))} (\lambda^T b^1 + (c^T - \lambda^T A_1)x) = \max_k (\lambda^T b^1 + (c^T - \lambda^T A_1)v^k)$$

□

Proof of (ii). It is from (i). □

12.6.2 Dantzig-Wolfe Reformulation

Definition 12.6.11 (Ray)**.** *Let* $a \in \mathbb{R}^n$. *We call* $[0, \infty)a$ *a Ray of* $\mathbb{R}^n$.

Definition 12.6.12 (Extreme Ray)**.** *Let* C *be a polyhedral cone. We call* $R \subset C$ *an extreme ray of* C *if* R *is an edge of* C.

Proposition 12.6.13 (Dantzig-Wolfe Reformulation)**.** *We take over notations and settings and assumptions in Theorem12.6.1. And*

(S9) $\{v^k\}_{k\in K}$ *is a finite subset of* $conv(Q)$.

(S10) We pick $\{v^k\}_{k\in K} \subset \mathbb{Q}^n$ *and* $\{u^h\}_{h\in H} \subset \mathbb{Q}^n$ *such that* $conv(Q) = conv(\{v^k\}_{k\in K}) + cone(\{u^h\}_{h\in H})$. *Remark that such* $\{v^k\}_{k\in K}$ *and* $\{u^h\}_{h\in H}$ *exist by Meyer's theorem.*

Then

(i) The problem

$$\max\{cx|A_1x \le b^1, x \in conv(Q)\}$$

is equal to the following problem.

$$\max(\sum_{k\in K}(cv^k)\lambda_k + \sum_{h\in H}(cr^h)\mu_h)$$
$$\sum_{k\in K}(A_1v^k)\lambda_k + \sum_{h\in H}(A_1r^h)\mu_h \le b^1$$
$$\sum_{k\in K}\lambda_k = 1$$
$$\lambda \in \mathbb{R}^{\#K}_{\ge 0}, \mu \in \mathbb{R}^{\#H}_{\ge 0}$$

Speaking of which, the following formulation is called the Dantzig-Wolfe reformulation of the original problem.

$$\max(\sum_{k\in K}(cv^k)\lambda_k + \sum_{h\in H}(cr^h)\mu_h)$$
$$\sum_{k\in K}(A_1v^k)\lambda_k + \sum_{h\in H}(A_1r^h)\mu_h \le b^1$$
$$\sum_{k\in K}\lambda_k = 1$$
$$\sum_{k\in K}(v^k)\lambda_k + \sum_{h\in H}(r^h)\mu_h \in \mathbb{Z}^n$$
$$\lambda \in \mathbb{R}^{\#K}_{\ge 0}, \mu \in \mathbb{R}^{\#H}_{\ge 0}$$

(ii) The dual of the Dantzig-Wolf reformulation is equivalent to the Lagragian Dual.

(iii) Let z_{DWR} the optimal value of the Dantzig-Wolfe reformulation. Then $z_{DWR} \in (-\infty, \infty)$ and

$$z_I \le z_{DWR} = z_{LD} \le z_{LR}$$

Proof of (i). By applying Meyer's Theorem to that, we get (i). □

Proof of (ii). The Dantzig-Wolf reformulation is equivalent to the following.

$$\max(\begin{pmatrix} c^Tv^1 & ... & c^Tv^{\#K} & c^Tu^1 & ... & c^Tu^{\#H}\end{pmatrix}\begin{pmatrix}\lambda \\ \mu\end{pmatrix})$$
$$\begin{pmatrix} A_1v^1 & ... & A_1v^{\#K} & A_1r^1 & ... & A_1r^{\#H} \\ 1 & ... & 1 & 0 & ... & 0 \\ -1 & ... & -1 & 0 & ... & 0 \end{pmatrix}\begin{pmatrix}\lambda \\ \mu\end{pmatrix} \le \begin{pmatrix} b^1 \\ 1 \\ -1\end{pmatrix}$$
$$\lambda \ge 0, \mu \ge 0$$

From Proposition12.1.7, the dual of Dantzig-Wolf reformulation is equivalent to the following.

