A study memo on Lie Group and Representation Theory

Contents

This note is the result of studying facts based on [1], [2].

1 Preliminary

1.1 Linear algebra

1.1.1 Some facts without proof

For the proof, see [5].

Theorem 1.1 (Hahn Banach Theorem1). *Let*

(S1) $(V, \{p_n\}_{n\in\mathbb{N}})$ *is a semi-normed space.*

(S2) $x, y \in V$ *such that* $x \neq y$.

Then there is real-valued continuous linear function f *such that* $f(x) \neq f(y)$.

1.1.2 Tensor Space

Clearly the following holds.

Proposition 1.1 (Tensor Space). *Here are the settings and assumptions.*

(S1) K *denotes one of* $\mathbb{Q}, \mathbb{R}, \mathbb{C}$.

(S2) V, W *are* K*-vector spaces.*

(S3) By $V^\vee, W^\vee$ *denote by the set of all* K*-linear functionals of* V, W*, respectively.*

(S4) For $v \in V, w \in W$*, we set*

$$v \otimes w(f, g) := f(v)g(w) \ (f \in V^\vee, g \in W^\vee)$$

Then, for any $v \in V, w \in W$*,* $v \otimes w \in B(V, W)$*. We set*

$$V^\vee \otimes W^\vee := \langle \{v \otimes w | v \in V, w \in W\} \rangle$$

Proposition 1.2. *Here are the settings and assumptions.*

(S1) K *denotes one of* $\mathbb{Q}, \mathbb{R}, \mathbb{C}$.

(S2) V, W *are* K*-vector spaces.*

(S3) $w_1, ..., w_m \in W$ *are linear independent.*

(S4) $v_1, ..., v_m \in V \setminus \{0\}$.

Then, $\{v_i \otimes w_i\}_{i=1}^m$ *are linear independent.*

By Hahn-Banach Theorem,

Proof. there are $f_1, ..., f_m \in W^\vee$ such that $f_i(w_j) = \delta_{i,j}$ $(\forall i, j)$ and there are $g_1, ..., g_m \in W^\vee$ such that $g_i(v_i) \neq 0$ $(\forall i)$

Let us fix any $a_1, ...a_m \in K$ such that $\sum_{i=1}^m a_i v_i \otimes w_i = 0$. Since $0 = \sum_{i=1}^m a_i v_i \otimes w_i(g_j, f_j) = a_j$ $(\forall j)$, $\{v_i \otimes w_i\}_{i=1}^m$ are linear independent. □

1.1.3 Kronecker Product

Definition 1.1 (Kronecker Product). *Let* K *denotes one of* $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ *and* $A \in M(m, n, K)$ *and* $B \in M(p, q, K)$*. Then*

$$A \otimes B = \{c_{i+k,j+l} := a_{i,j} b_{k,l}\}_{i,j,k,l\in\mathbb{N}} = \begin{pmatrix} a_{1,1}b_{1,1} & ... & a_{1,1}b_{1,q} & a_{1,2}b_{1,1} & ... & a_{1,2}b_{1,q} & ... & a_{1,n}b_{1,1} & ... & a_{1,n}b_{1,q} \\ a_{1,1}b_{2,1} & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ a_{1,1}b_{p,1} & ... & a_{1,1}b_{p,q} & a_{1,2}b_{p,1} & ... & a_{1,2}b_{p,q} & ... & a_{1,n}b_{1,q} & ... & a_{1,n}b_{p,q} \\ a_{2,1}b_{1,1} & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ a_{2,1}b_{p,1} & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ a_{m,1}b_{1,1} & ... & a_{m,1}b_{1,q} & a_{m,2}b_{1,1} & ... & a_{m,2}b_{1,1} & ... & a_{m,n}b_{1,1} & ... & a_{m,n}b_{p,q} \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ a_{m,1}b_{p,1} & ... & a_{m,1}b_{p,q} & a_{m,2}b_{p,1} & ... & a_{m,2}b_{p,q} & ... & a_{m,n}b_{p,1} & ... & a_{m,n}b_{p,q} \end{pmatrix}$$

We call $A \otimes B$ *the Kronecker Product of* A *and* B.

Proposition 1.3. *Here are the settings and assumptions.*

(S1) $K = \mathbb{Q}, \mathbb{R}, \mathbb{C}$.

(S2) $A \in M(m_1, m_2, K), B \in M(m_3, m_4, K), C \in M(n_1, n_2, K), D \in M(n_3, n_4, K)$.

(A1) $m_2 = n_1, m_4 = n_3$.

Then

$$(A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$$

Proof. For any i_1, i_2, j_1, j_2,

$$(A \otimes B) \cdot (C \otimes D)_{(i_1,i_2),(j_1,j_2)} = \sum_{k_1,k_2} (A \otimes B)_{(i_1,i_2),(k_1,k_2)} (C \otimes D)_{(k_1,k_2),(j_1,j_2)} = \sum_{k_1,k_2} a_{i_1,k_1} b_{i_2,k_2} c_{k_1,j_1} b_{k_2,j_2}$$

$$= \sum_{k_1} a_{i_1,k_1} c_{k_1,j_1} \sum_{k_2} b_{i_2,k_2} b_{k_2,j_2} = (A \cdot C)_{i_1,j_1} (B \cdot D)_{i_2,j_2} = ((A \cdot C)_{\otimes} (B \cdot D))_{(i_1,i_2),(j_1,j_2)}$$

□

Proposition 1.4. *Here are the settings and assumptions.*

(S1) $A \in M(m, \mathbb{C}), B \in M(n, \mathbb{C})$.

(S2) $\lambda_1, ..., \lambda_m$ *are the eigenvalues of* A.

(S3) $\mu_1, ..., \mu_n$ *are the eigenvalues of* B.

(A1) $\lambda_i \mu_j$ $(i = 1, 2, ..., m, j = 1, 2, ..., n)$ *are distinct.*

Then

$$\lambda_i \mu_j \ (i = 1, 2, ..., m, j = 1, 2, ..., n)$$

are the all eigenvalues of $A \otimes B$.

Proof. Let x_i denote an eigenvector of A with respect to λ_i $(i = 1, 2, ..., m)$ and y_j denote an eigenvector of B with respect to μ_j $(j = 1, 2, ..., n)$. By Proposition1.3, the vector $x_i \otimes y_j$ is an eigenvector of $A \otimes B$ with respect to $\lambda_i \mu_j$ $i = 1, 2, ..., m, j = 1, 2, ..., n$. □

Proposition 1.5. *Here are the settings and assumptions.*

(S1) $A \in M(m, \mathbb{C}), B \in M(n, \mathbb{C})$.

Then

$$det(A \times B) = det(A)det(B)$$

Proof. By applying triangulization of matrices, we can show that there are $\{A_i\}_{i=1}^{\infty} \subset M(m, \mathbb{C}), \{B_i\}_{i=1}^{\infty} \subset M(m, \mathbb{C})$ such that A_i, B_i satisfies the settings and the assumptions in Proposition1.4 for any i and

$$\lim_{i \to \infty} A_i = A, \lim_{i \to \infty} B_i = B$$

So,

$$\lim_{i \to \infty} det(A_i \otimes B_i) = det(A \otimes B), \lim_{i \to \infty} det(A_i) = det(A), \lim_{i \to \infty} det(B_i) = det(B)$$

By Proposition1.4,

$$det(A_i \otimes B_i) = det(A_i)det(B_i) \ (\forall i)$$

Consequently,

$$det(A \otimes B) = det(A)det(B)$$

□

1.2 Topological space

Proposition 1.6. *Let X and Y are topological space and $i : X \to Y$ is homeomorphism. And let $U \subset X$ ando $V := i(u)$. Then $i|U : U \to V$ is homeomorphisim.*

Proof. For any closed set in X A and any closed set in Y B, $i^{-1}(B \cap V) = i^{-1}(B) \cap V$ and $i(A \cap U) = i(A) \cap V$. So$i^{-1}(B \cap V)$ is closed set of X and $i(A \cap U)$ is closed set of Y. □

Proposition 1.7. *Let X is a topological space and $U \subset U' \subset X$. Let us assume the topology of U' is the relative topology respect to X. The relative topology of U respect to U' is equal to the relative topology of U respect to X.*

Proof. Because for any open set A in X $A \cap U = A \cap U' \cap U$, the Proposition holds. □

Proposition 1.8. *Let X be a Housdorff space and $C \subset X$ be a compact subset. Then C is a closed subset of X.*

Proof. Let us fix any $x \in X \setminus C$. For each $y \in C$, there are U_y and V_y such that U_y is an open neighborhood of x and V_y is an open neighborhood of y and $U_y \cap V_y = \phi$. Because C is compact, there are $V_{y_1}, ..., V_{y_m}$ such that $C \subset \cup_{i=1}^m V_{y_i}$. Because $\cap_{i=1}^m U_{y_i}$ is an open neighborhood of x and $\cap_{i=1}^m U_{y_i} \cap \cup_{i=1}^m V_{y_i} = \phi$, $x \notin \bar{C}$. Consequently, C is a closed subset. □

Definition 1.2 (Locally path-connected space)**.** *Here are the settings and assumptions.*

(S1) X is a topological space.

We say X is locally path-connected if for any $U \in \mathcal{O}(X)$ and $x \in U$, there is V such that V is a path-connected open neighborhood of x and $V \subset U$.

The following clearly holds.

Proposition 1.9. *Any topological manifold is locally path-connected.*

Definition 1.3 (Covering Space)**.** *Here are the settings and assumptions.*

(S1) E, B are path-connected and locally connected topological space.

(S2) $p : E \to B$ is a surjective continuous map.

We say (E, B, p) is a covering space if for any $b \in B$ there is U such that U is an open neighborhood of b and any connected component of $\pi^{-1}(U)$ V satisfies $\pi|V : V \to \pi(V)$ is a homeomorphism. We call E the total space, B the base space, p the projection.

Definition 1.4 (Finite covering Space)**.** *Here are the settings and assumptions.*

(S1) (E, B, p) is a covering space.

We say (E, B, p) is a finite covering space if there is $m \in \mathbb{N}$ such that for any $b \in B$ $\#p^{-1}(b) = m$. We call m the covering degree of (E, B, p).

1.3 Hilbert Space

Proposition 1.10. *Here are the settings and assumptions.*

(S1) V is an inner product space.

(A1) $\{v \in V | ||v|| = 1\}$ is compact.

Then $dimV < \infty$.

Proof. Let us assume $dimV = \infty$. Then there is a orthonormality $\{v_i\}_{i=1}^\infty \subset V$. Because there is no subsequence of $\{v_i\}_{i=1}^\infty$ which converges in V, $\{v \in V | ||v|| = 1\}$ is not compact. This is contradiction. □

Proposition 1.11 (Bessel Inequality)**.** *Let*

(S1) V is a inner product space.

(S2) $\{v_i\}_{i=1}^N$ is a orthonormal system of V.

Then for any $u \in V$,

$$\sum_{i=1}^N |(u, v_i)|^2 \leq ||u||^2$$

Proof. By (S2),

$$0 \leq ||u - \sum_{i=1}^{N}(u, v_i)v_i||^2 = ||u||^2 - \sum_{i=1}^{N}|(u, v_i)|^2$$

This impliese the above inequality. □

Proposition 1.12. *Let*

(S1) V is a separable Hilbert space.

(S2) $\{v_i\}_{i=1}^{\infty}$ is a complete orthonormal system of V.

Then

(i) If $u \in V$ and $(u, v_i) = 0$ $(\forall i)$, then $u = 0$.

(ii) For any $u \in V$, $\sum_{i=1}^{\infty}(u, v_i)v_i$ converges and

$$u = \sum_{i=1}^{\infty}(u, v_i)v_i$$

(iii) Any complete orthonormal system of V is countable.

Proof of (i). We set $W := \sum_{i=1}^{\infty} \mathbb{C}v_i$. There is a sequcen $\{w_i\}_{i=1}^{\infty} \subset W$ such that $\lim_{i\to\infty} w_i = u$. So,

$$||u||^2 = \lim_{i\to\infty}(u, w_i) = 0$$

This implies $u = 0$. □

Proof of (ii). By bessel inequality, $\{\sum_{i=1}^{N}(u, v_i)v_i\}_{N\in\mathbb{N}}$ is a cauchy sequence in V. Because V is complete, $\sum_{i=1}^{\infty}(u, v_i)v_i$ converges. Because $(u - \sum_{i=1}^{\infty}(u, v_i)v_i, v_j) = 0$ $(\forall j)$, by (i), (ii) holds. □

Proof of (iii). Let us fix $\{w_\alpha\}_{\alpha\in\Lambda}$ which is any complete orthonormal system of V. For each $m, n \in \mathbb{N}$, there is a finite subset $\Lambda_{m,n} \subset \Lambda$ such that

$$d(v_m, \sum_{\alpha\in\Lambda_{m,n}} \mathbb{C}w_\alpha) < \frac{1}{n}$$

We set $\Lambda^* := \cup_{m,n}\Lambda_{m,n}$. Clearly Λ^* is at most countable and $\{w_\alpha\}_{\alpha\in\Lambda^*}$ is a complete orthonormal system of V. So, $\Lambda^* = \Lambda$. □

Proposition 1.13 (Projection Theorem)**.** *Let*

(S1) V is a Hilbert space.

(S2) W is a closed subspace of V.

then

$$V = W \oplus W^{\perp}$$

So, for each $v \in V$, there is a unique $w \in W$ such that $v - w \in W^{\perp}$. We call w is the orthogonal projection of v. We set $p_W : V \to W$ by

$$p_M : V \ni v \mapsto w \in W s.t. v - w \in W^{\perp}$$

We call p_W is the orthogonal projection of W.

Proof in general case. Let us fix any $v \in W$. We set

$$d := d(v, W)$$

Then there is $\{w_i\}_{i=1}^{\infty} \subset W$ such that

$$\lim_{n\to\infty} ||v - w_i|| = d$$

We will show $\{w_i\}_{i=1}^{\infty}$ is a cauchy sequence. For any $m, n \in \mathbb{N}$,

$$||w_m - w_n||^2 = ||v_m - w||^2 - 2Re(w_m - w, w_n - w) + ||w_n - w||^2$$

And

$$2Re(w_m - w, w_n - w) = ||(w_m - w) + (w_n - w)||^2 - ||w_m - w||^2 - ||w_n - w||^2$$

So,

$$||w_m - w_n||^2 + 4||\frac{w_m + w_n}{2} - w||^2 = 2||w_m - w||^2 + 2||w_n - w||^2$$

Because

$$||w_m - w_n||^2 + 4||\frac{w_m + w_n}{2} - w||^2 \geq ||w_m - w_n||^2 + 4d^2$$

$$||w_m - w_n||^2 \leq 2||w_m - w||^2 + 2||w_n - w||^2 - 4d^2$$

So, $\{w_i\}_{i=1}^{\infty}$ is a cauchy sequence. Because V is Hilbert space,

$$w := \lim_{n \to \infty} w_n$$

exists. Because W is closed, $w \in W$.

$$||v - w||^2 = ||v - w_n + w_n - w||^2 = ||v - w_n||^2 + 2Re(v - w_n, w_n - w) + ||w_n - w||^2$$

So,

$$||v - w||^2 = d^2$$

We set

$$u := v - w$$

Let us assume $u \notin W^{\perp}$. Then there is $w_0 \in W$ such that $(u, w_0) > 0$. So, for any $\delta > 0$

$$d^2 \leq ||u - \delta w_0||^2 = d^2 - 2\delta Re(u, w_0) + \delta^2 ||w_0||^2$$

This implies

$$2Re(u, w_0) \leq \delta ||w_0||^2$$

So, if we take $\delta < \dfrac{2Re(u, w_0)}{||w_0||^2}$, a contradiction arises. So $u \in W^{\perp}$. □

Proof in case W is separable. Because W is separable, by Gram-Schmit orthogonalization method, there a $\{w_i\}_{i=1}^{\infty}$ which is a complete orthonormal system of W. Let us fix any $u \in V$. By the same argument as the proof of Proposition1.12, $w := \sum_{i=1}^{\infty} (u, w_i) w_i$ converges. Because W is closed, $w \in W$. Clearly $u - w \perp W$. □

By the argument in the proof of Propoisition1.13, the following holds.

Proposition 1.14. *Let*

(S1) V is a pre Hilbert space.

(S2) W is a subspace of V.

(S3) $v \in V$.

(S4) $\{v_n\}_{n \in \mathbb{N}} \subset V$ such that

$$\lim_{n \to \infty} ||v - v_n|| = \inf_{u \in W} ||v - u||$$

then $\{v_n\}_{n \in \mathbb{N}}$ is a cauchy space.

Proposition 1.15. *Let*

(S1) V is a Hilbert space.

(S2) W is a closed subspace of V.

(A1) $p : V \to W$ is a surjective self adjoint linear operator such that $p^2 = p$.

then p is the orthogonal projection of W.

Proof. Let us set p_W the orthogonal projection of W. Let us fix any $v \in V$ and $w := p_W(v)$. Then, firstly, $p(v) - w \in W$ and there is $v' \in V$ such that $p(v') = w$.

$$p(v) - w = p(v) - p(v') = p(v) - p^2(v') = p(v) - p(w) = p(v - w)$$

Because $v - w \in W^{\perp}$, for any $w' \in W$,

$$(p(v) - w, w') = (p(v - w), w') = (v - w, p^* w') = (v - w, p(w')) = 0$$

So, $p(v) - w \in W^{\perp}$. These imply $p(v) = w$. □

By Proposition1.15, the following holds.

Proposition 1.16. *Let*

(S1) V *is a Hilbert space.*

(S2) $W_1, ..., W_m$ *are closed subspace of* V *and* $W_i \perp W_j$ $(\forall i \neq \forall j)$.

(A1) $p_i : V \to W_i$ *is the orthogonal projection to* W_i $(i = 1, 2, ..., m)$.

then

$$p := \sum_{i=1}^{m} p_i$$

is the orthogonal projection of $\oplus_{i=1}^{m} W_i$.

Proposition 1.17. *Let*

(S1) V *is a Hilbert space.*

(S2) $\{W_i\}_{i \in I}$ *is a family of closed subspaces of* V.

(A1) $W_i \perp W_j$ $(\forall i \neq \forall j)$.

(A2) $V = \oplus_{i \in I} W_i$.

(S3) We denote the orthogonal projection of W_i *by* p_i $(i \in I)$.

then for any $v \in V$

$$\inf\{||v - \sum_{j \in J} P_j v|| \ |J \subset I : finite\} = 0$$

Proof. Let us fix any $v \in V$ and $\epsilon > 0$. By (A2), there are $J \subset I$:finite and $\{v_i\}_{i \in J}$ such that $v_i \in W_i$ $(\forall i \in J)$ and $||v - \sum_{i \in J} v_i|| < \epsilon$. We set $p := \sum_{i \in J} P_i$. By Proposition1.16, p is the orthogonal projection of $\oplus_{i \in J} W_i$. By the proof of Projection theorem, $||v - p(v)|| \leq ||v - \sum_{i \in J} v_i||$. So, $||v - \sum_{j \in J} P_j v|| < \epsilon$. □

Proposition 1.18 (Riez representation theorem)**.** *Let*

(S1) V *is a Hilbert space.*

(S2) $f \in V^*$.