$$\min(\begin{pmatrix}(b^1)^T & 1 & -1\end{pmatrix}\begin{pmatrix} x \\ z_1 \\ z_2\end{pmatrix})$$
$$\begin{pmatrix} (v^1)^TA_1^T & 1 & -1 \\ ... & & \\ (v^{\#K})^TA_1^T & 1 & -1 \\ (r^1)^TA_1^T & 0 & 0 \\ ... & & \\ (r^{\#K})^TA_1^T & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ z_1 \\ z_2\end{pmatrix} \ge \begin{pmatrix} c^Tv^1 \\ ... \\ c^Tv^{\#K} \\ c^Tr^1 \\ ... \\ c^Tr^{\#H}\end{pmatrix}$$
$$x \ge 0, z_1 \ge 0, z_2 \ge 0$$

By setting $z := z_1 - z_2$, the dual of Dantzig-Wolf reformulation is equivalent to the following.

$$\begin{gathered}
\max((b^1)^T\lambda + z) \\
z \geq c^T v^1 - (v^1)^T (A_1)^T \lambda, \\
\dots \\
z \geq c^T v^{\#K} - (v^{\#K})^T (A_1)^T \lambda \\
(u^1)^T (c - (A_1)^T \lambda) \leq 0, \\
\dots \\
(u^{\#H})^T (c - (A_1)^T \lambda) \leq 0 \\
\lambda \geq 0, z \in \mathbb{R}
\end{gathered}$$

Then the dual is equivalent to the following.

$$\max_{\lambda \geq 0, (u^h)^T (c - (A_1)^T \lambda) \leq 0 \ (\forall h)} ((b^1)^T \lambda + \min_{k \in K} (c^T v^k - (v^k)^T (A_1)^T x))$$

From (ii) in Proposition12.6.10, the problem is equivalent to Lagragian dual. □

12.6.3 Column Generation

Proposition 12.6.14 (Dantzig-Wolfe Reformulation and Column Generation)**.** *We take over notations and settings and assumptions in Proposition12.6.13.*

(S1) $K' \subset K, H' \subset H$.

(S2) We call the following problem master problem.

$$\begin{gathered}
\max(\sum_{k \in K'} (cv^k)\lambda_k + \sum_{h \in H'} (cr^h)\mu_h) \\
\sum_{k \in K'} (A_1 v^k)\lambda_k + \sum_{h \in H'} (A_1 r^h)\mu_h \leq b^1 \\
\sum_{k \in K'} \lambda_k = 1 \\
\sum_{k \in K'} (v^k)\lambda_k + \sum_{h \in H} (r^h)\mu_h \in \mathbb{Z}^n \\
\lambda \in \mathbb{R}^{\#K'}_{\geq 0}, \mu \in \mathbb{R}^{\#H'}_{\geq 0}
\end{gathered}$$

Then

(i) The master problem is unbounded, the Dantzig-Wolfe relaxation is also unbounded.

(ii) The dual of the master problem is the following.

$$\begin{gathered}
\min(\pi b^1 + \sigma) \\
\pi(A_1 v^k) + \sigma \geq cv^k, k \in K' \\
\pi(A_1 r^h) \geq cr^h, h \in H' \\
\pi \in \mathbb{R}^m_{\geq 0}, \sigma \in \mathbb{R}
\end{gathered}$$

(iii) Let us assume the master problem has an optimal solution, and $(\bar{\pi}, \bar{\sigma})$ *is an solution of the dual problem. We set*

$$\begin{gathered}
\bar{c}_k := c^T v^k - \bar{\pi}^T (A_1 v^k) - \bar{\sigma} \ (k \in K), \\
\bar{c}_h := c^T r^h - \bar{\pi}^T (A_1 r^h) \ (h \in H)
\end{gathered}$$

If $\bar{c}_k \leq 0$ $(\forall k \in K)$ *and* $\bar{c}_h \leq 0$ $(\forall h \in H)$*, then* $(\bar{\pi}, \bar{\sigma})$ *is an optimal solution of the Dantzig-Wolfe relaxation.*

(iv) *We take over notations and settings and assumptions in (iii). We call the following problem the pricing problem.*

$$\zeta := -\bar{\sigma} + \max_{x \in Q}(c^T - \bar{\pi}^T A_1)x$$

Then

(a) *ζ is unbounded if and only if there is $h \in H$ such that $\bar{c_h} > 0$.*

(b) *ζ is bounded and $\zeta > 0$ if and only if there is $k \in K$ such that $\bar{c_k} > 0$.*