Then there is $u \in V$ *such that*

$$f(\cdot) = (\cdot, u)$$

Proof. We set $W := Ker(f)$. We can assume $f \neq 0$. Let us take $w_0 \in W^{\perp} \setminus \{0\}$. We can assume $f(w_0) = 1$. Let us fix $v \in V$ and $u := v - f(v)w_0$. Clearly $u \in W$, so $u \perp w_0$. This implies

$$(v, w_0) = f(v)||w_0||^2$$

□

Proposition 1.19. *Let*

(S1) V *is a Hilbert space.*

(S2) $\{v_i\}_{i=1}^{\infty} \subset \{v \in V | ||v|| = 1\}$.

Then there is subsequence $\{v_{\varphi}(i)\}_{i=1}^{\infty}$ *and* $v \in V$ *such that for any* $f \in V^*$

$$\lim_{i \to \infty} f(v_{\varphi(i)}) = f(v)$$

We denote this by

$$w - \lim_{i \to \infty} v_{\varphi(i)} = v$$

Proof. Because $(v_i, v_j) \in \mathbb{T}_1 (\forall i, j)$ and $\mathbb{T}_1$ is compact, then there are subsequences $\{v_{\varphi_n(k)}\}_{k=1}^{\infty}$ $(n = 1, 2, ...)$ such that for each $n \in \mathbb{N}$ $\{v_{\varphi_n(k)}\}_{k=1}^{\infty}$ is a subsequence of $\{v_{\varphi_{n+1}(k)}\}_{k=1}^{\infty}$ and $\lim_{k\to\infty}(v_{\varphi_n(k)}, v_n)$ exists. We set

$$\psi(n) := \varphi_n(n) \ (n \in \mathbb{N})$$

Then for any $n \in \mathbb{N}$, $f(v_n) := (\lim_{k\to\infty}(v_n, v_{\psi(k)})$ exists. We set V_0 be the minimum sublinear space which contains $\{v_i\}_{i=1}^{\infty}$ and $V_1 := \bar{V_0}$. Let us fix any $w \in \bar{V_1}$. Then there is $\{w_i\}_{i=1}^{\infty} \subset V_0$ such that $\lim_{i\to\infty} w_i = w$. Let us fix any $\epsilon > 0$. Then there is $n_0 \in \mathbb{N}$ for any $m, n \geq n_0$ $||w_m - w_n|| \leq \epsilon$. $|f(w_m) - f(w_n)| = |f(w_m - w_n)| \leq ||w_m - w_n|| \leq \epsilon$. So, $f(w) := \lim_{n\to\infty} f(w_n)$ exists. Clearly $||f|| \leq 1$. So $f \in V_1^*$. By Riez representation theorem, there is $v \in V_1$ such that $f = (\cdot, v)$. Let us fix any $u \in \bar{V_1}$ and $\epsilon > 0$. Then there is $u' \in V_0$ such that $||u - u'|| < \frac{\epsilon}{2}$. There is $n_0 \in \mathbb{N}$ such that for any $k \geq n_0$ $|(u', v_{\psi(k)}) - (u', v)| \leq \frac{\epsilon}{2}$. So $|(u, v_{\psi(k)}) - (u, v)| \leq \epsilon$. This means

$$\lim_{k\to\infty}(u, v_{\psi(k)}) = (u, v) \tag{1.3.1}$$

Let us fix any $g \in V^*$. Then $g|V_1 V_1^*$. By Riez representation theorem, there is $u_g \in V_1$ such that $g|V_1 = (\cdot, u_g)$. So,

$$\lim_{k\to\infty} g(v_{\psi(k)}) = g(v) \tag{1.3.2}$$

□

The following clearly holds.

Proposition 1.20. *Any finite linear subspace of a Hilbert space is closed.*

1.4 Topological group and representation

Definition 1.5 (Topological group)**.** *We call G is a topological group if G is a housdorff space and G is a group and $G \times G \ni (x, y) \mapsto xy \in G$ is continuous and $G \ni x \mapsto x^{-1} \in G$ is continuous.*

Proposition 1.21. *Let G is a topological group. Then the followings hold.*

(i) $i : G \ni x \mapsto x^{-1} \in G$ is isomorphism.

(ii) For any $g \in G$, $L_g : G \ni x \mapsto gx \in G$ is isomorphism.

(iii) For any $g \in G$, $R_g : G \ni x \mapsto xg \in G$ is isomorphism.

Proof of (i). For any open set U in G, $i(U) = i^{-1}(U)$. Because i is continuous, i is open map. So i is isomorhism. □

Proof of (ii). For any open set U in G, $L_g(U) = L_{(g^{-1})^{-1}}(U)$. Because $L_{g^{-1}}$ is continuous, L_g is open map. So L_g is isomorhism. □

Proof of (iii). It is possible to show (iii) by the approach which is similar to (ii). □

Proposition 1.22 (Semidirectproduct of groups)**.** *Let*

(i) G, H are groups.

(ii) $\sigma : G \to Aut(H)$ is a homomorphism of group.

(iii) We set

$$(g_1, h_1) \cdot (g_2, h_2) := (g_1 g_2, h_1 \sigma(g_1)(h_2)) \ (g_1, g_2 \in G, h_1, h_2 \in H)$$

Then $G \times H$ is a group with $\cdot$. We denote this group by $G \ltimes_\sigma H$.

Proof. Clearly $(1_G, 1_H)$ is the unit element of $G \ltimes_\sigma H$. Let us fix any $g_1, g_2, g_3 \in G$ and $h_1, h_2, h_3 \in H$.

$$\begin{aligned}
&(g_1, h_1) \cdot ((g_2, h_2) \cdot (g_3, h_3)) = (g_1, h_1) \cdot (g_2 g_3, h_2 \sigma(g_2)(h_3)) = (g_1 g_2 g_3, h_1 \sigma(g_1)(h_2 \sigma(g_2)(h_3))) \\
&= (g_1 g_2 g_3, h_1 \sigma(g_1)(h_2) \sigma(g_1)(\sigma(g_2)(h_3)))) = (g_1 g_2 g_3, h_1 \sigma(g_1)(h_2) \sigma(g_1 g_2)(h_3))) = (g_1 g_2, h_1 \sigma(g_1)(h_2))(g_3, h_3) \\
&= ((g_1, h_1) \cdot (g_2, h_2)) \cdot (g_3, h_3)
\end{aligned}$$

So, the associativity of $\cdot$ holds. For every $(g, h) \in G \ltimes_\sigma H$, $(g^{-1}, \sigma(g)(h)^{-1} h^{-1})$ is the inverse element of (g, h). Consequently, $G \ltimes_\sigma H$ is a group. □

Definition 1.6 (Representation of group)**.** *Let G be a group and V be a vector space on a field K. We call $\pi : G \to End_K(V)$ a representation of G if $\pi(1_G) = id_V$ and $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$ $(\forall g_1, g_2 \in G)$.*

Definition 1.7 (Continuous Representation of Group)**.** *Let G be a topological group and V be a Hilbert space on a field K. We call $\pi : G \to End_K(V)$ a continuous representation of G if (π, V) is a representation of G and $G \times V \ni (g, v) \mapsto \pi(g)v \in V$ is continuous.*

Definition 1.8 (Unitary Representation of Group)**.** *Let G be a group and V be a Hilbert space on a field K. We call $\pi : G \to End_K(V)$ a unitary representation of G if (π, V) is a representation of G and $\pi(g)$ is a unitary operator for any $g \in G$.*

Definition 1.9 (Subrepresentation)**.** *Let (π, V) be a continuous unitary representation of a topological group G and W be an invariant closed subspace of G. We call $(\pi|W, W)$ is a subrepresentation of π. We denote $\pi|W$ by π_1. We denote this by $\pi_1 < \pi$. And let (π_2, V_2) be a continuous unitary representation of a topological group G. We denote $\pi_2 \prec \pi$ if π_2 is isomorphic to a subrepresentation of G as continuous unitary representations.*

Proposition 1.23. *Let*

(S1) G is a topological group.

(S2) (π, V) is a finite dimensional continuous representations of G.

Then

$$G \ni g \mapsto \pi(g) \in GL(V)$$

is continuous.

Proof. Let us take $\{v_i\}_{i=1}^r$ such that $\{v_i\}_{i=1}^r$ is a orthonormal basis of V. For any $g_1, g_2 \in G$ and i, j

$$||(\pi(g_1)v_i, v_j) - (\pi(g_2)v_i, v_j)|| \leq ||\pi(g_1)v_i - \pi(g_2)v_i||$$

So, $(\pi(\cdot)v_i, v_j)$ is continuous. □

Proposition 1.24. *Let*

(S1) V is a vector space on $K := \mathbb{R}$ or $\mathbb{C}$.

(S2) $A \in End_K(V)$.

(S3) $A^*(f)(u) := f(Au)$ $(f \in V^*, u \in V)$.

Then $A^* \in End_K(V^*)$.

Proof. For any $a, b \in K$ and $f, g \in V^*$ and $u \in V$,

$$A^*(af + bg)(u) = (af + bg)(Au) = af(Au) + bg(Au) = a(A^*f)(u) + b(A^*g)(u) = (a(A^*f) + b(A^*g))(u)$$

□

Proposition 1.25 (Contragredient representation)**.** *Let*

(S1) G is a topological group.

(S2) (π, V) is a representations of G.

Then

(i) The following π^* is a homomorphism as groups.

$$\pi^* : G \ni g \mapsto \pi(g^{-1})^* \in GL_{\mathbb{C}}(V)$$

We call π^* a the contragredient representation of π.

(ii) If (π, V) is a finite dimensional continuous representations of G, then π^* is continuous.

Proof of (i). For any $g, h \in G$ and $f \in V^*$ and $u \in V$,

$$\pi^*(gh)f(u) = f(\pi(gh)^{-1}u) = f(\pi(h)^{-1}\pi(g)^{-1}u) = (\pi^*(h)f)(\pi(g)^{-1}u) = \pi^*(g)(\pi^*(h)f)(u)$$

□

Proof of (ii). Let us fix $\{v_1, .., v_m\}$ an orhonormal basis of V. We set $f_i := (\cdot, v_i)$ $(i = 1, 2, ..., m)$.

$$\pi(g)f(u) = f(\sum_{i=1}^{m} \pi(g^{-1}(u, v_i)v_i) = \sum_{i=1}^{m}(u, v_i)f(\pi(g^{-1})v_i) = \sum_{i=1}^{m} f(\pi(g^{-1})v_i)f_i(u)$$

So, π^* is continuous. □

Definition 1.10 (Intertwining operator, G-linear map.)**.** *Let*

(S1) G is a topological group.

(S2) (π_1, V_1) and (π_2, V_2) are representations of G.

We say $T : V_1 \to V_2$ is an intertwining operator or a G-linear map if T is a linear and

$$T \circ \pi_1 = \pi_2 \circ T$$

If π_1 and π_2 are continuous representations of G, we denote the set of all continous G-linear mapping from π_1 to π_2 by

$$Hom_G(V_1, V_2) \text{ or } Hom_G(\pi_1, \pi_2)$$

Definition 1.11 (Equivalent between two continuous representations of G)**.** *Let*

(S1) G is a topological group.

(S2) (π_1, V_1) and (π_2, V_2) are continuous representations of G.

We say π_1 and π_2 are equivalent if there is $T : V_1 \to V_2$ such that T is a bijective continuous G-linear and T^{-1} is a continuous G-linear.

Definition 1.12 (Equivalent between two unitary representations of G)**.** *Let*

(S1) G is a topological group.

(S2) (π_1, V_1) and (π_2, V_2) are unitary representations of G.

We say π_1 and π_2 are equivalent if there is $T : V_1 \to V_2$ such that T is a bijective unitary G-linear.

Definition 1.13 (G-linear map.)**.** *Let*

(S1) G is a topological group.

(S2) (π_1, V_1) and (π_2, V_2) are representations of G.

We say $T : V_1 \to V_2$ is an intertwining operator or a G-linear map if T is a linear and

$$T \circ \pi_1 = \pi_2 \circ T$$

The following is clear.

Proposition 1.26. *Let*

(S1) G is a topological group.

(S2) (π, V) is a continuous unitary representations of G.

(S2) W is a G-invariant subspace of V.

then $W^\perp$ is also a G-invariant subspace of V.

Definition 1.14 (Completely reducible)**.** *Let*

(S1) G is a topological group.

(S2) (π, V) is a continuous representations of G.

We say (π, V) is completely reducible if for any invariant subspace W_1 there is an invariant subspace W_2 such that $V = W_1 + W_2$.

Proposition 1.27. *Let*

(S1) G is a topological group.

(S2) (π, V) is a continuous unitary representations of G.

Then (π, V) is completely reducible.

Proof. Because of (S2), for any invarian subspace of W, $W^{\perp}$ is an invariant subspace. So, (π, V) is completely reducible. $\square$

By Proposition1.26, the following holds.

Proposition 1.28. *Let*

(S1) G is a topological group.

(S2) (π, V) is a finite dimensional continuous unitary representations of G.

then (π, V) has an irreducible decomposition.

Proposition 1.29 (Shur Lemma)**.** *Let*

(S1) G is a compact Lie group.

(S2) (π_i, V_i) is a continous irreducible representation of G on $\mathbb{C}$ $(i = 1, 2)$.

(A1) Either V_1 or V_2 is finite dimensional.

(S2)

Then

$$Hom_G(V_1, V_2) = \begin{cases} \{0\} & (\pi_1 \not\simeq \pi_2) \\ \mathbb{C}T & (\pi_1 \simeq \pi_2) \end{cases}$$

Here T is an G-isomorphism from V_1 to V_2.

STEP1. Proof of $Hom_G(V_1, V_2) = \{0\}$ $(\pi_1 \not\simeq \pi_2)$. Let us assume $Hom_G(V_1, V_2) \neq \{0\}$. There is $f \in Hom_G(V_1, V_2) \setminus \{0\}$. Because $Ker(f)$ is closed G-invariant, $Ker(f) = \{0\}$. Because of (A1), $Im(f)$ is finite dimensional. By Proposition1.20, $Im(f)$ is closed G-invariant subspace of V_2. Becuase π_2 is irreducible, $Im(f) = V_2$. So, V_2 is finite dimensional and f is bijective. Then V_1 is finite dimensional. By Proposition1.20, $f^{-1} \in Hom_G(V_2, V_2)$. So, f is an G-isomorphism from V_1 to V_2. $\square$

STEP2. Proof of $Hom_G(V_1, V_2) = \mathbb{C}T(\pi_1 \simeq \pi_2)$. Let us fix any $f \in Hom_G(V_1, V_2) \neq \{0\}$. By STEP1, f is an G-isomorphism from V_1 to V_2.

By (A1), V_1 and V_2 are finite dimensional. So, becuase $T \circ f$ has a eingenvalue λ, $Ker(T^{-1} \circ f - \lambda id) \neq \{0\}$. Because π_1 is irreducible, $Ker(T^{-1} \circ f - \lambda id) = V_1$. So, $f = \lambda T$. $\square$

Proposition 1.30. *Let*

(S1) G is a commutative topological group.

(S2) (π, V) is a continous finite dimensional irreducible representation of G on $\mathbb{C}$.

then $\dim \pi = 1$.

Proof. Let us fix $v, w \in V \setminus \{0\}$. Because π is irreducible, $\pi(G)v = V$. So, there is $g \in G$ such that $\pi(g)v = w$. Because G is commutative, $A : V \ni u \mapsto \pi(g)u \in V$ is continuous G-linear and $ImA \neq \{0\}$. So, by Shur Lemma, there is $\lambda \in \mathbb{C}$ such that $A = \lambda id_V$. So, $w = \lambda v$. $\square$

1.5 Homotopy and Fundamental group

Definition 1.15 (Path)**.** *Let*

(S1) X be a topological space.

We call each element of $C([0, 1], X)$ a path. For each $c \in C([0, 1], X)$, we call $c(0)$ the start point of c and $c(1)$ the end point of c. If $c(0) = c(1)$ then we call c a loop.

Definition 1.16 (Homotop of continuous maps)**.** *Let*

(S1) X, Y be a topological space.

(S2) $f, g \in C(X, Y)$.

We say f and g are homotop or homotopy equivalent if there is $\Phi \in C([0, 1] \times X, Y)$ such that $\Phi(0, \cdot) = f$ and $\Phi(1, \cdot) = g$.

Definition 1.17 (Homotopy equivalent of continuous maps)**.** *Let*

(S1) X, Y *be a topological space.*

(S2) $f, g \in C(X, Y)$*.*

We say f *and* g *are homotop or homotopy equivalent if there is* $\Phi \in C([0,1] \times X, Y)$ *such that* $\Phi(0, \cdot) = f$ *and* $\Phi(1, \cdot) = g$*. We call* Φ *a homotopy.*

Clearly, the following holds.

Proposition 1.31. *We succeed notations in Definition1.17. Homotop on* $C(X, Y)$ *is an equivalent relation on* $C(X, Y)$*.*

Definition 1.18 (Homotopy equivalent of topological spaces)**.** *Let*

(S1) X, Y *be a topological space.*

We say X *and* Y *are homotopy equivalent if there is* $\Phi \in C([0,1] \times X, Y)$ *such that* $\Phi(0, \cdot) = f$ *and* $\Phi(1, \cdot) = g$*. We call* Φ *a homotopy.*

Then, clearly, the followings hold.

Proposition 1.32 (Fundamental group)**.** *Let*

(S1) X *be a topological space.*

(S2) $x_0 \in X$*.*

(S3) Define

(i) Set

$$[([0,1], \partial I), (X, x_0)] := \{c \in C(I, X) | c(\partial I) \subset \{x_0\}\}$$

Here, $I := [0,1]$*.*

(ii) For each $c_1, c_2 \in [(I, \partial I), (X, x_0)]$*,*

$$c_1 \sim c_2$$

if there is a homotopy Φ *from* c_1 *to* c_2 *such that* $\Phi(t, \cdot) \in [(I, \partial I), (X, x_0)]$ $(\forall t \in I)$*.*

(iii) For each $c_1, c_2 \in [(I, \partial I), (X, x_0)]$*,*

$$c_2 \cdot c_1(t) = \begin{cases} c_1(2t) & (0 \leq t < \frac{1}{2}) \\ c_2(2t-1) & (\frac{1}{2} \leq t \leq 1) \end{cases}$$

(iii) Set

$$\pi_1(X, x_0) := [(I, \partial I), (X, x_0)] / \sim$$

(iv) For each $[c_1], [c_2] \in \pi_1(X, x_0)$

$$[c_2] \cdot [c_1] = [c_2 \cdot c_1]$$

Then $\sim$ *is a equivalent relation on* $[(I, \partial I), (X, x_0)]$ *and* $\cdot$ *on* $\pi_1(X, x_0)$ *is well-defined and* $\pi_1(X, x_0)$ *is a group with respect to* $\cdot$*. We call* $\pi_1(X, x_0)$ *the fundamental group of* X *with base point* x_0*. If* X *is path-connected and* $\pi_1(X, x_0) = \{e\}$*, we say* X *is simply connected.*

Proposition 1.33 (n-th Homotopy group)**.** *Let*

(S1) X *be a topological space.*

(S2) $x_0 \in X$*.*

(S3) $n \in \mathbb{N}$*.*

(S4) Define

(i) Set

$$[(I^n, \partial I^n), (X, x_0)] := \{c \in C(I^n, X) | c(\partial I^n) \subset \{x_0\}\}$$

Here, $I^n := [0,1]^n$*.*

(ii) For each $c_1, c_2 \in [(I^n, \partial I), (X, x_0)]$*,*

$$c_1 \sim c_2$$

if there is a homotopy Φ *from* c_1 *to* c_2 *such that* $\Phi(t, \cdot) \in [(I^n, \partial I), (X, x_0)]$ $(\forall t \in I)$*.*

(iii) For each $c_1, c_2 \in [(I := [0,1], \partial I), (X, x_0)]$,

$$c_2 \cdot c_1(t) = \begin{cases} c_1(2t_1, t_2, ..., t_n) & (0 \le t_1 < \frac{1}{2}) \\ c_2(2t_1 - 1, t_2, ..., t_n) & (\frac{1}{2} \le t_1 \le 1) \end{cases}$$

(iii) Set

$$\pi_n(X, x_0) := [(I^n, \partial I^n), (X, x_0)]/\sim$$

(iv) For each $[c_1], [c_2] \in \pi_n(X, x_0)$

$$[c_2] \cdot [c_1] = [c_2 \cdot c_1]$$

Then $\sim$ *is a equivalent relation on* $[(I^n, \partial I), (X, x_0)]$ *and* $\cdot$ *on* $\pi_n(X, x_0)$ *is well-defined and* $\pi_n(X, x_0)$ *is a group with respect to* $\cdot$*. We call* $\pi_n(X, x_0)$ *the* n*-th homotopy group of* X *with base point* x_0*.*

1.6 Fiber bundle

Definition 1.19 (Topological transformation group)**.** *Let* G *be a topological group. And let* Y *be a topological space. If* $\eta : G \times Y \to Y$ *satisfies the following conditions, we say* G *is a topological transformation group of* Y *respects to* η*.*

(i) $\eta(e, \cdot) = id_Y$.