(c) *ζ is bounded and $\zeta \leq 0$ if and only if there is $k \in K$ such that $\bar{c_h} \leq 0, \bar{c_k} \leq 0$ $(\forall h \in H, \forall k \in K)$.*

Proof of (i). Since the feasible region of the master problem is a subset of the feasible region of the Dantzig-Wolfe relaxation, (i) holds. □

Proof of (ii). It is from the proof of (ii) in Proposition12.6.13. □

Proof of (iii). It is from the proof of (ii) in Proposition12.6.13. □

Proof of (iv). From Meyer's theorem,

$$\zeta = -\bar{\sigma} + \max_{\lambda \geq 0, \sum_k \lambda_k = 1, \mu \geq 0}(\sum_k (c^T - \bar{\pi}^T A_1)\lambda_k + \sum_h (c^T - \bar{\pi}^T A_1)\mu_h) \tag{12.6.1}$$

(iv) is clear from the equation. □

12.6.4 Benders Decomposition

Theorem 12.6.15. *Here are the settings and assumptions.*

(S1) $A \in M(m, n; \mathbb{R})$.

(S2) $B \in M(m, p; \mathbb{R})$.

(S3) $b \in \mathbb{R}^m$.

(A1) *There is a polytope, denoted by Q, and a finite generated cone, denoted by $C := cone(\{r^h\}_{h=1}^H)$ such that*

$$P = Q + C$$

Then

$$proj_x(P) = \{x \in \mathbb{R}^n | (r^h)^T Ax \leq (r^h)^T b \ (h = 1, 2, ..., H)\}$$

Proof. □

Theorem 12.6.16 (Benders Theorem)**.** *Here are the settings and assumptions.*

(S1) $c \in \mathbb{Q}^n$.

(S2) $h \in \mathbb{Q}^p$.

(S3) $A \in M(m, n, \mathbb{Q})$.

(S4) $G \in M(m, p, \mathbb{Q})$.

(S5) $F := \{(x, y) \in \mathbb{Z}^n_{\geq 0} \times \mathbb{Q}^p_{\geq 0} | Ax + Fy \leq b\}$.

(S6) *We call*

$$\max_{(x,y) \in F}\{cx + hy\}$$
$$\iff$$
$$\max\{cx + hy\}$$
$$Ax + Gy \leq b$$
$$x \in \mathbb{Z}^n_{\geq 0}$$
$$y \in \mathbb{Q}^p_{\geq 0}$$

the original problem.

(S7) $Q := \{u \in \mathbb{R}^m_{\geq 0} | u^T G \geq h\}$.

(S8) *We pick* $\{u^k\}_{k \in K} \subset \mathbb{Q}^m$ *such that* $conv(Q) = conv(\{u^k\}_{k \in K})$. *Remark that such* $\{u^k\}_{k \in K}$ *exist by Meyer's theorem.*

(S9) $C := \{u \in \mathbb{R}^m_{\geq 0} | u^T G \geq 0\}$.

(S10) *We pick* $\{r^J\}_{j \in J} \subset \mathbb{Q}^m$ *and* $\{r^j\}_{j \in J} \subset \mathbb{Q}^m$ *such that* $C = cone(\{r^j\}_{j \in J})$. *Remark that such* $\{r^j\}_{j \in J}$ *and* $\{r^j\}_{j \in J}$ *exist by Meyer's theorem.*

Then the original problem is equal to the following problem.

$$\max\{\eta + cx\}$$
$$\eta \leq u^k(b - Ax) \ (\forall k \in K)$$
$$(r^j)^T(b - Ax) \geq 0 \ (\forall j \in J)$$
$$x \in \mathbb{Z}^n_{\geq 0}$$
$$\eta \in \mathbb{R}$$

12.7 Semidefinite Bounds

Semidefinite Problem is a generalization of linear programming. In a certain assumption, the problem can be solved in polynomial time. And a relaxation to a semidefinite problem may provide a tighter bound than linear programming relaxations. The method started from [49] on which [51] is based. The method was generalized to mixed 0-1 linear program in [50].

Definition 12.7.1 (Semidefinite Problem)**.** *The followings are settigs and assumptions.*

(S1) $C, A_1, ..., A_m$ *are* $n \times n$ *symmetric matrices.*

(S2) $b \in \mathbb{R}^m$.

We set the problem

$$\max(C, X)$$
$$(A_i, X) = b_i \ (i = 1, 2, ..., m)$$
$$X \text{ is positive semidefinite matrix}$$

We call it a semidefinite problem.