(ii) $\eta(g_2, \eta(g_1, \cdot)) = \eta(g_2 g_1, \cdot)$ $(\forall g_1, g_2 \in G)$.

If is clear what η *is, we denote* $gy := \eta(g, y)$ *.*

Definition 1.20 (Effective topological transformation group)**.** *Let* G *be a topological transformation group of a topological space* Y *respects to* η*. We say that* G *is effective if* $\eta(g, \cdot) = id_Y$ *only if* $g = e$*.*

Definition 1.21 (Coordinate bundle)**.** *We call*

$$\mathfrak{B} := (B, X, Y, p, \{V_j\}_{j \in J}, \{\phi_j\}_{j \in J}, G)$$

a coordinate bundle if

(i) B, X, Y *are toplogical spaces.* B *is called a bundle space or total space.* X *is called a base space.* Y *is called a fibre.*

(ii) $p : B \to X$ *is a surjective and continuous map.* p *is called a projection.*

(iii) G *is a topological transformation group of* Y *respects to* η *and* G *is effective.*

(iii) $\{V_j\}_{j \in J}$ *is an open covering of* X*. We call each* V_j *a coordinate neighborhood.*

(iv) $\phi_j : V_j \times Y \to p^{-1}(V_j)$ *is an isomorphism. We call* $\phi_j^{-1} : p^{-1}(V_j) \to V_j \times Y$ *a local trivialization or a coordinate function. For each* $x \in V_j$*, we call* $Y_x := p^{-1}(x)$ *a fiber on* x*.*

(v) $p \circ \phi_j(x, y) = x$ $(\forall j \in J, \forall x \in V_j, \forall y \in Y)$

(vi) If $V_i \cap V_j \neq \phi$*, for each* $x \in V_i \cap V_j$*, we define* $\phi_{i,x} : Y \to Y$ *by*

$$\phi_{i,x}(y) := \phi_i(x, y)$$

Then there is the unique $g_{j,i}(x) \in G$ *such that*

$$\phi_{j,x}^{-1} \circ \phi_{i,x}(\cdot) = \eta(g_{j,i}(x), \cdot)$$

is an isomorphism.

(vii) $g_{j,i} : V_i \cap V_j \to G$ *is continuous.*

Memo 1.1. *I think, roughly speaking, a coordinate bundle is a pair* (B, X, Y, p) *with local trivializations* $(\{V_i\}_{i \in I}, \{\phi_i\}_{i \in I}$ *which induce a system of coordinate transformations* $\{g_{i,j}\}_{i,j \in I}$*. Steenrod Theorem, which is showed later, states a system of coordinate transformations induces a local trivializations.*

Definition 1.22 (Equivalent in the strict sense between two coordinate bundles)**.** *Let*

$$\mathfrak{B}_1 := (B_1, X_1, Y, p_1, \{V_{1,j}\}_{j \in J_1}, \{\phi_j\}_{j \in J_1}, G)$$

and

$$\mathfrak{B}_2 := (B_2, X_2, Y, p_2, \{V_{2,j}\}_{j \in J_2}, \{\phi_j\}_{j \in J_2}, G)$$

are coordinate bundles. We say that $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *are equivalent in the strict sense if* *(i)* $B_1 = B_2, X_1 = X_2, Y_1 = Y_2, G_1 = G_2$.

(ii) *Fix any* $j_1 \in J_1$ *and* $j_2 \in J_2$ *such that* $V_{1,j_1} \cap V_{2,j_2} \neq \phi$. *For any* $x \in V_{1,j_1} \cap V_{2,j_2}$, *there is unique* $g_{j_2,j_1}(x) \in G$ *such that*

$$g_{j_2,j_1}(x) = \phi_{2,x}^{-1} \circ \phi_{1,x}$$

and

$$g_{j_2,j_1} : V_{1,j_1} \cap V_{2,j_2} \to G$$

is continuous.

Proposition 1.34. *The relation in Definition1.22 is equivalent relation.*

Definition 1.23 (Fibre bundle)**.** *We define that a fibre bundle is a equivalent class by strict sense equivalent of coordinate bundles.*

Clearly the following holds.

Proposition 1.35. *Let*

(S1)

$$\mathfrak{B} := (B, X, Y, p, \{V_j\}_{j \in J}, \{\phi_j\}_{j \in J}, G)$$

(S2) X, Y, G *are* C^∞*-class manifolds.*

(A1) *Multiple operations and inverse operation of* G *are* C^∞*-class.*

(A2) *The action of* G *on* X *is* C^∞*-class.*

Then B *is a* C^∞*-class manifold. We call* B *a smooth corrdinate bundle.*

Definition 1.24 (Bundle map)**.** *Let*

$$\mathfrak{B}_1 := (B_1, X_1, Y, p_1, \{V_{1,j}\}_{j \in J_1}, \{\phi_{1,j}\}_{j \in J_1}, G)$$

and

$$\mathfrak{B}_2 := (B_2, X_2, Y, p_2, \{V_{2,j}\}_{j \in J_2}, \{\phi_{2,j}\}_{j \in J_2}, G)$$

are coordinate bundles. We call $(h, \bar{h})$ *a bundle map from* $\mathfrak{B}_1$ *to* $\mathfrak{B}_2$ *if*

(i) $h : B_1 \to B_2$ *is a continuous map.*

(ii) $\bar{h} : X_1 \to X_2$ *is a continuous map.*

(iii) *For each* $x \in X$, $x' := h(x)$ *and* $Y_x := p^{-1}(x)$ *and* $Y_{x'} := p^{-1}(x')$ *and* $h_x := h|Y_x$. *Then* $h_x : Y_x \to Y_{x'}$ *is an homeomorphism.*

(iv) *For any* $x \in V_{1,j} \cap \bar{h}^{-1}(V_{2,k})$, *there is unique* $\bar{g}_{k,j}(x) \in G$ *such that*

$$\phi_{2,\bar{h}(x)}^{-1} \circ h_x \circ \phi_{1,x} = \bar{g}_{k,j}(x) \cdot$$

(iv) $\bar{g}_{k,j} : V_{1,j} \cap \bar{h}^{-1}(V_{2,k}) \to G$ *is continuous. We call* $\bar{g}_{k,j}$ *a mapping transformation.*

We also call h *itself a bundle map and call* $\bar{h}$ *a map induced by* h *or call* $\bar{h}$ *the induced map from* h.

Proposition 1.36. *The followings hold.*

(i) *The identity map of any coordinate bundle is a bundle map.*

(ii) *The composition of any two bundle maps is a bundle map.*

Proof of (i). This is clear because of the definition of coordinate bundle. □

Proof of (ii). Let

$$\mathfrak{B}_i := (B_i, X_i, Y, p_i, \{V_{i,j}\}_{j \in J_i}, \{\phi_j\}_{j \in J_i}, G) \ (i = 1, 2, 3)$$

be corrdinate bundles and $(h_1, \bar{h_1})$ be a bundle map from $\mathfrak{B}_1$ to $\mathfrak{B}_2$ and $(h_2, \bar{h_2})$ be a bundle map from $\mathfrak{B}_2$ to $\mathfrak{B}_3$. We set $h_3 := h_2 \circ h_1$ and $\bar{h_3} := \bar{h_2} \circ \bar{h_1}$. Clearly, h_3 and $\bar{h_3}$ are continuous. For any $x \in X$, clearly,

$$h_{3,x} = h_{2,\bar{h_1}(x)} \circ h_{1,x}$$

So, $h_{3,x}$ is a homeomorphism from Y_x to $Y_{h_3(x)}$.

Let us fix any $x \in V_{1,j} \cap \bar{h_3}^{-1}(V_{3,k})$. Clearly

$$\bar{h_3}^{-1}(V_{3,k}) = \bar{h_1}^{-1}(\bar{h_2}^{-1}(V_{3,k}))$$

This implies

$$\bar{h_1}(x) \in \bar{h_2}^{-1}(V_{3,k})$$

Because $\{V_{2,j}\}_{j\in J_2}$ is an open covering of X, there is $j \in J_2$ such that

$$\bar{h}_1(x) \in V_{2,j}$$

So,

$$\begin{aligned} & \phi^{-1}_{3,\bar{h_3}(x)} \circ h_{3,x} \circ \phi_{1,x} \\ = \; & \phi^{-1}_{3,\bar{h_2}(\bar{h_1}(x))} \circ h_{2,x} \circ h_{2,x} \circ \phi_{1,x} \\ = \; & \phi^{-1}_{3,\bar{h_2}(\bar{h_1}(x))} \circ h_{2,x} \circ \phi_{2,\bar{h_1}(x)} \circ \phi^{-1}_{2,\bar{h_1}(x)} \circ h_{2,x} \circ \phi_{1,x} \\ = \; & \bar{g}_{2,k,j}(\bar{h}_1(x))\bar{g}_{1,j,i}(x) \end{aligned}$$

Clearly $\bar{g}_{2,k,j}(\bar{h}_1(\cdot))\bar{g}_{1,j,i}(\cdot)$ is continuous on $V_{1,j} \cap \bar{h_3}^{-1}(V_{3,k}) \cap \bar{h}_1^{-1}(V_{2,j})$. □

Definition 1.25 (Equivalent between two coordinate bundles)**.** *Let*

$$\mathfrak{B}_1 := (B_1, X_1, Y, p_1, \{V_{1,j}\}_{j\in J_1}, \{\phi_j\}_{j\in J_1}, G)$$

and

$$\mathfrak{B}_2 := (B_2, X_2, Y, p_2, \{V_{2,j}\}_{j\in J_2}, \{\phi_j\}_{j\in J_2}, G)$$

are coordinate bundles. We say that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are equivalent if there is h such that (h, id_X) is a bundle map from $\mathfrak{B}_1$ to $\mathfrak{B}_2$.

The following is clear from the definition of bundle map.

Proposition 1.37. *Let*

$$\mathfrak{B}_1 := (B_1, X_1, Y, p_1, \{V_{1,j}\}_{j\in J_1}, \{\phi_{1,j}\}_{j\in J_1}, G)$$

and

$$\mathfrak{B}_2 := (B_2, X_2, Y, p_2, \{V_{2,j}\}_{j\in J_2}, \{\phi_{2,j}\}_{j\in J_2}, G)$$

are coordinate bundles. And $(h, \bar{h})$ is a bundle map from $\mathfrak{B}_1$ to $\mathfrak{B}_2$. Then the followings hold.

$$\bar{g}_{j,i}(x)g_{i,k}(x) = \bar{g}_{j,k}(x) \ (\forall x \in V_{1,i} \cap V_{1,k} \cap \bar{h}^{-1}(V_{2,j})) \tag{1.6.1}$$

$$g_{j,i}(\bar{h}(x))\bar{g}_{i,k}(x) = \bar{g}_{j,k}(x) \ (\forall x \in V_{1,k} \cap \bar{h}^{-1}(V_{1,i} \cap V_{2,j})) \tag{1.6.2}$$

Lemma 1.1. *Let*

$$\mathfrak{B}_1 := (B_1, X_1, Y, p_1, \{V_{1,j}\}_{j\in J_1}, \{\phi_{1,j}\}_{j\in J_1}, G)$$

and

$$\mathfrak{B}_2 := (B_2, X_2, Y, p_2, \{V_{2,j}\}_{j\in J_2}, \{\phi_{2,j}\}_{j\in J_2}, G)$$

are coordinate bundles. And let us assume $\bar{h}$ is a continous map from X_1 to X_2. and there is $\{\bar{g}_{i,j}\}_{i,j\in J}$ such that for each $i, j \in J$ $\bar{g}_{i,j} \in C(V_j \cap \bar{h}^{-1}(V_i), G)$ and the followings hold.

$$\bar{g}_{j,i}(x)g_{i,k}(x) = \bar{g}_{j,k}(x) \ (\forall x \in V_{1,i} \cap V_{1,k} \cap \bar{h}^{-1}(V_{2,j}))$$

$$g_{j,i}(\bar{h}(x))\bar{g}_{i,k}(x) = \bar{g}_{j,k}(x) \ (\forall x \in V_{1,k} \cap \bar{h}^{-1}(V_{1,i} \cap V_{2,j}))$$

Then there is a bundle map h from $\mathfrak{B}_1$ to $\mathfrak{B}_2$ such that $\bar{h}$ is the induced map from h and for each $i, j \in J$ $\bar{g}_{i,j}$ is a mapping transformations of h.

Proof. For each $i \in J_1$ and $j \in J_2$ such that $(V_{1,i} \cap \bar{h}^{-1}(V_{2,j})) \times Y \neq \phi$, we set

$$h(\phi_{1,i}(x,y)) = \phi_{2,j}(\bar{h}(x), \bar{g}_{j,i}(x)y) \ ((x,y) \in (V_{1,i} \cap \bar{h}^{-1}(V_{2,j})) \times Y)$$

We will show h is well-defined. Let us assume $(x,y) \in (V_{1,i} \cap \bar{h}^{-1}(V_{2,j})) \times Y$ and $(x',y') \in (V_{1,i'} \cap \bar{h}^{-1}(V_{2,j'})) \times Y$ and

$$\phi_{1,i}(x,y) = \phi_{1,i'}(x',y')$$

Then

$$x = p \circ \phi_{1,i}(x,y) = p \circ \phi_{1,i'}(x',y') = x'$$

So, $\phi_{1,i}(x,y) = \phi_{1,i'}(x,y')$. This implies

$$g_{i',i}(x)y = y'$$

So,

$$\bar{g}_{j,i}(x)y = \bar{g}_{j,i}(x)g_{i,i'}(x)y' = \bar{g}_{j,i'}(x)y'$$

So,

$$\begin{aligned} & \phi_{2,j}(\bar{h}(x), \bar{g}_{j,i}(x)y) = \phi_{2,j,\bar{h}(x)}(\bar{g}_{j,i}(x)y) = \phi_{2,j',\bar{h}(x)} \circ \phi^{-1}_{2,j',\bar{h}(x)} \circ \phi_{2,j,\bar{h}(x)}(\bar{g}_{j,i'}(x)y') = \phi_{2,j',\bar{h}(x)}(g_{j',j}(\bar{h}(x))\bar{g}_{j,i'}(x)y') \\ = \ & \phi_{2,j',\bar{h}(x)}(\bar{g}_{j',i'}(x)y') = \phi_{2,j'}(\bar{h}(x), \bar{g}_{j',i'}(x)y') \end{aligned}$$

Consequently, h is well-defined. Clearly, h is continuous. Also, clearly, for any $x \in V_{1,i} \cap \bar{h}^{-1}(V_{2,j})$, $h|Y_x$ is an homeomorphism from Y_x to $Y_{\bar{h}(x)}$ and

$$\phi^{-1}_{2,j,\bar{h}(x)} \circ h \circ \phi_{1,i,x} = \bar{g}_{j,i}(x)$$

□

Lemma 1.2. *The followings are the settings and assumptions.*

(S1)

$$\mathfrak{B}_1 := (B_1, X_1, Y, p_1, \{V_{1,j}\}_{j \in J_1}, \{\phi_{1,j}\}_{j \in J_1}, G)$$

and

$$\mathfrak{B}_2 := (B_2, X_2, Y, p_2, \{V_{2,j}\}_{j \in J_2}, \{\phi_{2,j}\}_{j \in J_2}, G)$$

are coordinate bundles.

(A1) $X_1 = X_2$.

(A2) There are $\bar{g}_{k,j} : V_j \cap V'_k \to G$*:continuous map(*$j \in J_1,\ k \in J_2$*) such that*

$$\bar{g}_{k,j}(x)g_{j,i}(x) = \bar{g}_{k,i}(x) \ (\forall x \in V_i \cap V_j \cap V'_k), g'_{l,k}(x)\bar{g}_{k,j}(x) = \bar{g}'_{l,j}(x) \ (\forall x \in V_i \cap V'_l \cap V'_k)$$

Then $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *are equivalent.*

Proof. It is from Proposition1.1. □

Lemma 1.3. *The followings are the settings and assumptions.*

(S1)

$$\mathfrak{B}_1 := (B_1, X_1, Y, p_1, \{V_j\}_{j \in J}, \{\phi_j\}_{j \in J_1}, G)$$

and

$$\mathfrak{B}_2 := (B_2, X_2, Y, p_2, \{V_j\}_{j \in J}, \{\phi_j\}_{j \in J_2}, G)$$

are coordinate bundles.

(A1) $X_1 = X_2$.

(A2) There are $\lambda_j : V_j \to G$*:continuous map(*$j \in J$*) such that*

$$g'_{i,j}(x) = \lambda_i(x)^{-1}g_{i,j}(x)\lambda_j(x) \ (\forall x \in V_i \cap V_j)$$

Then $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *are equivalent.*

Proof. We set

$$\bar{g}_{i,j}(x) := \lambda_i(x)^{-1} g_{i,j}(x) \ (x \in V_i \cap V_j)$$

Then

$$\bar{g}_{i,j}(x) g_{j,k}(x) = \lambda_i(x)^{-1} g_{i,j}(x) g_{j,k}(x) = \lambda_i(x)^{-1} g_{i,k}(x) = \bar{g}_{i,k}(x)$$

and

$$g'_{k,i}(x)\bar{g}_{i,j}(x) = \lambda_k(x)^{-1} g_{k,i}(x)\lambda_i(x)\lambda_i(x)^{-1} g_{i,j}(x) = \lambda_k(x)^{-1} g_{k,i}(x) g_{i,j}(x) = \lambda_k(x)^{-1} g_{k,j}(x) = \bar{g}_{k,j}(x)$$

So, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are equivalent from Lemma1.2. □

Definition 1.26 (System of coordinate transformations). *Let*

(S1) G is a topological group.

(S2) X is a topological space.

We call $(\{V_j\}_{j \in J}, \{g_{i,j}\}_{i \in J})$ a system of coordinate transformations in X with values in G if

(i) $\{V_j\}_{j \in J}$ is an open covering of X.

(ii) $g_{j,i} \in C(V_j \cap V_i, G)$ $(\forall i, j \in J)$.

(iii) $g_{k,j} \circ g_{j,i} = g_{k,i}$ in $V_k \cap V_j \cap V_i$ $(\forall i, j, k \in J)$.

Clearly the following holds.

Proposition 1.38. *Let*

(S1) G is a topological group.

(S2) X is a topological space.

(S3) $(\{V_j\}_{j \in J}, \{g_{i,j}\}_{i \in J})$ is a a system of coordinate transformations in X with values in G.

Then the followings hold.

(i) $g_{i,i} = e$ $(\forall i \in J)$.

(ii) $g_{i,j} = g_{j,i}^{-1}$ $(\forall i, j \in J)$.

Theorem 1.2 (Steenrod's theorem). *Let*

(S1) G is a topological group.

(S2) X is a topological space.

(S3) $(\{V_j\}_{j \in J}, \{g_{i,j}\}_{i \in J})$ is a system of coordinate transformations in X with values in G.

(S4) Y is a topological space.

(S5) G is a topological transformation group of Y.

(A1) The action of G on Y is effective.

Then

(i) There is $B, p, \{\phi_j\}_{j \in J}$ such that $(B, X, p, \{V_j\}_{j \in J}, Y, \{\phi_j\}_{j \in J})$ is a coordinate bundle and for any $j, i \in J$ such that $V_i \cap V_j \neq \phi$, for any $x \in V_i \cap V_j$, in $V_i \cap V_j$,

$$\phi_{j,x}^{-1} \circ \phi_{i,x} = g_{j,i}$$

(ii) If B_1 and B_2 are topological spaces which individually satisfy (i), $(B_1, X, p, \{V_j\}_{j \in J}, Y, \{\phi_j^1\}_{j \in J})$ and $(B_2, X, p, \{V_j\}_{j \in J}, Y, \{\phi_j^2\}_{j \in J})$ are equivalent.

STEP1. Construction of B and $\{\phi_j\}_{j \in J}$. Hereafter, let us assume the topology of J is the discrete topology. We set

$$T := X \times Y \times J$$

We define the relation of T by

$$(x, y, j) \sim (x', y', k) :\Longleftrightarrow x = x' \text{ and } y' = g_{k,j}(x) y$$

We will show $\sim$ is a equivalent relation of T. Because $g_{i,i} = e$, the reflexivity of $\sim$ holds. Because $g_{i,i} = e$, by (S5), the reflexivity of $\sim$ holds. Because $g_{i,j} = g_{j,i}^{-1}$, by (S5), the symmetry of $\sim$ holds. Because $g_{k,j} \circ g_{j,i} = g_{k,i}$, by (S5), the transitivity of $\sim$ holds. So $\sim$ is a equivalent relation.

We set

$$B := T/\sim$$

and

$$q : T \ni (x, y, j) \mapsto [x, y, j] \in B$$

and

$$p : B \ni [x, y, j] \mapsto x \in X$$

By the definition of $\sim$, p is well-defined. And, clearly, p is surjective. Let us assume that the topology of B is the final topology of B induced by q. For any $O \in \mathcal{O}(X)$,

$$q^{-1}(p^{-1}(O)) = O \times Y \times \{j \in J | V_j \cap O \neq \phi\}$$

In this equation, the right side is an open set of T. So, p is continuous.

We define $\phi_j : V_j \times Y \to B$ by

$$\phi_j(x, y) = [x, y, j]$$

Clearly, ϕ_j is continuous and

$$\phi_j : V_j \times Y \subset B$$

and

$$p \circ \phi_j = id_{V_j}$$

□

STEP2. Proof of that ϕ_j is an isomorphism. By STEP1, it is enough to show that ϕ_j is bijective and an open map. We will show that $\phi_j : V_j \times Y \to p^{-1}(V_j)$ is surjective. Let us fix any $[x, y, k] \in p^{-1}(V_j)$. Clealy $x \in V_k$ and

$$(x, y, k) \sim (x, g_{j,k}(x)y, j)$$

So,

$$[x, y, k] = \phi_j(x, g_{j,k}(x)y)$$

So ϕ_j is surjective.

Nextly, we will show that ϕ_j is injective. Let us fix any $(x, y), (x', y') \in V_j \times Y$ such that $[x, y, j] = [x', y', j]$. Then $x = x'$ and

$$g_{j,j}(x)y = y'$$

Because $g_{j,j}(x) = id_{V_j}$, $y = y'$. So ϕ_j is injective.

Lastly, we will show that ϕ_j is an open map. Let us fix $W_1 \times W_2 \subset V_j \times Y$ which is an open set. For any $k \in J$ such that $V_k \cap V_j \neq \phi$, we set $r_{j,k} : (V_k \cap V_j) \times Y \to (V_k \cap V_j) \times Y$ by

$$r_{j,k}(x, y) := (x, g_{j,k}(x)y)$$

By (S5), $r_{j,k}$ is continuous.

We will show for any $W \in \mathcal{O}(V_j \times Y)$,

$$q^{-1}(\phi_j(W)) = \bigcup_{k \in J, V_k \cap V_j \neq \phi} r_{j,k}^{-1}(W) \times \{k\} \tag{1.6.3}$$

Let us fix any $(x, y) \in (V_j \cap V_k) \times Y$ such that $r_{j,k}(x, y) \in W$. Because

$$\phi_j(x, g_{j,k}(x)y) = [r_{j,k}(x), j] = q(x, y, k) \tag{1.6.4}$$

in (1.6.3), the right side is containd the left side. By (1.6.4), it is clear that in (1.6.3), the left side is containd the right side. So, (1.6.3) holds. Clearly, in (1.6.3), the right side is an open set. So, ϕ_j is an open map. □

STEP3. Proof of (i). By STEP1 and STEP2, it is enough to show that for any $i, j \in J$ such that $V_i \cap V_j \neq \phi$ and any $x \in V_i \cap V_j \neq \phi$

$$\phi_{j,x}^{-1} \circ \phi_{i,x} = g_{j,i} \tag{1.6.5}$$

For any $y \in Y$

$$\begin{aligned} & \phi_{j,x}^{-1} \circ \phi_{i,x}(y) \\ = \ & \phi_{j,x}^{-1}([x, y, i]) \\ = \ & \phi_{j,x}^{-1}([x, g_{j,i}(x)y, j]) \\ = \ & g_{j,i}(x)y \end{aligned}$$

So (1.6.5) holds. □

STEP3. Proof of (ii).

$$\phi_{1,j,x}^{-1} \circ \phi^{1,i,x} = g_{j,i}(x) = \phi_{2,j,x}^{-1} \circ \phi_{2,i,x} \ (\forall i, j, \forall x \in V_i \cap V_j)$$

When we set $\lambda_i(x) := e \ (\forall i, \forall x \in V_I)$, $\{\lambda_i\}_i$ satisfies the conditions of Lemma1.3. So, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are equivalent. □

Proposition 1.39 (Tangent bundle)**.** *The following are settings and assumptions.*

- *(S1)* $\{M, \{(U_i, \psi_i)\}_{i \in I}\}$ *is a n-dimensional* C^∞*-class manifold.*
- *(S2)* $B := \cup_{x \in M}\{x\} \times T_x(M))$.
- *(S3)* $p : B \ni (x, X) \mapsto x \in M$.
- *(S4)* $Y := \mathbb{R}^n$.
- *(S5)* $\phi_i : U_i \times Y \ni (x, v) \mapsto (x, \sum_{j=1}^n v_j \left(\frac{\partial}{\partial \psi_i^j}\right)_x) \in B$.

Then $\{B, p, M, \mathbb{R}^n, \{(U_i, \phi_i)\}_{i \in I}, GL(n, \mathbb{R})\}$ *is a coordinate bundle. We call the fibre bundle of the coordinate bundle tangent bundle of* M*.*

Proof. Clearly,

$$p \circ \phi_i(x, v) = x \ (\forall i \in I, \forall x \in U_i, v \in Y)$$

and

$$\phi_i(U_i \times Y) = p^{-1}(U_i)$$

and ϕ_i is injective and ϕ_i is C^∞-class and ϕ_i^{-1} is C^∞-class. So, ϕ_i is a local trivialization. And

$$\phi_{j,x}^{-1} \circ \phi_{i,x}(v) = J(\phi_{j,x}^{-1} \circ \phi_{i,x})(v)$$

and

$$U_i \cap U_j \ni x \mapsto J(\phi_{j,x}^{-1} \circ \phi_{i,x}) \in GL(n, \mathbb{R})$$

is C^∞-class. So, $\{J(\phi_{j,x}^{-1} \circ \phi_{i,x})\}_{x \in U_i \cap U_j}$ is a system of coordinate transformations. Consequently,

$$\{B, p, M, \mathbb{R}^n, \{(U_i, \phi_i)\}_{i \in I}, GL(n, \mathbb{R})\}$$

is a coordinate bundle. □

Definition 1.27 (Cross section)**.** *Let*

$$\mathfrak{B} := (B, X, Y, p, \{V_j\}_{j \in J}, \{\phi_j\}_{j \in J}, G)$$

is a coordinate bundle. We say $s : X \to B$ *is a cross-section if* s *is continuous and* $p \circ s = id|X$*.*

Definition 1.28 (Vector Bundle)**.** *Let*

$$\mathfrak{B} := (B, X, Y, p, G)$$

be a fibre bundle. We say $\mathfrak{B}$ *is a vector bundle if* $Y = \mathbb{R}^n$ *and* $G = GL(n, \mathbb{R})$ *and* G *acts on* Y *with* $g \cdot v = gv \ (g \in G, v \in Y)$*.*

Definition 1.29 (Principal Bundle)**.** *Let*

$$\mathfrak{B} := (B, X, Y, p, G)$$

be a fibre bundle. We say $\mathfrak{B}$ *is a principal bundle if* $Y = G$ *and* G *acts on* Y *with* $g \cdot h = gv \ (g, h \in G)$ *in* $\mathfrak{B}$*.*

2 Lie group and Lie algebra

2.1 Lie group

Definition 2.1 (Locally isomorphism between two topological groups)**.** *Let G and H are topological groups. We say G and H are locally isomorphic if there is $U \subset G$ and $V \subset H$ and isomoriphism $i : U \to G$ such that U is a neighborhood of 1_G and V is a neighborhood of 1_H and the followings hold.*

(i) For any $x, y \in U$ such that $xy \in U$, $i(xy) = i(x)i(y)$.

(ii) For any $x, y \in U$, $xy \in U \iff i(x)i(y) \in V$.

Example 2.1. *$\mathbb{R}$ and $\mathbb{T}$ are locally isomorphic.*

Definition 2.2 (Lie subgroup of $GL(n, \mathbb{C})$)**.** *We say G is a Lie subgroup of $GL(n, \mathbb{C})$ if the followings hold.*

(i) G is a subgroup of $GL(n, \mathbb{C})$

(ii) G is a topological group

(iii) There is a neighborhood of e in G V such that

- *(iii-1) The topology of V is relative topology of $GL(n, \mathbb{C})$*
- *(iii-2) There is a neighborhood of e in $GL(n, \mathbb{C})$ U such that if $x_j \in V$ $(j \in \mathbb{N})$ and $x_j \to x \in U$ then $x \in V$.*
- *(iii-3) G has at most countable connected components.*

Proposition 2.1. *Let*

(S1) G is a subgroup of $GL(n, \mathbb{C})$.

(A1) G is a topological group.

(A2) G has at most countable connected components.

Then the followings are hold.

(i) G is a Lie subgroup of $GL(n, \mathbb{C})$

(ii) There is V which is a neighborhood of 1_G and is a closed subset of $GL(n, \mathbb{C})$ and the topology of V is relative topology of $GL(n, \mathbb{C})$

Proof of that (ii) $\implies$ (i). We set $U := G$. V and U satisfies the condition (iii) in Definition2.2. ☐

Proof of that (i) $\implies$ (ii). By the condition (iii-1) in Definition2.2, there is W such that W is an open subset of $GL(n, \mathbb{C})$ and $V^\circ = V \cap W$. Clearly W is an open neighborhood of $1_{GL(n,\mathbb{C})}$. There is W_0 such that W_0 is an open subset of $GL(n, \mathbb{C})$ and $1_G \in W_0 \subset \bar{W}_0 \subset U \cap W$. We set $V' := \bar{W}_0 \cap V$. By the condition (iii-1) in Definition2.2, there is Z such that Z is an open subset of G and $V \cap W_0 = V \cap Z$. So $V' = \bar{W}_0 \cap V$ is a neighborhood of 1_G in G. Because $\bar{W}_0 \subset U$, by the condition (iii-2) in Definition2.2, V' is closed subset of $GL(n, \mathbb{C})$. ☐

Proposition 2.2. *Let*

(S1) G is a Lie subgroup of $GL(n, \mathbb{C})$.

Then, for any W which is a neighborhood of 1_G in G, there is V' such that V' is a closed subset of $GL(n, \mathbb{C})$ and V' is a neighborhood of 1_G.

Proof. There is $\epsilon > 0$ such that $B(1_G, 4\epsilon) \cap V \subset W \cap V$. Because $V \subset G$,

$$\overline{B(1_G, 2\epsilon)} \cap V \subset B(1_G, 4\epsilon) \cap V \subset W$$

Clearly $\overline{B(1_G, 2\epsilon)} \cap V$ is a closed subset of $GL(n, \mathbb{C})$.

There is Z such that Z is an open subset of G and $1 \in Z$ and $Z \subset V$. By Proposition1.6, $Z \cap B(1_G, \epsilon)$ is an open subset of Z. So, there is open subset of G O such that $Z \cap B(1_G, \epsilon) = Z \cap O$. So $Z \cap O$ is an open subset of G and $1 \in Z \cap O \subset \overline{B(1_G, 2\epsilon)} \cap V$. So, $\overline{B(1_G, 2\epsilon)} \cap V$ is a neighborhood of 1_G. By Proposition1.6, The topology of $\overline{B(1_G, 2\epsilon)} \cap V$ is the relative topology to $GL(n, \mathbb{C})$. ☐

Example 2.2. *Let λ be a irrelational number. Let $G := exp(i2\pi\lambda\mathbb{Z}) \subset GL(1,\mathbb{C})$. Le us assume G is a topological group respects to the discrete topology. $V := \{1\}$ is a neighborhood of 1 on G and V is a closed subset of $GL(1,\mathbb{C})$. So, G is a Lie subgroup of $GL(1,\mathbb{C})$. Because $\mathbb{T}$ is compact, there is subsequence $\{exp(i2\pi\lambda\varphi(m))\}_{m=1}^{\infty}$ and $x \in \mathbb{T}$ such that*

$$\lim_{m\to\infty} exp(i2\pi\lambda\varphi(m)) = x$$

Because λ is irrelational, $x \notin G$. So, G is not closed subset of $GL(1,\mathbb{C})$.

Definition 2.3 (Linear Lie group of $GL(n,\mathbb{C})$)**.** *We call $G \in GL(n,\mathbb{C})$ is a Linear Lie group of $GL(n,\mathbb{C})$ if G is closed subgroup of $GL(n,\mathbb{C})$*

Proposition 2.3. *If $G \in GL(n,\mathbb{C})$ is a Linear Lie group of $GL(n,\mathbb{C})$ then G is a Lie subgroup of $GL(n,\mathbb{C})$*

Proof. Clearly G satisfies Definition2.2. Because $GL(n,\mathbb{C})$ satisfies the second countable axiom, G satisfies the second countable axiom. So G has at most countable connected components. □

Definition 2.4 (General Lie group)**.** *We say G is a Lie group if G is a topological group such that there is a Lie subgroup of $GL(n,\mathbb{C})$ which is locally isomorphic to G.*

Proposition 2.4. *Let*

- *(S1) G_1 is a Lie group which is isomorphic to a Lie subgroup G_2 of $GL(n,\mathbb{C})$.*
- *(S2) V which is a neighborhood of 1_{G_2} in G_2 and U which is a neighborhood of 1_{G_1} in G_1 and isomorphism $i : U \to V$ satisfying the conditions in Definition2.1..*
- *(S3) $U' \subset U$ and $V' := i(U')$.*

Then $i|U'$ satisfying the conditions in Definition2.1.

Proof of condition(i). It is trivial. □

Proof of condition(ii). Let us fix any $x, y \in U'$. Let us assume $xy \in U'$. Then by condition(i), $i(x)i(y) = i(xy) \in V'$. Let us assume $i(x)i(y) \in U'$. Then $xy \in U$. $i(xy) = i(x)i(y) \in U'$. So $xy \in V'$. □

Proposition 2.5. *Let*

- *(S1) G_1 is a Lie group which is isomorphic to a Lie subgroup G_2 of $GL(n,\mathbb{C})$.*

Then there is $V := G \cap B(1_{G_2}, \epsilon)$ for some $\epsilon > 0$ which is a compact neighborhood of 1_{G_2} in G_2 and U which is a compact neighborhood of 1_{G_1} in G_1 and isomorphism $\tau : U \to V$ satisfying the conditions in Definition2.1.

Proof. Let us fix U and V and $\tau : U \to V$ such that U is a neighborhood of 1_{G_1} and V is a neighborhood of 1_{G_2} and $\tau : U \to V$ is isomorphism satisfying the conditions in Definition2.1. There is a open set B_1 of $GL(n,\mathbb{C})$ such that $V^\circ = G_2 \cap B_1$. There is $\epsilon > 0$ such that $B(1_{G_2}, 2\epsilon) \subset B_1$. We set $V_2 := \overline{B(1_{G_2}, \epsilon)} \cap G_2$ and $U_1 := \tau^{-1}(V_2)$. Because $\tau^{-1}(G \cap B(1_{G_2}, \epsilon)$ is open set in the relative topology with G and subset of U_1, U_1 is the neighborhood of 1_{G_1}. We set $\eta := \tau^{-1}$. Because $G_2 \cap \overline{B(1_{G_2}, \epsilon)} \subset G_2 \cap B_1 \subset V$, $V_2 = V \cap \overline{B(1_{G_2}, \epsilon)}$. So V_2 is a closed subset of V and U_1 is a closed subset of U.

By Proposition1.7 and Propositon1.6, $\tau|U_1$ is homeomorphism. So U_1 is compact. Also, by Proposition2.4.1, $\tau|U_1$ satisfies conditions in Definition2.1. □

In this note, unless otherwise stated, U and V are assumed to be the neighborhoods obtained in Proposition2.5.

Proposition 2.6. *Let*

- *(S1) G_1 is a Lie group which is isomorphic to a Lie subgroup G_2 of $GL(n,\mathbb{C})$.*
- *(S2) V which is a neighborhood of 1_{G_2} in G_2 and U which is a neighborhood of 1_{G_1} in G_1 and isomorphism $i : U \to V$ satisfying the conditions in Definition2.1..*

Then $j := i^{-1}$ satisfying the conditions in Definition2.1.

Proof of condition(i). Let us fix any $z, w \in V$. Let us assume $zw \in V$. Then $i(j(z))i(j(w)) \in V$. So $j(z)j(w) \in U$. By condition(i), $i(j(z)j(w)) = i(j(z))i(j(w)) = zw$. So $j(z)j(w) = j(zw)$. □

Proof of condition(ii). Let us fix any $z, w \in V$. Let us assume $zw \in V$. By the proof of condition(i), $j(z)j(w) \in U$.