Definition 12.7.2 (Lovasz-Schrijver Relaxation)**.** *The followings are settigs and assumptions.*

(S1) $A \in M(m, n + p, \mathbb{Q})$.

(S2) $b \in \mathbb{R}^m$.

(S3) $P := \{x \in \mathbb{R}^{n+p}_{\geq 0} | Ax \geq b\}$.

(S4) $S := P \cup \{0, 1\}^n \times \mathbb{R}^n_{\geq 0}$.

Then

STEP1. *We set*

$$T := \{x \in [0, 1]^n \times \mathbb{R}^n_{\geq 0} | P_j(x) := x_j(Ax - b) \geq 0, Q_j(x) := (1 - x_j)(Ax - b) \geq 0 \ (j = 1, 2, ..., n)\}$$

STEP2. *We set*

$$P_j(x) \mapsto R_j(x, y)$$

and

$$Q_j(x) \mapsto S_j(x, y)$$

by

$$x_i x_j \mapsto y_{i,j} \ (j < i, i = 1, 2, ..., n + p)$$

and

$$x_i \mapsto x_i \ (i = 1, 2, ..., n + p)$$

and

$$x_j^2 \mapsto x_j \ (j = 1, 2, ..., n)$$

and

$$y_{0,0} := 1, y_{0,j} = y_{j,0} = y_{j,j} = x_j \ (j = 1, 2, ..., n)$$

Remark that the number of variables of (x, y) *is* $\frac{1}{2}n(n+1) + np$. *We set*

$$M_+(P) := \{(x, y) \in \mathbb{R}_{\geq 0}^{\frac{1}{2}(n(n+1)+np)} | R_j(x, y) \geq 0, S_j(x, y) \geq 0 \ (j = 1, 2, ..., n), \\ Y := (y_{i,j})_{(n+1)\times(n+1)} \text{ is positive semidefinite.}\}$$

STEP3. We set

$$M_+(P) \ni (x, y) \mapsto x \in \mathbb{R}_{\geq 0}^n$$

and Let $N_+(P)$ *denote the image of the above map.*

Definition 12.7.3. *We take over the notations of Definition12.7.2. Then*

STEP1. It is the same as the STEP1 of Definition12.7.2.

STEP2. It is the same as the STEP1 of Definition12.7.2. We set

$$M(P) := \{(x, y) \in \mathbb{R}_{\geq 0}^{\frac{1}{2}(n(n+1)+np)} | R_j(x, y) \geq 0, S_j(x, y) \geq 0 \ (j = 1, 2, ..., n)\}$$

STEP3. We set

$$M(P) \ni (x, y) \mapsto x \in \mathbb{R}_{\geq 0}^n$$

and Let $N(P)$ *denote the image of the above map.*

Proposition 12.7.4 (Lovasz-Schrijver Relaxation)**.** *We take over the notations of Definition12.7.2 and Definition12.7.3. Then*

$$conv(S) \subset N_+(P) \subset N(P) \subset P$$

Chapter 13

Event graph analysis

13.1 Max-plus algebra

Definition 13.1.1 (Semi-ring)**.** *Here are the settings.*

(S1) R is a set.

(S2) $\oplus, \otimes$ are binomial operators on R.

We say $(R, \oplus, \otimes)$ is a semi ring if

(i) For any $x, y, z \in R$,

$$(x \oplus y) \oplus z = x \oplus (y \oplus z)$$

$$(x \otimes y) \otimes z = x \otimes (y \otimes z)$$

(ii) For any $x, y, z \in R$,

$$x \oplus y = y \oplus x$$

(iii) For any $x, y, z \in R$,

$$x \otimes (y \oplus z) = x \otimes y \oplus x \otimes z$$

(iv) R has the unit element ϵ with respect to $\oplus$.

(v) R has the unit element e with respect to $\oplus$.

(vi) $\epsilon \otimes x = x \otimes \epsilon = \epsilon$.

We say R is commutative if $\otimes$ is commutative. We say R is idempotent if $\otimes$ is idempotent.

Definition 13.1.2 ($\mathbb{R}_{max}$)**.** *Here are the settings.*

(S1) $\mathbb{R}_{max} := \mathbb{R} \cup \{-\infty\}$. We set $\epsilon := -\infty$ and $e := 0$.