Inversely, let us assume $j(z)j(w) \in U$. Then by condition(ii), $zw = i(j(z))i(j(w) \in V$. □

2.2 Matrix exponential

Definition 2.5 (Operator Norm). *For $X \in M(n,\mathbb{C})$,*

$$||X||_{op} := ||X|| := \sup_{||v||=1,\ v\in\mathbb{C}^n} |Xv|$$

Definition 2.6. *For $X \in M(n,\mathbb{C})$,*

$$||X||_\infty := \sup\{|x_{i,j}||i,j \in \{1,2,...,n\}\}$$

Proposition 2.7. *For $X \in M(n,\mathbb{C})$,*

$$||X||_\infty \leq ||X||_{op} \leq \sqrt{n}||X||_\infty$$

Proof of $||X||_\infty \leq ||X||_{op}$. For any $i,j \in \{1,2,...,n\}$, $|x_{i,j}| \leq |Xe_j| \leq ||X||$. □

Proof of $||X||_{op} \leq \sqrt{n}||X||_\infty$. We set $x_i := (x_{i,j})_{j=1}^n$ for each i. For any $u \in \mathbb{C}^n$ such that $|u| = 1$, by Schwartz's inequality,

$$|Xu| \leq |((x_1|u),...,(x_n|u))| \leq \sqrt{n} \sup_{i=1,2,...,n} |x_i| \leq \sqrt{n}||X||_\infty$$

□

Proposition2.7 implies the following.

Proposition 2.8. *$M(n,\mathbb{C})$ is banach space with the operator norm.*

Proposition 2.9. *Let*

(S1) $X \in M(n,\mathbb{C})$

Then for any eigenvalue λ of X

$$|\lambda| \leq ||X||$$

Proposition 2.10. *Let*

(S1) $M := \{X \in M(n,\mathbb{C})|\ X$ is diagonalizable $\}$

Then M is dense in $M(n,\mathbb{C})$

Proof. Because M is triangularisable(See [12]), there is $P \in GL(n,\mathbb{C})$ such that

$$P^{-1}MP := \begin{pmatrix} \alpha_1 & & * \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}$$

We set for each $0 \leq s << 1$

$$E(s) := \begin{pmatrix} s & & 0 \\ & \ddots & \\ 0 & & s^n \end{pmatrix}$$

Because $P^{-1}MP+E(s)$ has not a duplicate eigenvalue, so $P^{-1}MP+E(s)$ is diagonalizable. So $M(s) := M+PE(s)P^{-1}$ is diagonalizable. $\lim_{s\to 0} M(s) = M$. □

Proposition 2.11. *(S1) $X \in M(n,\mathbb{C})$*

(S2) f is a power series whose radius of convergence is not less than $R > 0$.
then

(i) If $||X|| < R$ then $f(X)$ exists.

(ii) $f(X)$ is a horomoriphic function for each variable $x_{i,j}$.

Proof of (i). We set $f(x) =: \sum_{i=1}^\infty c_i X^i$. By the definition of the radius of convergence,

$$\sum_{i=1}^\infty |c_i|||X||^i < \infty$$

This implies that $\{\sum_{i=1}^n c_i X_i\}_{n=1}^\infty$ is a cauchy sequence. By Proposition2.8, $f(X)$ exists. □

Proof of (ii). We set $f_n(X) := \Sigma_{i=1}^n c_i X^i$ for each $n \in \mathbb{N}$. By Proposition2.7, for any $K \in (0, R)$, $\{X \in M(n,\mathbb{C}) | ||X|| \leq K\}$ is compact. And,

$$\begin{aligned} & \sup_{||X|| \leq K} ||f_n(X) - f(X)|| \\ = & \sup_{||X|| \leq K} || \sum_{i=n+1}^{\infty} c_i X^i || \\ = & \sum_{i=n+1}^{\infty} |c_i| K^i \\ \to & \ 0 \ (n \to \infty) \end{aligned} \tag{2.2.1}$$

So $\{f_n\}_{n=1}^{\infty}$ uniformly converges to f on compact sets. By Weierstrass's theorem(See [6]), bhis implies that f is homomorphic. □

Proposition 2.12. *Let*

- *(S1)* $X \in M(n,\mathbb{C})$
- *(S2)* f, h *are power series whose radius of convergence is not less than* $R > 0$.
- *(S3)* u *is a power series whose radius of convergence is not less than* $R' > 0$.
- *(A1)* $||X|| < R$.

then the followings hold

- *(i) If* $u = f + h$ *and* $R = R'$ *then* $u(X) = f(X) + h(X)$.
- *(ii) If* $u = fh$ *and* $R = R'$ *then* $u(X) = f(X)h(X)$.
- *(iii) If* $||f(X)|| < R'$ *then* $u \circ f(X) = u(f(X))$.

Proof. By Proposition2.9, clearly these Propositions hold in M.

By Proposition2.11, $u, f + h, fh, u \circ f, u(f(\cdot))$ are continuous on $M(n,\mathbb{C})$. So, by Proposition2.10, these Propositions hold at X. □

Proposition 2.13. *For any* $X \in M(n,\mathbb{C})$

$$det(exp(X)) = exp(tr(X)) \tag{2.2.2}$$

Proof. Because $det(exp(\cdot))$ and $exp(tr(\cdot))$ are continuous, by Proposition2.10, it is enough to show (2.2.2) for any $X \in M(n,\mathbb{C})$ such that X is diagonizable. Let us fix $X \in M(n,\mathbb{C})$ such that X is diagonizable. There is $P \in GL(n,\mathbb{C})$ such that $PXP^{-1} = \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n \end{pmatrix}$. And $exp(PXP^{-1}) = \begin{pmatrix} exp(\lambda_1) & 0 & ... & 0 \\ 0 & exp(\lambda_2) & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & exp(\lambda_n) \end{pmatrix}$ So

$$\begin{aligned} det(exp(X)) &= det(Pexp(X)P^{-1}) \\ &= det(exp(PXP^{-1})) \\ &= exp(\lambda_1)exp(\lambda_2)...exp(\lambda_n) \\ &= exp(\sum_{i=1}^{n} \lambda_i) \\ &= exp(tr(PXP^{-1})) \\ &= exp(tr(X)) \end{aligned} \tag{2.2.3}$$

□

Proposition 2.14 (Exponential and Logarithm of matrix)**.** *Let*

- *(S1)* $log(X) := \Sigma_{i=1}^{\infty} \frac{(-1)^{i-1}(X - E_n)^i}{i!}$ *for* $X \in M(n,\mathbb{C})$ *such that* $||X|| < 1$.

then

- *(i)* $exp(log(X)) = X$ *for any* $X \in M(n,\mathbb{C})$ *such that* $||X|| < 1$.

(ii) $log(exp(X)) = X$ for any $X \in M(n,\mathbb{C})$ such that $||X|| < 1$ such that $||X|| < log2$.

Proof. By (iii) of Proposition, (i) and (ii) hold. □

The following Proposition says exponential map is locally isomorphism.

Proposition 2.15.

(i) $exp(\cdot)$ is C^∞ isomorphism to some open set in some neighborhood of O.

(ii) $log(E+\cdot)$ is C^∞ isomorphism to some open set in some neighborhood of E.

Proof. See the corollary of inverse mapping theorem in [13] □

Proposition 2.16 (Basic properties about Exponential of matrix)**.**

(i) $exp(X+Y) = exp(X)exp(Y)$ for any $X, Y \in M(n,\mathbb{C})$ such that $XY = YX$.

(ii) $exp(X)^m = exp(mX)$ for any $X \in M(n,\mathbb{C})$ and $m \in \mathbb{N}$.

(iii) $exp(tX) = \Sigma_{i=0}^{K} \frac{t^i X^i}{i!} + O(t^{K+1})$ $(t \to 0)$ for any $X \in M(n,\mathbb{C})$ and $K \in \mathbb{N}$.

(iv) $\frac{d}{dt} exp(tX) = exp(tX)X = Xexp(tX)$

proof of (i).

$$\begin{aligned} exp(X+Y) &= \Sigma_{j=0}^{\infty}\Sigma_{i=0}^{j} \frac{{}_jC_i X^i Y^{j-i}}{j!} \\ &= \Sigma_{j=0}^{\infty}\Sigma_{i=0}^{j} \frac{{}_jP_i}{i!} \frac{X^i Y^{j-i}}{j!} \\ &= \Sigma_{j=0}^{\infty}\Sigma_{i=0}^{j} \frac{j!}{(j-i)!i!} \frac{X^i Y^{j-i}}{j!} \end{aligned}$$

For any $M \in \mathbb{N}$

$$\begin{aligned} ||\Sigma_{i=0}^{M} \frac{X^i}{i!} \Sigma_{j=0}^{M} \frac{Y^j}{j!} &- \Sigma_{j=0}^{M}\Sigma_{i=0}^{j} \frac{j!}{(j-i)!i!} \frac{X^i Y^{j-i}}{j!}|| \\ &= ||\Sigma_{0\le i\le M, 0\le j\le M, i+j>M} \frac{X^i Y^j}{i!j!}|| \\ &\le \Sigma_{0\le i\le M, 0\le j\le M, i+j>M} \frac{||X||^i ||Y||^j}{i!j!} \\ &= ||\Sigma_{i=0}^{M} \frac{||X||^i}{i!} \Sigma_{j=0}^{M} \frac{||Y||^j}{j!} \\ &\quad -\Sigma_{j=0}^{M}\Sigma_{i=0}^{j} \frac{j!}{(j-i)!i!} \frac{||X||^i ||Y||^{j-i}}{j!}|| \end{aligned}$$

Because

$$\lim_{M\to\infty} ||\Sigma_{i=0}^{M} \frac{||X||^i}{i!} \Sigma_{j=0}^{M} \frac{||Y||^j}{j!} = exp(||X||)exp(||Y||)$$

and

$$\lim_{M\to\infty} \Sigma_{j=0}^{M}\Sigma_{i=0}^{j} \frac{j!}{(j-i)!i!} \frac{||X||^i ||Y||^{j-i}}{j!}|| = exp(||X|| + ||Y||)$$

and $exp(||X||)exp(||Y||) = exp(||X|| + ||Y||)$, the following holds.

$$\lim_{M\to\infty} \Sigma_{i=0}^{M} \frac{||X||^i}{i!} \Sigma_{j=0}^{M} \frac{||Y||^j}{j!} - \Sigma_{j=0}^{M}\Sigma_{i=0}^{j} \frac{j!}{(j-i)!i!} \frac{||X||^i ||Y||^{j-i}}{j!} = 0$$

So

$$exp(X+Y) = \lim_{M\to\infty} \Sigma_{i=0}^{M} \frac{X^i}{i!} \Sigma_{j=0}^{M} \frac{Y^j}{j!} = exp(X)exp(Y)$$

□

proof of (ii). It is easy to show (ii) from (i) □

proof of (iii).

$$\begin{aligned} ||exp(tX) - \Sigma_{i=0}^{K}\frac{t^i X^i}{i!}|| &\leq ||\Sigma_{i=K+1}^{\infty}\frac{t^i X^i}{i!}|| \\ &= |t|^{K+1}||\Sigma_{i=K+1}^{\infty}\frac{t^{i-K+1}X^i}{i!}|| \\ &\leq |t|^{K+1}||X||^{K+1}\Sigma_{i=K+1}^{\infty}\frac{|t|^{i-K+1}||X||^{i-K+1}}{i!} \\ &\leq |t|^{K+1}||X||^{K+1}\Sigma_{i=K+1}^{\infty}\frac{|t|^{i-K+1}||X||^{i-K+1}}{(i-K-1)!} \\ &= |t|^{K+1}||X||^{K+1}exp(|t|||X||) \qquad (2.2.4) \end{aligned}$$

□

proof of (iv). By (i), for any $t_0 \in \mathbb{R}$

$$\begin{aligned} exp(tX) - exp(t_0X) &= exp(t_0X)(exp((t-t_0)X) - E) \\ &= (exp((t-t_0)X) - E)exp(t_0X) \end{aligned}$$

By (iii),

$$exp((t-t_0)X) - E = X + o(t-t_0)$$

So (iv) holds. □

Proposition 2.17.

$$exp(tX)exp(tY) = exp(t(X+Y) + \frac{t^2[X,Y]}{2} + o(t^2))$$

Proof.

$$\begin{aligned} exp(tX)exp(tY) &= (E + tX + \frac{1}{2}t^2X^2 + O(t^3))(E + tY + \frac{1}{2}t^2Y^2 + O(t^3)) \\ &= E + t(X+Y) + \frac{1}{2}t^2(X^2+Y^2+2XY) + o(t^3) \end{aligned}$$

So

$$\begin{aligned} log(exp(tX)exp(tY)) &= t(X+Y) + \frac{1}{2}t^2(X^2+Y^2+2XY) + O(t^3) \\ &\quad -\frac{1}{2}(t(X+Y) + \frac{1}{2}t^2(X^2+Y^2+2XY) + O(t^3))^2 \\ &\quad +O(t^3) \\ &= t(X+Y) + \frac{1}{2}t^2(X^2+Y^2+2XY) - \frac{1}{2}t^2(X+Y)^2 \\ &\quad +O(t^3) \\ &= t(X+Y) + \frac{1}{2}t^2(XY-YX) + O(t^3) \end{aligned}$$

By Proposition2.16,

$$exp(tX)exp(tY) = exp(t(X+Y) + \frac{1}{2}t^2(XY-YX) + O(t^3))$$

□

Proposition implies the following.

Proposition 2.18.

$$exp(tX)exp(tY)exp(-tX)exp(-tY) = exp(\frac{t^2[X,Y]}{2} + o(t^2))$$

2.3 Lie algebra

2.3.1 Definition of Lie algebra

Definition 2.7 (Lie algebra of Lie subgroup). *Let G is a Lie subgroup of $GL(n, \mathbb{C})$. We set*

$$Lie(G) := \{X \in M(n, \mathbb{C}) | exp(tX) \in G \ (\forall t \in \mathbb{R})\}$$

We call $Lie(G)$ Lie algebra of G.

Definition 2.8 (Lie algebra of Lie group). *Let G_1 is a Lie group and G_2 is a Lie subgroup of $GL(n, \mathbb{C})$ such that G_1 is locally isomorphic to G_2. We set $Lie(G_1) := Lie(G_2)$.*

By Proposition2.32, $Lie(G_1)$ is well-defined.

Definition 2.9 (General Lie algebra). *Let*

(i) K be a field.

(ii) L be a vector space on K.

(iii) L has operation $[\cdot, \cdot]$ which satisfies the followings.

(a)Alternativity. $[X, X] = 0$ *for any* $X \in L$.

(b)Jacobi's Rule. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ *for any* $X,\ Y,\ Z \in L$.

(c)Bilinearity. $[aX + bY, cZ + dW] = ac[X, Z] + ad[X, W] + bc[Y, Z] + bd[Y, W]$ *for any* $X,\ Y,\ Z,\ W \in L$ *and* $a, b, c, d \in K$.

then we call L a Lie algebra on K.

Clearly, the followings hold.

Proposition 2.19. *For any Lie albegra L,*

$$[X, Y] = -[Y, X] \ (\forall X, Y \in L)$$

Definition 2.10 (Lie subalgebra, ideal). *Let L be a Lie algebra. We call $L' \subset L$ a Lie subalgebra of L if L' is a subvectorspace of L and $[L', L'] \subset L'$. And, if L' is a Lie subalgebra and $[L, L'] \subset L'$ then we call L' is an ideal of L. We call $\{0\}$ and L are trivial ideals.*

The following clearly holds.

Proposition 2.20. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are ideals of $\mathfrak{g}$. We denote the minimam ideal containing $\mathfrak{h}_1$ and $\mathfrak{h}_2$ by $\langle [\mathfrak{h}_1, \mathfrak{h}_2] \rangle$.*

Proposition 2.21. *Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{z} := \{X \in \mathfrak{g} | [X, Y] = 0 \ (\forall Y \in \mathfrak{g})\}$*

Definition 2.11 (Simple Lie algebra). *Let $\mathfrak{g}$ be a Lie algebra. We call $\mathfrak{g}$ is a simple Lie algebra if $\mathfrak{g}$ has no non-trivial ideals and $\mathfrak{g}$ is not abelian.*

By Proposition2.24, the following clearly holds.

Proposition 2.22. *Let $\mathfrak{g}$ be a simple Lie algebra. Then $\langle [\mathfrak{g}, \mathfrak{g}] \rangle = \mathfrak{g}$.*

Definition 2.12 (Direct sum of Lie algebras). *Let L be a Lie algebra. And let $\mathfrak{g}_1, ..., \mathfrak{g}_k$ be ideals of L and $L = \oplus_{i=1}^k \mathfrak{g}_i$. Then we say L is the direct sum of $\mathfrak{g}_1, ..., \mathfrak{g}_k$.*

Definition 2.13 (Abelian Lie algebra). *Let $\mathfrak{g}$ be a Lie algebra. We call $\mathfrak{g}$ is an abelian Lie algebra if $[\mathfrak{g}, \mathfrak{g}] = 0$.*

Proposition 2.23. *Let $\mathfrak{z}$ is the center of a Lie algebra and fix any $X \in \mathfrak{z}$. Then $\langle X \rangle$ is an ideal of $\mathfrak{g}$ and irreducible.*

By Proposition2.24, the following clearly holds.

Proposition 2.24. *Let $\mathfrak{g}$ is a Lie algebra which is the direct sum of $\mathfrak{g}_1, ..., \mathfrak{g}_k$ which are ideals of $\mathfrak{g}$. Then if $i \neq j$ then*

$$[\mathfrak{g}_i, \mathfrak{g}_j] = \{0\}$$

Proposition 2.25. *Let $\mathfrak{g}$ is a Lie algebra which is the direct sum of $\mathfrak{g}_1, ..., \mathfrak{g}_k$ which are ideals of $\mathfrak{g}$. Let us fix any $i \in \{1, 2, ..., k\}$. For any $\mathfrak{h}$ which is an ideal of $\mathfrak{g}_i$, $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.*

Proof. Let us fix any $X \in \mathfrak{g}$ and $Y \in \mathfrak{g}_i$. There are $X_j \in \mathfrak{g}_j$ $(j = 1, 2, ., , , .k)$ such that $X = \sum_{j=1}^{k} X_j$. By Proposition2.24,

$$XY = X_i Y \in \mathfrak{g}_i$$

□

Definition 2.14 (Semisimple Lie algebra)**.** *Let $\mathfrak{g}$ be a Lie algebra. We call $\mathfrak{g}$ is a semisimple Lie algebra if $\mathfrak{g}$ is a direct sum of finite simple Lie algebras.*

Definition 2.15 (Reductive Lie algebra)**.** *Let $\mathfrak{g}$ be a Lie algebra. We call $\mathfrak{g}$ is a reductive Lie algebra if $\mathfrak{g}$ is a direct sum of finite simple Lie algebras and an abelian Lie algebras.*

Proposition 2.26 (quotient Lie algebra)**.** *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ be an ideal of $\mathfrak{g}$. Let $\mathfrak{g}/\mathfrak{h}$ be the quotient vector space. We set for each $X, Y \in \mathfrak{g}$*

$$[X + \mathfrak{h}, Y + \mathfrak{h}] = [X, Y] + \mathfrak{h}$$

$[\cdot, \cdot]$ is the well-defined Lie bracket on $\mathfrak{g}/\mathfrak{h}$. So $\mathfrak{g}/\mathfrak{h}$ is a Lie algebra.