(S2) For $x, y \in \mathbb{R}_{max}$

$$x \oplus y := \max\{x, y\}$$

$$x \otimes y := x + y$$

We call $\mathcal{R}_{max} := (\mathbb{R}_{max}, \oplus, \otimes)$ the max-plus algebra.

Clearly the following holds.

Proposition 13.1.3. *$\mathcal{R}_{max}$ is a commutative and idempotent semi ring.*

13.2 Petri net and Event graph

Definition 13.2.1 (Petri net, place, transition). *Here are the settings.*

(S1) $(\mathcal{N}, \mathcal{A})$ *is a directed graph.*

We say $(\mathcal{N}, \mathcal{A})$ *is a petri net if there is* $(\mathcal{P}, \mathcal{Q})$ *which is a pair of disjoint subsets of* $\mathcal{N}$ *satisfying the following two conditions.*

(i) $\mathcal{N} = \mathcal{P} \cup \mathcal{Q}, \mathcal{P} \cap \mathcal{Q} = \phi$.

(ii) $\mathcal{A} \subset \mathcal{P} \times \mathcal{Q} \cup \mathcal{Q} \times \mathcal{P}$.

We denote this petri net by $(\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A})$.

We call each element of $\mathcal{P}$ *a place and call each element of* $\mathcal{Q}$ *a transistion. Let us fix* $p \in \mathcal{P}$ *and* $q \in \mathcal{Q}$. *We say* p *is the input place of the transition* q *and* q *is the output place of the transition* p *if* $(p, q) \in \mathcal{A}$. *We say* p *is the output place of the transition* q *and* q *is the input place of the transition* p *if* $(p, q) \in \mathcal{A}$.

We denote the set of all input place of q *by* $\pi(q)$ *and denote the set of all input transition of* p *by* $\pi(p)$.

We denote the set of all output place of q *by* $\sigma(q)$ *and denote the set of all output transition of* p *by* $\sigma(p)$.

Definition 13.2.2 (Event graph). *Here are the settings.*

(S1) $(\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A})$ *is a petri net.*

We say this petri net is an event graph if for each $p \in \mathcal{P}$ *there is the unique* $q_1 \in \mathcal{Q}$ *such that* $(p, q_1) \in \mathcal{A}$ *and there is the unique* $q_2 \in \mathcal{Q}$ *such that* $(q_2, p) \in \mathcal{A}$.

Definition 13.2.3 (Enability and Firing in petri net). *Here are the settings.*

(S1) $(\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A})$ *is a petri net.*

(S2) $w : \mathcal{A} \to \mathbb{N}_{\geq 1}$. *We call* $w(a)$ *is the weight of* $a \in \mathcal{A}$.

(S3) $M_1 : \mathcal{P} \to \mathbb{Z}_{\geq 0}$. *For each* $p \in \mathcal{P}$, *we say* p *is marked with* $M_1(p)$ *tokens.*

(S4) $q \in \mathcal{Q}$.

Then

(i) We say q *is enable if each input place* p *of* q *is marked with at least* $w(p, q)$ *tokens.*

(ii) Let us assume q *is enable. We set for each* $p \in \mathcal{P}$

$$M_1(p) := M_0(p) + \chi_{\sigma(q)}(p)w(q, p) - \chi_{\pi(q)}(p)w(p, q)$$

We call M_1 *the firing of* M_0 *with respect to* q.

Definition 13.2.4 (Liveness, Autonomous, Time event graph). *Here are the settings.*

(S1) $G := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_0)$ *is an event graph with weight and token.*

Then

(i) We say G *is liveness if for any cycle* c *of* G *there is* $p \in \mathcal{P}$ *whose output transition is enable.*

(ii) For each $q \in \mathcal{Q}$, q *is a supplier transition if* $\pi(q) = \phi$.

(iii) We say G *is autonomous if* G *is no supplier transitions.*

(iv) Let $\tau : \mathcal{P} \to \mathbb{Z}_{\geq 0}$ *and* $\gamma : \mathcal{A} \cap \mathcal{P} \times \mathcal{Q} \to \mathbb{Z}_{\geq 0}$ *such that*

$$\gamma(p, q) \leq \tau(p)$$

Then (G, τ, γ) *with time event graph.*

Definition 13.2.5 (Enability and Firing in Time event graph). *Here are the settings.*

(S1) $G := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_0, \tau, \gamma_0)$ *is a time event graph.*

(A1) For any $q_1, q_2 \in \mathcal{Q}$, *there is at most one* $p \in \mathcal{P}$ *such that* $(q, p), (p, q) \in \mathcal{A}$.