Proof. For any $X, Y \in \mathfrak{g}$ and $h_X, h_Y \in \mathfrak{h}$,

$$[X + h_X, Y + h_Y] = [X, Y] + [X, h_Y] - [Y + h_Y, h_X]$$

So $[X + h_X, Y + h_Y] \in [X, Y] + \mathfrak{h}$. This means that $[\cdot, \cdot]$ is the well-defined Lie bracket on $\mathfrak{g}/\mathfrak{h}$. □

Proposition 2.27 (Adjoint representation of a Lie algebra)**.** *Let $\mathfrak{g}$ be a Lie algebra. We set for each $X \in \mathfrak{g}$*

$$ad(X)Y = [X, Y] \ (Y \in \mathfrak{g})$$

Then

$$ad(aX + bY) = a \cdot ad(X) + b \cdot ad(Y) \ (\forall a, \forall b \in \mathbb{R}, \forall X \in \mathfrak{g}, \forall Y \in \mathfrak{g}) \tag{2.3.1}$$

and

$$ad([X, Y]) = [ad(X), ad(Y)] \ (\forall X \in \mathfrak{g}, \forall Y \in \mathfrak{g}) \tag{2.3.2}$$

We call ad the adjoint representation of $\mathfrak{g}$.

Proof. By linearlity of Lie bracket, (2.3.1) holds. And for any $X, Y, Z \in \mathfrak{g}$

$$\begin{aligned} & [[X, Y], Z] \\ = & -[Z, [X, Y]] \\ = & [X, [Y, Z]] + [Y, [Z, X]] \\ = & [X, [Y, Z]] - [Y, [X, Z]] \\ = & (ad(X)ad(Y) - ad(Y)ad(X))Z \\ = & [ad(X), ad(Y)]Z \end{aligned}$$

So (2.3.2) holds. □

2.3.2 Examples of Lie group and Lie algebra

Example 2.3 ($\mathbb{R}^\times$)**.** *Clearly $\mathbb{R}^\times$ is Linear Liegroup of $GL(1, \mathbb{C})$. So $\mathbb{R}^\times$ is Lie subgroup of $GL(1, \mathbb{C})$. And clearly $Lie(\mathbb{R}^\times) = \mathbb{R}$.*

Example 2.4 ($\mathbb{C}^\times$)**.** *Clearly $\mathbb{C}^\times$ is Linear Liegroup of $GL(1, \mathbb{C})$ and $Lie(\mathbb{C}^\times) = \mathbb{C}$.*

Clearly $G := \{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} | a, b \in \mathbb{R} such that a^2 + b^2 \neq 0\}$ is Lie subgroup $GL(2, \mathbb{R}$ and G is isomorphic to $\mathbb{C}^\times$ and $Lie(G) = \{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} | a, b \in \mathbb{R}\}$. Clearly the right side is subset of the left side. We will show the proof of the inverse in below.

Proof. Let us fix any $X \in Lie(G)$.

$$exp(tX) = E + tX + O(t^2) \ (t \to 0)$$

We define

$$M(t) := \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} := exp(tX) - tX$$

So there is $C > 0$ such that $||M(t)|| \leq C|t|^2$ for any $t \in \mathbb{R}$. We assume $|x_{1,1} - x_{2,2}| \neq 0$.
We pick $t \neq 0$ such that

$$|t| < \frac{|x_{1,1} - x_{2,2}|}{2(C+1)}$$

Because $X \in Let(G)$

$$|t(x_{1,1} - x_{2,2})| = |a(t) - d(t)|$$

Because for any $t \in [-1, 1]$ $|a(t) - d(t)| \leq 2C|t|^2 < |t||x_{1,1} - x_{2,2}|$,

$$|t(x_{1,1} - x_{2,2})| < |t||x_{1,1} - x_{2,2}|$$

So $1 < 1$. It implies contradiction. □

Example 2.5 ($SL(n, \mathbb{R})$, $SL(n, \mathbb{C})$)**.** *By Proposition2.13,*

$$Lie(SL(n, \mathbb{R})) = \{X \in M(n, \mathbb{R}) | tr(X) = 0\}$$

Example 2.6 ($O(n)$, $U(n)$)**.**

$$Lie(O(n)) = \{X \in M(n, \mathbb{R} | X^T = -X\} \quad (2.3.3)$$

$$Lie(U(n)) = \{X \in M(n, \mathbb{C} | X^* = -X\} \quad (2.3.4)$$

proof of (2.3.3). Let us fix any $X \in M(n, \mathbb{R})$ such that $X^T = -X$. Then for any $t \in \mathbb{R}$ $exp(tX)exp(tX)^T = exp(tX)exp(tX^T) == exp(tX)exp(-tX) = E$. So the right side is subset of $Lie(O(n))$. Nextly let us fix any $X \in Lie(O(n))$. Because for any $t \in \mathbb{R}$ $exp(tX) \in M(n, \mathbb{R})$. By the argument similar to Example2.4, $X \in M(n, \mathbb{R})$. By Proposition2.2, $E = exp(tX)exp(tX)^T = exp(t(X + X^T) + O(t^2))$. By the argument similar to Example2.4, $X + X^T = O$. □

proof of (2.3.4). It is similar to the proof of (2.3.3). □

$$Lie(SL(n, \mathbb{R})) = \{X \in M(n, \mathbb{R}) | tr(X) = 0\}$$

$$Lie(SL(n, \mathbb{C})) = \{X \in M(n, \mathbb{C}) | tr(X) = 0\}$$

Example 2.7 ($\mathbb{R}$)**.** *Because $i : \mathbb{R} \ni t \mapsto exp(t) \in (0, \infty)$ is isomorphism of topological groups, $\mathbb{R}$ is a Lie group. Clearly $Lie(\mathbb{R}) = \{a + n\pi i | a \in \mathbb{R}, n \in \mathbb{Z}\}$.*

Example 2.8 ($\mathbb{C}$)**.** *By inverse function theorem about holomorphic function, $i : \mathbb{R} \times (-\pi, \pi) \ni (a, b) \mapsto exp(a)exp(ib)\mathbb{R}$ is isomorphism of topological spaces. Clearly $i|\mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$ is isomorphism in Definition2.1. So $\mathbb{C}$ is a Lie group. Clearly $Lie(\mathbb{C}) = \mathbb{C}$.*

2.3.3 Basic properties of Lie algebra

Lemma 2.1. *Let*

- *(S1) $A : \mathbb{N} \ni n \mapsto A(n) \in M(n, \mathbb{C})$ and $B : \mathbb{N} \ni n \mapsto B(n) \in M(n, \mathbb{C})$.*
- *(A1) $B(m) = O(\frac{1}{m^2})$*
- *(A2) $S := sup_{m \in \mathbb{N}} ||A(m)||^m < \infty$*

then

$$\{A(m)(E + B(m))\}^m = A(m)^m + O(\frac{1}{m})$$

Proof.

$$\begin{aligned} \{A(m)(E + B(m))\}^m &= A(m)(E + B(m))A(m)(E + B(m))...A(m)(E + B(m)) \\ &= A(m)^m + \sum_{k=1}^{m} C_k(m) \end{aligned}$$

Here, for each $k \in \{1, 2, ..., m\}$

$$C_k(m) := \sum_{i_1 < i_2 < ... < i_k} A(m)^{i_1} B(m) A(m)^{i_2} B(m) ... A(m)^{i_k} B(m) A(m)^{m - i_1 - i_2 ... - i_k}$$

Then $||C_k(m)|| \leq {}_mC_k ||A(m)||^m ||B(t)||^k \leq \frac{S}{k!} m^k O(\frac{1}{m^2 k}) = O(\frac{1}{m^k})$.
So $\sum_{k=1}^m ||C_k(m)|| = ||C_1(m)|| + \sum_{k=2}^m ||C_k(m)|| \leq O(\frac{1}{m}) + mO(\frac{1}{m^2}) = O(\frac{1}{m})$. □

Proposition 2.28. *Let G is a Lie sub group of $GL(n,\mathbb{C})$. Then $Lie(G)$ is a $\mathbb{R}$-vector space and for any $X,Y \in Lie(G)$ $[X,Y] \in Lie(G)$.*

Proof. There is W such that W is an open subset of $GL(n,\mathbb{C})$ and $1_G \in W$ and $W \cap G \subset V$.

By the definition of $Lie(G)$, For any $X \in Lie(G)$ and $a \in \mathbb{R}$, $aX \in Lie(G)$.

Let us fix any $X,Y \in Lie(G)$. By Proposition2.2,

$$exp(sX)exp(sY) = exp(s(X+Y)+O(s^2)) = exp(s(X+Y))(E+O(s^2)) \ (s \to 0)$$

So

$$\{exp(\frac{t}{m}(X+Y))(E+O(\frac{1}{m^2}))\}^m = exp(t(X+Y))+O(\frac{1}{m})$$

This implies

$$exp(t(X+Y))+O(\frac{1}{m}) = \{exp(\frac{t}{m}X)exp(\frac{t}{m}Y)\}^m \ (m \to \infty)$$

There is $\delta > 0$ such that $exp(s(X+Y) \in W \ (\forall s \in (-\delta,\delta))$. Let us fix $s \in (-\delta,\delta)$. So for sufficient larget $m \in \mathbb{N}$ $exp(s(X+Y))+O(\frac{1}{m}) \in W \cap G$. So $exp(s(X+Y))+O(\frac{1}{m}) \in V$,

Because V is closed set, $exp(t(X+Y)) \in V$. Consequently $X+Y \in Lie(G)$.

Also, by similar argument to the above one,

$$exp(t[X,Y]) = \lim_{m\to\infty}\{exp(\frac{t}{m}X)exp(\frac{t}{m}Y)exp(\frac{-t}{m}X)exp(\frac{-t}{m}Y)\}^m$$

Consequently $[X,Y] \in Lie(G)$. □

From the proof of Proposition, the following holds.

Proposition 2.29. *Let G is a Lie subgroup of $GL(n,\mathbb{C})$ and V is a closed subset of $GL(n,\mathbb{C})$ and V is a neighborhood of 1_G. And we set*

$$\mathfrak{g}_V := \{X \in M(n,\mathbb{C}|exp(tX) \in V, \ |t| \ll 1\}$$

Then $\mathfrak{g}_V$ is a $\mathbb{R}$-vector space and for any $X,Y \in \mathfrak{g}_V$ $[X,Y] \in \mathfrak{g}_V$.

2.4 The structure of C^ω-class manifold of Lie group

2.4.1 Local coordinate system of Lie group

Lemma 2.2. *For $X_1, X_2, ..., X_m \in M(n,\mathbb{C})$,*

$$exp(X_1)exp(X_2)...exp(X_m) = E + X_1 + X_2 + ... + X_m + o(\sum_{i=1}^m ||X_i||)$$

Proof. For any i,

$$o(||X_i||) = o(\sum_{i=1}^m ||X_i||)$$

So, by the definition of exponential of matrix and Lemma2.4.1

$$\begin{aligned}
& exp(X_1)exp(X_2)...exp(X_m) \\
= \ & (E+X_1+o(||X_1||))(E+X_2+o(||X_2||))...(E+X_m+o(||X_m||)) \\
= \ & E+X_1+X_2+...+X_m \\
& + \sum_{2\le k\le m, i_1<i_2<...<i_k} X_{i_1}X_{i_2}...X_{i_k} + o(\sum_{i=1}^m ||X_i||) \\
= \ & E+X_1+X_2+...+X_m \\
& + \sum_{2\le k\le m, i_1<i_2<...<i_k} o(X_{i_1}) + o(\sum_{i=1}^m ||X_i||) \\
= \ & E+X_1+X_2+...+X_m \\
& + \sum_{2\le k\le m, i_1<i_2<...<i_k} o(\sum_{i=1}^m ||X_i||) + o(\sum_{i=1}^m ||X_i||) \\
= \ & E+X_1+X_2+...+X_m + o(\sum_{i=1}^m ||X_i||)
\end{aligned}$$

□

Lemma 2.3. *Let us fix any subvectorspace $V1$ and $V2$ of $\mathbb{C}^n$ such that $V_1 \oplus V_2 = \mathbb{C}^n$. Then V_1 and V_2 are closed subset.*

Proof. There is $P \in GL(n,\mathbb{C})$ such that $V_1 = P\{w \in \mathbb{C}^n | w_j = 0\ (j = 1, 2, ..., dimV_1)\}P^{-1}$ and $V_2 = P\{w \in \mathbb{C}^n | w_j = 0\ (j = dimV_1 + 1, ..., n)\}P^{-1}$ □

Lemma 2.4. *Let*

(S1) $G = GL(n,\mathbb{C})$.

(S2) $\mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_m$ *are vector subspaces of* $Lie(G)$ *such that*

$$Lie(G_2) = \oplus_{i=1}^m \mathfrak{g}_i$$

(S3) $\mathfrak{g}_i(\epsilon) := \{X \in Lie(G) | ||X|| < \epsilon\}\ (i = 1, 2, ..., m, \epsilon > 0)$.

$$\begin{array}{rcl} i : \oplus_{i=1}^m \mathfrak{g}_i(\epsilon) & \to & G \\ \Cup & & \Cup \\ (X_1, X_2, ..., X_m) & \mapsto & exp(X_1)exp(X_2)...exp(X_m) \end{array}$$

then there is $\epsilon > 0$ such that $i(\oplus_{i=1}^m \mathfrak{g}_i(\epsilon))$ is an open set and $i| \oplus_{i=1}^m \mathfrak{g}_i(\epsilon)$ is C^ω-class isomorphism.

Proof. We set

$$\begin{array}{rcl} j : G & \to & M(n,\mathbb{C}) \\ \Cup & & \Cup \\ y & \mapsto & log(y) \end{array}$$

By Lemma2.2,

$$j \circ i(X_1, X_2, ..., X_m) = X_1 + X_2 + ... + X_m + o(||X_1|| + ||X_2|| + ... + ||X_m||)$$

So, the jacobian of $j \circ i$ at O is non-singular. By inverse function theorem(see [13]), the proposition holds. □

Lemma 2.5. *Let*

(S1) G_2 *is a Lie subgroup of* $GL(n,\mathbb{C})$.

Then for sufficient small $\epsilon > 0$,

$$G_2 \cap exp(B(O,\epsilon)) = exp(Lie(G_2) \cap B(O,\epsilon))$$

Proof of the right side $\subset$ the left side. It is trivial. □

Proof of the left side $\subset$ the right side. There is a vector subspace $\mathfrak{q}$ such that $M(n,\mathbb{C}) = Lie(G) \oplus \mathfrak{q}$. Proposition2.4, $i : Lie(G) \oplus \mathfrak{q} \ni (X, Y) \mapsto exp(X)exp(Y)$ is locally homeomorphism. Let us assume there is $\{\epsilon_k\}_{k=1}^\infty \subset (0,1)$ such that $\lim_{k\to\infty} \epsilon = 0$ and for each ϵ_k the left side $\subsetneq$ the right side. By Lemma2.4, there are $Z_k \in B(O,\epsilon_k)$ and $X_k \in Lie(G_2)$ and $Y_k \in \mathfrak{q}$ $(k = 1, 2, ...)$ such that for any k

$$exp(Z_k) = exp(X_k)exp(Y_k)$$

and

$$\lim_{k\to\infty} ||X_k|| = 0,\ \lim_{k\to\infty} ||Y_k|| = 0$$

and

$$||Y_k|| \neq 0$$

We can assume $||Y_k|| \leq 1$ for any k. Because $\overline{B(O,1)}$ is compact, there is a subsequence $\{Y_{\varphi(k)}\}_{k=1}^\infty$ such that $\lim_{k\to\infty} \lceil \frac{1}{||Y_\varphi(k)||} \rceil Y_{\varphi(k)} = Y$. Clearly $||Y|| = 1$. By Proposition2.3, $Y \in \mathfrak{q}$. So $Y \notin Lie(G)$.

Because V is a neighborhood of 1_{G_2}, there is $\epsilon > 0$ such that $exp(B(O,\epsilon)) \cap G_2 \subset V$. Let us fix any $t \in (0,\epsilon)$.

$$exp(tY) \quad = \quad \lim_{k\to\infty} exp(t\lceil \frac{1}{||Y_\varphi(k)||} \rceil Y_{\varphi(k)})$$

Because $r_k := \lceil \frac{1}{||Y_\varphi(k)||} \rceil \to \infty$, $t = \lim_{k \to \infty} \frac{\lceil tr_k \rceil}{r_k}$. So

$$\begin{aligned} exp(tY) &= \lim_{k\to\infty} exp(\frac{\lceil tr_k \rceil}{r_k} r_k Y_{\varphi(k)}) \\ &= \lim_{k\to\infty} exp(Y_{\varphi(k)})^{\lceil tr_k \rceil} \end{aligned}$$

For any k,

$$exp(\lceil tr_k \rceil Y_{\varphi(k)}) = \{exp(-X_{\varphi(k)})exp(Z_{\varphi(k)})\}^{\lceil tr_k \rceil} \in G_2 \cap exp(B(O,\epsilon)) \subset V$$

Because V is closed set, $exp(tY) \in V$. So for any $t \in \mathbb{R}$

$$exp(tY) = exp(\frac{t}{\lceil \frac{t}{\delta} \rceil + 1} Y)^{\lceil \frac{t}{\delta} \rceil + 1} \in G_2$$

So $Y \in Lie(G_2)$. This is contradiction. □

Proposition 2.30. *Let G be a topological group and G_0 be a connected component of G which contains 1_G. Then G_0 is closed normal subgroup of G.*

Proof. Because $\bar{G_0}$ is connected, $\bar{G_0} = G_0$. So G_0 is closed. Because $x \mapsto x^{-1}$ is isomorphism, G_0^{-1} is connected and $1_G \in G_0^{-1}$. So $G_0^{-1} \subset G_0$. Because $x \mapsto gx$ is isomorphism, for any $g \in G_0$, gG_0 is connected and contains 1_G. So for any $g \in G_0$, $gG_0 \subset G_0$. This implies that G_0 is subgroup of G. And for any $g \in G_0$, gG_0g^{-1} is connected and contains 1_G.. So for any $g \in G_0$, $gG_0g^{-1} \subset G_0$. This implies that G_0 is a normal subgroup of G. □

Proposition 2.31. *Let*

- *(S1) G_1 is a connected Lie group which is locally isomorphic to a Lie subgroup of $GL(n, \mathbb{C})$ G_2.*
- *(S2) G_0 is a connected component of G_1 which contains 1_{G_1}.*
- *(A1) N is a connected open neighborhood of 1_{G_1}.*
- *(S3) $N_m := \{n_1 n_2 ... n_m | n_i \in N,\ i = 1, 2, ..., m\}$ for each $m \in \mathbb{N}$.*

then

- *(i) G_0 is closed and open subset of G_1.*
- *(ii) $G_0 = \cup_{i=1}^{\infty} N_i$.*
- *(iii) Any connected component of G_1 is closed and open subset of G_1.*
- *(iv) G_1 satisfies the second axiom of countability. Specially, G_1 is paracompact.*
- *(v) G_1 is separable.*
- *(vi) G_1 is σ-compact.*

Proof of (i) and (ii). By Lemma2.5, we can assume $N = \eta(exp(Lie(G_2) \cap B(O, \epsilon)))$ for some $\epsilon > 0$ and $N = N^{-1}$. We set $H := \cup_{i=1}^{\infty} N_i$. By continuity of multiple operation in G_1, for each $i \in \mathbb{N}$, N_i is connected. Because $1_{G_1} \in N_i$ for any $i \in \mathbb{N}$, H is connected. So,

$$H \subset G_0$$

Because N_m is an open subset for each $m \in \mathbb{N}$, H is an open subset. Let us fix any $g \in H^c$. If we assume $gN \cap H \neq \phi$, then there is $m \in \mathbb{N}$ and there are $n_0 \in N$ and $n_1, n_2, ..., n_m \in N$ such that $gn_0 = n_1 n_2 .. n_m$. So $g \in N_m N^{-1} = N_m N = N_{m+1}$. This implise $g \in H$. This is a contradiction. So $gN \cap H = \phi$. This means H is a closed subset of G_1. Because $G_0 \subset H \cup H^c$ and H is open and H^c is open and G_0 is connected and $G_0 \cap H \neq \phi$, $G_0 \cap H^c = \phi$. This means

$$G_0 \subset H$$

So $G_0 = H$. □

Proof of (iii). Let us fix and set any connected component of G_1 C. And let us fix $g_0 \in C$. Clearly $C = g_0 G_0$. Because L_{g_0} is isomorphism, C is open and closed. □

Proof of (iv)(v). In the proof of (ii), we set $N' := \eta(exp(Lie(G_2) \cap \overline{B(O,\epsilon)}))$. By (ii), $G_0 = \cup_{n=1}^{\infty} N'_n$. Because N'_n is compact for any $n \in \mathbb{N}$, clearly, G_0 satisfies the second axiom of countability. Because $\overline{B(O,\epsilon)})$ is separable, N' is separable. Because N'_n is separable for any $n \in \mathbb{N}$, clearly, G_0 is separable. And, by (S1) and (iii), G_1 satisfies the second axiom of countability and G_1 is separable. □

Proof of (vi). Let $\{X_i\}_{i=1}^{\infty}$ is a sequence of all connected components of G. Let fix $\{x_i\}_{i=1}^{\infty}$ such that $x_i \in X_i$ $(\forall i)$. In (A1), we can assume that N is relative compact. Then $G = \cup_{m=1}^{\infty} \cup_{k=1}^{m} x_k \bar{N}_m$ and $\cup_{k=1}^{m} x_k \bar{N}_m$ is compact $(\forall m \in \mathbb{N})$. So, G is σ-compact. □

From the proof of Lemma2.5, by Proposition2.2, the following holds.

Lemma 2.6. *Let*

- *(S1)* G_2 *is a Lie subgroup of* $GL(n,\mathbb{C})$.
- *(A1)* W *is a neighborhood of* 1_{G_2} *in* G_2.
- *(S2)* $\mathfrak{g}_W := \{X \in M(n,\mathbb{C}|exp(tX) \in W \ |t| \ll 1\}$.

Then for sufficient small $\epsilon > 0$,

$$W \cap exp(B(O,\epsilon)) = exp(\mathfrak{g}_W \cap B(O,\epsilon))$$

Proposition 2.32. *Let* G *is a Lie subgroup of* $GL(n,\mathbb{C})$ *and* W *is a neighborhood of* 1_G. *Then*

$$Lie(G) = \{X \in M(n,\mathbb{C})|exp(tX) \in W \ (0 \leq t << 1)\}$$

Proof. By Proposition2.2, there is V such that V is a closed subset of $GL(n,\mathbb{C})$ and V is a neighborhood of 1_G and $V \subset W$. Clearly $\mathfrak{g}_V \subset \mathfrak{g}_W$ and $\mathfrak{g}_V \subset Lie(G)$. We assume that there is $X \in Lie(G) \setminus \mathfrak{g}_V$. By Proposition2.29, $\langle X \rangle \cap \mathfrak{g}_V = \{0\}$. By Lemma2.4, there is $\delta > 0$ such that

$$(-\delta,\delta) \times (B(O,\delta) \cap \mathfrak{g}_V) \ni (t,Y) \to exp(tX)exp(Y) \in GL(n,\mathbb{C})$$

is injective. By Lemma2.6, $\{exp(tX)exp(\mathfrak{g}_V \cap B(O,\delta))\}_{t\in(-\delta,\delta)}$ is a family of neighborhood of some point of G. Because $\{exp(tX)exp(\mathfrak{g}_V \cap B(O,\delta))\}_{t\in(-\delta,\delta)}$ are disjoint, G does not satisfy the second axiom. This contradicts with Proposition2.31. □

By Lemma2.6 and Proposition2.32, the following holds.

Lemma 2.7. *Let*

- *(S1)* G_2 *is a Lie subgroup of* $GL(n,\mathbb{C})$.
- *(A1)* W *is a neighborhood of* 1_{G_2} *in* G_2.
- *(S2)* $\mathfrak{g}_W := \{X \in M(n,\mathbb{C}|exp(tX) \in W \ |t| \ll 1\}$.

Then for sufficient small $\epsilon > 0$,

$$W \cap exp(B(O,\epsilon)) = exp(Lie(G_2) \cap B(O,\epsilon)) \tag{2.4.1}$$

Theorem 2.1 (von Neumann-Cartan's theorem I)**.** *Let*

- *(S1)* G_1 *is a Lie group which is isomorphic to a Lie subgroup* G_2 *of* $GL(n,\mathbb{C})$.
- *(S2)* $\mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_m$ *are vector subspaces of* $Lie(G_2)$ *such that*

$$Lie(G_2) = \oplus_{i=1}^{m} \mathfrak{g}_i \tag{2.4.2}$$

- *(S3)* $\mathfrak{g}_i(\epsilon) := \{X \in Lie(G_2)|||X|| < \epsilon\}$ $(i = 1,2,...,m, \epsilon > 0)$.
- *(S4)* *For any* $x \in G_2$

$$\begin{array}{rcl} i_x : \oplus_{i=1}^{m} \mathfrak{g}_i(\epsilon) & \to & G_2 \\ \cup & & \cup \\ (X_1, X_2, ..., X_m) & \mapsto & xexp(X_1)exp(X_2)...exp(X_m) \end{array} \tag{2.4.3}$$

- *(S5)* $\psi := i_e$
- *(S6)* $\phi := exp(\cdot)$

then

(i) *G_1 is a C^ω-manifold and $\{\eta_z \circ \phi\}_{z\in G_1}$ is a local coordinate system.*

(ii) *$\{\eta_z \circ \psi\}_{z\in G_1}$ is a local corrdinate system which is equivalent to $\{\eta_z \circ \phi\}_{z\in G_1}$.*

(iii) *There are open neighborhood of 1_{G_1} U and open neighborhood of 1_{G_2} V and $\tau : U \to V$ is a C^ω-class homeomorphism.*

STEP1. Showing i_x is locally injective. We set

$$\begin{array}{rcl} j_x : G_2 & \to & M(n, \mathbb{C}) \\ \Psi & & \Psi \\ y & \mapsto & log(x^{-1}y) \end{array} \tag{2.4.4}$$

By Lemma2.2,

$$j_x \circ i_x(X_1, X_2, ..., X_m) = X_1 + X_2 + ... + X_m + o(||X_1|| + ||X_2|| + ... + ||X_m||) \tag{2.4.5}$$

So, the jacobian of $j_x \circ i_x$ at O is non-singular. By inverse function theorem(see [13]), i_x is locally injective. □

STEP2. Constructing local corrdinates system of G_2. By Lemma2.7, there is $\epsilon > 0$ such that

$$V_\epsilon := exp(Lie(G) \cap B(O, \epsilon)) = V \cap exp(B(O, \epsilon)) \tag{2.4.6}$$

Clearly V_ϵ is an open neighborhood of 1_{G_2}. By (2.4.6), for any $X_0 \in Lie(G) \cap B(O, \epsilon)$ and $\delta > 0$ such that $B(X_0, \delta) \subset B(O, \epsilon)$,

$$exp(Lie(G) \cap B(X_0, \delta)) = V \cap exp(B(X_0, \delta)) \tag{2.4.7}$$

Because the topology of V is equal to the relative topology respect to $GL(n, \mathbb{C})$, $i_e : Lie(G) \cap B(O, \epsilon) \to G_2 \cap exp(B(O, \epsilon))$ is an continous and open map. By STEP1, i_e is a homeomorphism.

And, for any $x \in G_2$, $i_x : Lie(G_2) \cap B(O, \epsilon) \to xV_\epsilon$ is homeomorphism. □

STEP3. Constructing local corrdinates system of G_1. There is $\delta > 0$ such that

$$V_\delta V_\delta^{-1} V_\delta \subset V_\epsilon \tag{2.4.8}$$

$U_\delta := \eta(V_\delta)$. For any $x' \in G_1$, $\phi'_x : Lie(G_2) \cap B(O, \delta) \ni X \mapsto x'\eta(exp(X)) \in x'U_\delta$. Clearly ϕ'_x is homeomorphism. By Proposition, U_ϵ and V_ϵ satisfy the conditions in Definition2.1. □

STEP4. Showing (i). Let us assume $zU_\delta \cap wU_\delta \neq \phi$ and let us fix any $X \in \phi_z^{-1}(zU_\delta \cap wU_\delta)$ and let us set $Y := \phi_w^{-1}(\phi_z(X))$. Then

$$Y = log(\tau(w^{-1}z\eta(exp(X))) \tag{2.4.9}$$

There are $u_x, u_y \in U_\delta$ and $v_x, v_y \in V_\delta$ such that

$$zu_x = wu_y$$

and

$$\eta(v_x) = u_x, \ \eta(v_y) = u_y$$

By (2.4.8),

$$v_x^{-1} \in V_\epsilon \tag{2.4.10}$$

So

$$\eta(v_x^{-1}) = \eta(v_x)^{-1}$$

This implies

$$u_y u_x^{-1} = \eta(v_y)\eta(v_x^{-1})$$

By (2.4.10),

$$\eta(v_y)\eta(v_x^{-1}) = \eta(v_y x_x^{-1})$$

So

$$Y = log(\tau(\eta(v_y x_x^{-1})\eta(exp(X)))) \tag{2.4.11}$$

Because $v_y x_x^{-1} exp(X) \in V_\epsilon$,

$$\eta(v_y x_x^{-1})\eta(exp(X)) = \eta(v_y x_x^{-1} exp(X))$$

So

$$Y = log(v_y x_x^{-1} exp(X)) \tag{2.4.12}$$

Consequently, $\phi_w^{-1} \circ \phi_z$ is C^ω-class. □

STEP5. Showing (iii). It is possible to show (iii) by from STEP1. to STEP4. □

STEP6. Showing $\psi^{-1} \circ \phi$ is locally C^ω-homeomorphism. It is possible to show STEP6 by STEP1. □

STEP7. Showing (ii). If $zU_\delta \cap wU_\delta \neq \phi$,

$$\phi^{-1} \circ \tau_w \circ \eta_z \circ \phi = \phi^{-1} \circ \psi \circ \psi^{-1} \circ \tau_w \circ \eta_z \circ \psi \circ \psi^{-1} \circ \phi$$

and

$$\psi^{-1} \circ \tau_w \circ \eta_z \circ \phi = \psi^{-1} \circ \tau_w \circ \eta_z \circ \psi \circ \psi^{-1} \circ \phi$$

So by STEP6, (iii) holds. □

Proposition 2.33. *Let G be a Lie group. Then there is an open neighborhood U such that U has no subgroups without $\{e\}$.*

Case when $Lie(G) = \{0\}$. By von-Neumann Cartan theorem, $\{e\}$ is an open neighborhood. □

Case when $Lie(G) \neq \{0\}$. There is $\epsilon > 0$ such that $Exp : Lie(G) \cap B(O, 2\epsilon) \ni X \mapsto Exp(X) \in Exp(Lie(G) \cap B(O, 2\epsilon))$ is a diffeomorphism and $Exp(Lie(G) \cap B(O, 2\epsilon))$ is an open subset of G. We set $U := Exp(Lie(G) \cap B(O, \epsilon))$. Let us any $Exp(X) \in U$ such that $X \in Lie(G) \cap B(O, \epsilon) \setminus \{0\}$. We set $g := Exp(\lfloor \frac{||X||}{\epsilon} \rfloor X)$. Then $\epsilon \leq \lfloor \frac{||X||}{\epsilon} \rfloor ||X|| < 2\epsilon$. So, $g \notin U$. This implies that U has no subgroups without $\{e\}$. □

2.4.2 Analycity of Lie group

Definition 2.16 (One-parameter group)**.** *We call $g \in C(\mathbb{R}, G)$ a one-parameter group of G if $g(s+t) = g(s)g(t)$ (for any $s, t \in \mathbb{R}$).*

Proposition 2.34. *Let G_1 be a Lie group which is isomorphic to a Lie subgroup G_2 of $GL(n, \mathbb{C})$. Let us assume τ is a local isomorphism from G_1 to G_2. And let $g \in C(\mathbb{R}, G)$ be a one-parameter group of G. Then there is $\epsilon > 0$ and such that there is the unique $X \in Lie(G_2)$ such that*

$$\tau(g(s)) = exp(sX) \ \forall s \in (-\epsilon, \epsilon) \tag{2.4.13}$$

Existence. Let us fix $\tau : U \to V$ is a local isomorphism and $\epsilon > 0$ and $i : Lie(G_2) \cap B(O, 2\epsilon) \to G_2 \cap exp(B(O, 2\epsilon))$ be a homeomorphism and $\delta > 0$ such that $g((-2\delta, 2\delta)) \subset U$. There is the one-parameter subgroup h such that $h|(-2\delta, 2\delta) = \tau \circ g|(-2\delta, 2\delta)$.

If $h \equiv 1_{G_2}$, then O satisfies (2.4.13). Else if $h \equiv 1_{G_2}$, there is $t_0 \in (0, \delta)$ and $X_1 \in Lie(G_2) \cap B(O, \epsilon)$ such that $1_{G_2} \neq h(t_0) = exp(X_1)$. We set $X_0 := \frac{X_1}{t_0}$.

There is $Y_1 \in Lie(G_2) \cap B(O, \epsilon)$ such that

$$h(\frac{t_0}{2}) = exp(Y_1)$$

Then $exp(X_1) = h(t_0) = exp(2Y_1)$. Because $2Y_1 \in Lie(G_2) \cap B(O, 2\epsilon)$, $X_1 = 2Y_1$. So,

$$h(\frac{t_0}{2}) = exp(\frac{1}{2}X_1)$$

And there is $Y_1 \in Lie(G_2) \cap B(O, \epsilon)$ such that

$$h(\frac{t_0}{4}) = exp(Y_2)$$

Then $exp(Y_1) = h(\frac{t_0}{2}) = exp(2Y_2)$. Because $2Y_2 \in Lie(G_2) \cap B(O, 2\epsilon)$, $Y_1 = 2Y_2$. So,

$$h(\frac{t_0}{4}) = exp(\frac{1}{2}Y_1) = exp(\frac{1}{4}X_1)$$

So, by mathematical induction,

$$h(\frac{t_0}{2^m}) = exp(\frac{1}{2^m}X_1) \ (\forall m \in \mathbb{N})$$

By calculating powers of both sides,

$$h(t_0\frac{k}{2^m}) = exp(t_0\frac{k}{2^m}X_0) \ (\forall k, m \in \mathbb{N})$$

Because $\{t_0\frac{k}{2^m}|k,m \in \mathbb{N}\ such\ that\ \frac{k}{2^m} \leq 1\}$ is dense in $[0,\delta]$,

$$h(t) = exp(tX_0)\ (\forall t \in (-\delta,\delta))$$

□

Uniqueness. Let us fix any $X, Y \in Lie(G_2)$ such that $exp(tX) = exp(tY)$ ($\forall t \in \mathbb{R}$). If there is $a \in \mathbb{R}$ such that $X = aY$, $exp(t(a-1)Y) = E$ ($\forall t \in \mathbb{R}$). By (i) of Theorem2.1, $a = 1$ or $Y = 0$.

If there is X and Y are linear independent, there are $Z_1, Z_2, ..., Z_r$ such that $Z_1, Z_2, ..., Z_r, X, -Y$ are the basis of $Lie(G_2)$. $exp(tX) = exp(tY)$ implies $exp(tX)exp(t(-Y)) = e$. This contradicts with (ii) of Theorem2.1. □

Theorem 2.2. *Let*

(S1) $G_{1,1}$ *be a Lie group which is isomorphic to a Lie subgroup* $G_{1,2}$ *of* $GL(n,\mathbb{C})$.

(S2) $G_{2,1}$ *be a Lie group which is isomorphic to a Lie subgroup* $G_{2,2}$ *of* $GL(n,\mathbb{C})$.

(A1) $\Phi \in C(G_{1,1}, G_{2,1})$ *is a homomorphism.*

then

(i) There is a homomorphisim of Lie algebras $\iota : Lie(G_{1,1}) \to Lie(G_{2,1})$ *such that*

$$\Phi(\eta_1(exp(tX)) = \eta_2(exp(t\iota(X)))\ (|t| \ll 1) \tag{2.4.14}$$

(ii) Φ *is* C^ω*-class.*

(iii) If Φ *is a local isomorphism, then* ι *is an isomorphism.*

STEP1. constructing ι*.* For each $X \in Lie(G_{1,1})$, by Proposition2.34, there is only one Y such that

$$\Phi(\eta_1(exp(tX))) = \eta_2(exp(tY))\ (any\ t\ such\ that\ |t| \ll 1)$$

We set $\iota(X) := Y$. □

STEP2. Showing ι *is a linear.* For any $X \in Lie(G_{1,1})$ and $a \in \mathbb{R}$, clearly $\iota(aX) := a\iota(X)$.