(A2) $w = 1$ *on* $\mathcal{A}$.

(S2) $q \in \mathcal{Q}$.

Then

(i) *We say q is enable if each input place p of q is marked with at least $w(p,q)$ tokens and $\tau(p) \leq \gamma(p,q)$. We denote the all enable transitions by $E(G)$.*

(ii) *Let us assume q is enable. We set for each $p \in \mathcal{P}$*

$$M_1(p) := M_0(p) + \chi_{\sigma(q)}(p)w(p,q) - \chi_{\pi(q)}(p)w(p,q), \gamma_1(p) := 0$$

We call (M_1, γ_1) the firing of (M_0, γ_0) with respect to q.

Clearly the following holds.

Proposition 13.2.6. *Here are the settings.*

(S1) *$G_0 := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_0, \tau, \gamma_0)$ is a time event graph.*

(A1) *For any $q_1, q_2 \in \mathcal{Q}$, there is at most one $p \in \mathcal{P}$ such that $(q,p), (p,q) \in \mathcal{A}$.*

(A2) *$w = 1$ on $\mathcal{A}$.*

(S3) *We set*

$$M_1(p) := M_0(p) + \chi_{E(G_0)}(q_1) - \chi_{E(G_0)}(q_2)$$

Here $q_1 \in \pi(p)$ and $q_2 \in \sigma(p)$. And

$$\gamma_1(p,q) := \begin{cases} \gamma_0(p,q) + 1 & M_0(p) > 0 \text{ and } q \text{ is not enable} \\ 0 & \text{otherwise} \end{cases}$$

(S4) *We set $G_1 := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_1, \tau, \gamma_1)$.*

Then G_1 is a time event graph.

Definition 13.2.7 (Firing time)**.** *Here are the settings.*

(S1) *$G_0 := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_0, \tau, \gamma_0)$ is a time event graph.*

(A1) *For any $q_1, q_2 \in \mathcal{Q}$, there is at most one $p \in \mathcal{P}$ such that $(q,p), (p,q) \in \mathcal{A}$.*

(A2) *$w = 1$ on $\mathcal{A}$.*

(S3) *We define $\{G_t\}_{t=0}^{\infty}$ inductively by the procedure defined in Proposition13.2.6.*

Then

$$x_q(k) := \{t_0 \in \mathbb{Z}_{\geq 0} | k = \#\{t \leq t_0 | q \in E(G_t)\}\} \ (q \in \mathcal{Q}, k \in \mathbb{N}_{\geq 1})$$

We call $x_q(k)$ the k-th firing time of q. We set

$$x(k) := (x_{q_1}(k), ..., x_{q_{\#\mathcal{Q}}})^T \ (k \in \mathbb{N}_{\geq 1})$$

Definition 13.2.8 (System Matrix)**.** *Here are the settings.*

(S1) *$\{G_t := (\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{A}, w, M_t, \tau, \gamma_t)\}_{t \in \mathbb{Z}_{\geq 0}}$ is a sequence of time event graphs by the procedure defined in Proposition13.2.6.*

(S2) *$\{x(k)\}_{k=1}^{\infty}$ is the sequence by Definition13.2.7.*

(S3) *We denote the maximum number of tokens at any one place in $\{G_t\}_{t \in \mathbb{Z}_{\geq 0}}$ by M.*

Then for each $m \in \{0, 1, ..., M\}$

$$[A_m]_{j,l} := \begin{cases} a_{j,l} & p_{j,l} \text{ exists and } p_{j,l} \text{ has } m \text{ tokens in } G_0 \\ \epsilon & \text{otherwise} \end{cases} \quad (j, l = 1, 2, ..., \#\mathcal{Q})$$

Here $p_{j,l}$ is the place such that $(q_j, p_{j,l}), (p_{j,l}, q_l) \in \mathcal{A}$.

Proposition 13.2.9. *We succeed notations in Definition13.2.8. And let us assume any G_t is autonomous. Then*

$$x(k) = A_0 \otimes x(k) \oplus A_1 \otimes x(k-1) \oplus ... \oplus A_M \otimes x(k-M) \ (k = M+1, M+2, ...)$$

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