For any $X,\ Y \in Lie(G_{1,1})$ and $t \in \mathbb{R}$ such that $|t| \ll 1$,

$$\begin{aligned}
& \Phi(\eta_1(exp(t(X+Y)))) \\
= & \Phi(\eta_1(\lim_{m\to\infty}(exp(\frac{t}{m}X)exp(\frac{t}{m}Y))^m)) \\
= & \Phi(\lim_{m\to\infty}\eta_1((exp(\frac{t}{m}X)exp(\frac{t}{m}Y))^m)) \\
= & \lim_{m\to\infty}\Phi(\eta_1((exp(\frac{t}{m}X)exp(\frac{t}{m}Y))^m)) \\
= & \lim_{m\to\infty}\Phi(\eta_1((exp(\frac{t}{m}X)exp(\frac{t}{m}Y)))^m) \\
= & \lim_{m\to\infty}\Phi(\eta_1((exp(\frac{t}{m}X)exp(\frac{t}{m}Y))))^m \\
= & \lim_{m\to\infty}\{\Phi(\eta_1(exp(\frac{t}{m}X)))\Phi(\eta_1(exp(\frac{t}{m}Y)))\}^m \\
= & \lim_{m\to\infty}\{\eta_2(exp(\frac{t}{m}\iota(X)))\eta_2(exp(\frac{t}{m}\iota(Y)))\}^m \\
= & \lim_{m\to\infty}\{\eta_2(exp(\frac{t}{m}\iota(X))exp(\frac{t}{m}\iota(Y)))\}^m \\
= & \lim_{m\to\infty}\eta_2(\{exp(\frac{t}{m}\iota(X))exp(\frac{t}{m}\iota(Y))\}^m) \\
= & \eta_2(\lim_{m\to\infty}\{exp(\frac{t}{m}\iota(X))exp(\frac{t}{m}\iota(Y))\}^m) \\
= & \eta_2(t(\iota(X)+\iota(Y)))
\end{aligned}$$

So

$$\iota(X+Y) = \iota(X) + \iota(Y)$$

□

STEP2. Showing (ii). Let ψ_i is the local corrdinate of $G_{i,2}$ in von Neumann-Cartan's theorem($i = 1, 2$). By (i), for any $x \in G_{1,1}$ and $X \in Lie(G_{1,1})$ such that $||X|| \ll 1$

$$\Phi(\eta_{x,1} \circ \psi_1^{-1}(X)) = \Phi(x)\eta_2(\psi_2^{-1}(\iota(X))$$

This implies

$$\psi_2(\tau_{\Phi(x),2}(\Phi(\eta_{x,1} \circ \psi_1^{-1}(X)))) = \iota(X)$$

Because ι is a linear mapping, Φ is C^ω. □

STEP3. Showing $\iota([X, Y]) = [\iota(X), \iota(Y)]$. By Proposition2.18, for any $X,\ Y \in Lie(G_{1,1})$ and $t \in \mathbb{R}$ such that $|t| \ll 1$,

$$\begin{aligned}
& \Phi(\eta_1(exp(t([X,Y])))\\
= \ & \Phi(\eta_1(\lim_{m\to\infty}(exp(\frac{\sqrt{t}}{m}X)exp(\frac{\sqrt{t}}{m}Y)exp(\frac{-\sqrt{t}}{m}X)exp(\frac{-\sqrt{t}}{m}Y))^m))\\
= \ & \Phi(\lim_{m\to\infty}\eta_1((exp(\frac{\sqrt{t}}{m}X)exp(\frac{\sqrt{t}}{m}Y)exp(\frac{-\sqrt{t}}{m}X)exp(\frac{-\sqrt{t}}{m}Y))^m))\\
= \ & \lim_{m\to\infty}\Phi(\eta_1((exp(\frac{\sqrt{t}}{m}X)exp(\frac{\sqrt{t}}{m}Y)exp(\frac{-\sqrt{t}}{m}X)exp(\frac{-\sqrt{t}}{m}Y))^m))\\
= \ & \lim_{m\to\infty}\Phi(\eta_1((exp(\frac{\sqrt{t}}{m}X)exp(\frac{\sqrt{t}}{m}Y)exp(\frac{-\sqrt{t}}{m}X)exp(\frac{-\sqrt{t}}{m}Y)))^m)\\
= \ & \lim_{m\to\infty}\Phi(\eta_1((exp(\frac{\sqrt{t}}{m}X)exp(\frac{\sqrt{t}}{m}Y)exp(\frac{-\sqrt{t}}{m}X)exp(\frac{-\sqrt{t}}{m}Y))^m))\\
= \ & \lim_{m\to\infty}\Phi(\eta_1((exp(\frac{\sqrt{t}}{m}X)exp(\frac{\sqrt{t}}{m}Y)exp(\frac{-\sqrt{t}}{m}X)exp(\frac{-\sqrt{t}}{m}Y))))^m\\
= \ & \lim_{m\to\infty}\{\Phi(\eta_1(exp(\frac{\sqrt{t}}{m}X)))\Phi(\eta_1(exp(\frac{\sqrt{t}}{m}Y)))\Phi(\eta_1(exp(-\frac{\sqrt{t}}{m}X)))\Phi(\eta_1(exp(-\frac{\sqrt{t}}{m}Y)))\}^m\\
= \ & \lim_{m\to\infty}\{\eta_2(exp(\frac{\sqrt{t}}{m}\iota(X)))\eta_2(exp(\frac{\sqrt{t}}{m}\iota(Y)))\eta_2(exp(-\frac{\sqrt{t}}{m}\iota(X)))\eta_2(exp(-\frac{\sqrt{t}}{m}\iota(Y)))\}^m\\
= \ & \lim_{m\to\infty}\{\eta_2(exp(\frac{\sqrt{t}}{m}\iota(X))exp(\frac{\sqrt{t}}{m}\iota(Y))exp(-\frac{\sqrt{t}}{m}\iota(X))exp(-\frac{\sqrt{t}}{m}\iota(Y)))\}^m\\
= \ & \lim_{m\to\infty}\eta_2(\{exp(\frac{\sqrt{t}}{m}\iota(X))exp(\frac{\sqrt{t}}{\iota}(Y))exp(-\frac{\sqrt{t}}{m}\iota(X))exp(-\frac{\sqrt{t}}{\iota}(Y))\}^m)\\
= \ & \eta_2(\lim_{m\to\infty}\{exp(\frac{\sqrt{t}}{m}\iota(X))exp(\frac{\sqrt{t}}{\iota}(Y))exp(-\frac{\sqrt{t}}{m}\iota(X))exp(-\frac{\sqrt{t}}{\iota}(Y))\}^m)\\
= \ & \eta_2(t[\iota(X),\iota(Y)])
\end{aligned}$$

□

Proposition 2.35. *Let*

- *(S1)* $G_{1,1}$ *be a Lie group which is isomorphic to a Lie subgroup* $G_{1,2}$ *of* $GL(n,\mathbb{C})$.
- *(S2)* $G_{2,1}$ *be a Lie group which is isomorphic to a Lie subgroup* $G_{2,2}$ *of* $GL(n,\mathbb{C})$.
- *(S3)* $G_{3,1}$ *be a Lie group which is isomorphic to a Lie subgroup* $G_{3,2}$ *of* $GL(n,\mathbb{C})$.
- *(A1)* $f : G_{1,1} \to G_{2,1}$ *is a homomorphism of Lie groups.*
- *(A2)* $g : G_{2,1} \to G_{3,1}$ *is a homomorphism of Lie groups.*
- *(S4) By Propositionprop:homomorphismanalytic, homomorphisms of Lie algebras derived from* $f \circ g, f, g,$ *respectively. We define* $\Phi(f \circ g), \Phi(f), \Phi(g)$ *are homomorphisms of Lie algebras derived from* $f \circ g, f, g,$ *respectively.*

then

$$\Phi(f \circ g) = \Phi(g) \circ \Phi(f) \tag{2.4.15}$$

Proof. Let us fix any $X \in Lie(G_{1,1})$. Because for $t \in \mathbb{R}$ such that $|t| \ll 1$

$$\begin{aligned}
& \eta_3(exp(t\Phi(g \circ f)X)\\
= \ & g \circ f(\eta_1(exp(tX)))\\
= \ & g(\eta_2(expt\Phi(f)X))\\
= \ & \eta_3(exp(t\Phi(g)\Phi(f)X))
\end{aligned}$$

$\Phi(f \circ g) = \Phi(g) \circ \Phi(f)$. □

By Theorem2.2, any inner automorphism of G_1 is C^ω-class. By von-Neumann Cartan's theorem, This implies the following two Proposition.

Proposition 2.36. *Let*

(S1) G_1 *is a Lie group which is isomorphic to a Lie subgroup* G_2 *of* $GL(n, \mathbb{C})$.

(S2) For sufficient small open neighborhood of 1_{G_2} V *and* $z \in G_1$, *we set* $\mu_z : V \ni g \mapsto gz \in G_1$.

then

(i) $\{\mu_z \circ \phi\}_{z \in G_1}$ *is a local corrdinate system of* G_1 *which is equivalent to* $\{\eta_z \circ \phi\}_{z \in G_1}$.

(ii) $\{\mu_z \circ \psi\}_{z \in G_1}$ *is a local corrdinate system of* G_1 *which is equivalent to* $\{\eta_z \circ \psi\}_{z \in G_1}$.

Proposition 2.37. *Let*

(S1) G_1 *is a Lie group which is isomorphic to a Lie subgroup* G_2 *of* $GL(n, \mathbb{C})$.

then for any $g \in G_1$,

(i) $l_g : G_1 \ni x \mapsto gx \in G_1$ *is* C^ω*-class homeomorphism.*

(ii) $r_g : G_1 \ni x \mapsto xg \in G_1$ *is* C^ω*-class homeomorphism.*

These Propositions imply the following theorem.

Theorem 2.3 (von Neumann-Cartan's theorem II)**.** *Let*

(S1) G_1 *is a Lie group which is isomorphic to a Lie subgroup* G_2 *of* $GL(n, \mathbb{C})$.

(S2) $\mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_m$ *are vector subspaces of* $Lie(G_2)$ *such that*

$$Lie(G_2) = \oplus_{i=1}^m \mathfrak{g}_i$$

(S3) $\mathfrak{g}_i(\epsilon) := \{X \in Lie(G_2) | ||X|| < \epsilon\}$ $(i = 1, 2, ..., m, \epsilon > 0)$.

(S4) For any $x \in G_2$

$$\begin{array}{rcl} i_x : \oplus_{i=1}^m \mathfrak{g}_i(\epsilon) & \rightarrow & G_2 \\ \cup & & \cup \\ (X_1, X_2, ..., X_m) & \mapsto & xexp(X_1)exp(X_2)...exp(X_m) \end{array}$$

then $G_1 \times G_1 \ni (x, y) \mapsto xy^{-1} \in G_1$ *is* C^ω*-class.*

Proposition 2.38 (Exponential mapping of Lie algebra)**.** *Let*

(S1) G_1 *is a Lie group which is isomorphic to a Lie subgroup* G_2 *of* $GL(n, \mathbb{C})$.

(S2) $\epsilon > 0$ *and* $exp(Lie(G_1) \cap B(O, \epsilon))$.

(S3) For each $X \in Lie(G_1)$, *set* $Exp(X) := \eta(exp(\frac{X}{m}))^m$ *for* $m \in \mathbb{N}$ *such that* $\frac{X}{m} \in B(O, \epsilon)$.

then the followings hold.

(i) *Exp is well-defined and continuous.*

Proof of (i). Let us fix any $m, m' \in \mathbb{N}$ such that $\frac{X}{m} \in B(O, \epsilon)$ and $\frac{X}{m'} \in B(O, \epsilon)$. Then $\frac{iX}{mm'} \in B(O, \epsilon)$ $i = 0, 1, ..., max(m, m')$. By the Definition of locally isomorphism(Definition2.1),

$$\eta(exp(\frac{t}{m}X))^m = \eta(exp(\frac{t}{mm'}X))^{mm'} = \eta(exp(\frac{t}{m'}X))^{m'}$$

So Exp is well-defined. Because η and exp are continuous and G_1 is topological group, Exp is continuous. □

2.5 Correspondence between Lie groups and Lie algebras

2.5.1 Tangent space of Lie Groups

Proposition 2.39. *Let*

(S1) G_1 is a Lie group which is isomorphic to a Lie subgroup G_2 of $GL(n,\mathbb{C})$.

(S2) For each $X \in Lie(G_1)$,

$$\iota(X)(f) := \frac{d}{dt}|_{t=0} f(\eta(exp(tX))) \ (f \in C^\infty(1_{G_1}))$$

then $\iota(Lie(G_1)) \subset T_{1_{G_1}}(G_1)$ and $\iota : Lie(G_1) \to T_{1_{G_1}}(G_1)$ is a isomorphism of vector spaces.

STEP0:Proof of $\iota(Lie(G_1)) \subset T_{1_{G_1}}(G_1)$. By Leibniz product rule in calculas, $\iota(Lie(G_1)) \subset T_{1_{G_1}}(G_1)$. □

STEP1:Proof of linearity of ι. Let us fix any $X \in Lie(G_1)$ and $a \in \mathbb{R}$. For the formula of the compositition of $f(\eta(exp(\cdot X)))$ and $a\cdot$, $\iota(aX) = a\iota(X)$

And let us fix any $Y \in Lie(G_1)$. By Lemma2.2,

$$f(\eta(\psi(t(X+Y)))) = f(\eta(\varphi(\varphi^{-1}\psi(t(X+Y))))) = f(\eta(\varphi((tX,tY)+o(t))))$$

By the chain rule, $\iota(X+Y)(f) = \frac{d}{dt}_{|t=0} f(\eta(\varphi(tX,tY)))$. By applying the chain rulte to the composition of $(u,w) \mapsto f(\eta(\varphi(uX,wY)))$ and $t \mapsto (tX,tY)$,

Because $f(\eta(exp(t(X+Y)))) = f(\eta(exp(tX)exp(tY)+o(t)))$,

$$\frac{d}{dt}_{|t=0} f(\eta(\varphi(tX,tY))) = \iota(X)(f) + \iota(X)(f)$$

□

STEP2:Proof of that ι is injective. Let us fin any $X \in Lie(G_1)$ such that $X \neq O$. By linearity of ι, it is enought to show $\iota(X) \neq 0$. There is $X_2, X_3, ..., X_r \in Lie(G_1)$ such that $X, X_2, X_3, ..., X_r$ is a basis of $Lie(G_1)$. Here, $r := Lie(G_1)$. Let us set $f_X(\eta(\psi(t_1,t_2,...,t_r))) := t_1$ for $|t_1| \ll 1, ..., |t_r| \ll 1$. Clearly $f_X \in C^\infty(1_{G_1})$ and $\iota(X)(f_X) = 1$. So $\iota(X) \neq 0$. □

STEP3:Proof of that ι is surjective. By Proposition2.1, $\dim T_{1_{G_1}} = Lie(G_1)$. By this and STEP1 and STEP2, ι is surjective. □

2.5.2 Homomorphism of Lie groups

Theorem 2.4. *Let*

(S1) $G_{1,1}$ be a Lie group which is isomorphic to a Lie subgroup $G_{1,2}$ of $GL(n,\mathbb{C})$.

(S2) $G_{2,1}$ be a Lie group which is isomorphic to a Lie subgroup $G_{2,2}$ of $GL(n,\mathbb{C})$.

(A1) $\Phi \in C(G_{1,1}, G_{2,1})$ is a homomorphism.

then

(i) $d\Phi_e(i_1(X)) = i_2(\iota(X))$ $(\forall X \in Lie(G_{1,1})$. Here, $i_i : Lie(G_{i,1}) \to T_e(G_{i,1})$ $(i = 1,2)$ are isomorphisms of two vector spaces.

(ii) $\Phi(Exp(X)) = Exp(i_2^{-1}(d\Phi_e(i_1(X))))$ $(\forall X \in Lie(G_{1,1}))$

STEP1. Showing (i). Let us fix any $X \in Lie(G_{1,1})$ and $f \in C^\infty(1_{G_{2,1}})$. Then

$$f(\Phi(\eta_1(exp(tX)))) = f(\eta_2(expt\iota(X))) \ (\forall t : |t| \ll 1)$$

Differentiating both sides by t and setting $t = 0$,

$$d\Phi_e(\iota_1(X))(f) = i_2(\iota(X))(f)$$

□

STEP2. Showing (ii). Let us fix any $X \in Lie(G_{1,1})$. For sufficient large $m \in \mathbb{N}$,

$$\begin{aligned}\Phi(Exp(X)) &= \Phi(Exp(\frac{1}{m}X))^m \\ &= \Phi(\eta_1(exp(\frac{1}{m}X)))^m \\ &= \eta_2(exp(\iota(\frac{1}{m}X)))^m \\ &= \eta_2(exp(i_2^{-1}(i_2(\iota(\frac{1}{m}X)))))^m \\ &= \eta_2(exp(d\Phi_e(i_1((\frac{1}{m}X)))))^m \\ &= Exp(d\Phi_e(i_1((\frac{1}{m}X))))^m \\ &= Exp(d\Phi_e(i_1((m\frac{1}{m}X)))) \\ &= Exp(d\Phi_e(i_1((X))))\end{aligned}$$

□

2.5.3 Invariant vector fields of Lie Groups

It is easy to show the following proposition.

Proposition 2.40 (Regular representation on $C^\infty(G)$)**.** *Let G_1 be a Lie group which is locally isomorphic to a linear Lie subgroup G_2. For $g \in G_1$ and $f \in C^\infty(G_1)$, we set*

$$\pi_L(g)f(x) := f(g^{-1}x),\ \pi_R(g)f(x) := f(xg),\ (x \in G_1) \tag{2.5.1}$$

Then π_L and π_R are representation of G_1. We call π_L the left regular representation of G_1 and π_R the right regular representation of G_1

Proof. By

$$\begin{aligned}& \pi_L(g_1)\pi_L(g_2)f(x) \\ =& \pi_L(g_2)f(g_1^{-1}x) \\ =& f(g_2^{-1}g_1^{-1}x) \\ =& f((g_1g_2)^{-1}x) \\ =& \pi_L(g_1g_2)f(x)\end{aligned}$$

and

$$\begin{aligned}& \pi_R(g_1)\pi_R(g_2)f(x) \\ =& \pi_R(g_2)f(xg_1) \\ =& f(xg_1g_2) \\ =& \pi_R(g_1g_2)f(x)\end{aligned}$$

π_L and π_R are representation of G_1. □

Definition 2.17 ($\mathscr{D}(M)$)**.** *Let M be a C^∞-class manifold. Denote the set of all C^∞-class vector fields by $\mathfrak{X}$. Denote the algebra on $\mathbb{R}$ generated by $C^\infty(M,\mathbb{R})$ and $\mathfrak{X}(M)$ with the operation of $End_{\mathbb{C}}(C^\infty(M))$ by $\mathscr{D}(M)$.*

Definition 2.18 (Invariant vector field on a Lie group)**.** *Let G_1 be a Lie group which is locally isomorphic to a Lie subgroup G_2. We call $P \in \mathscr{D}(G_1)$ an left invariant differential operation if $\pi_L(g)P = P\pi_L(g)$ for any $g \in G_1$. We call $P \in \mathscr{D}(G_1)$ an right invariant differential operation if $\pi_R(g)P = P\pi_R(g)$ for any $g \in G_1$. If $P \in \mathfrak{X}(G_1)$ then we call P a left invariant vector field on G_1 by $\mathfrak{X}_L(G_1)$. If $P \in \mathfrak{X}(G_1)$ then we call P a right invariant vector field on G_1. We denote the set of all left invariant differential fields on G_1 by by $\mathfrak{X}_L(G_1)$. We denote the set of all right invariant differential fields on G_1 by by $\mathfrak{X}_R(G_1)$.*

The following clearly holds.

Proposition 2.41. *Let G_1 be a Lie group which is locally isomorphic to a Lie subgroup G_2. Then $\mathfrak{X}_L(G_1)$ and $\mathfrak{X}_R(G_1)$ are algebras on $\mathbb{R}$.*

Proposition 2.42. *Let*

(S1) G_1 is a Lie group which is isomorphic to a Lie subgroup G_2 of $GL(n,\mathbb{C})$.

(S2) For each $X \in Lie(G_1)$,

$$\iota_L(X)(f)(x) := \frac{d}{dt}|_{t=0} f(x\eta(exp(tX)))\ (f \in C^\infty(1_{G_1},\ x \in G_1)) \tag{2.5.2}$$

and

$$\iota_R(X)(f)(x) := \frac{d}{dt}|_{t=0} f(\eta(exp(-tX))x)\ (f \in C^\infty(1_{G_1},\ x \in G_1)) \tag{2.5.3}$$

then the followings hold.

(i) ι_L is an isomorphism of Lie algebras between $Lie(G_1)$ and $\mathfrak{X}_L(G_1)$. In particular, for anly $X, Y \in Lie(G_1)$

$$[\iota_L(X), \iota_L(Y)] = \iota_L([X,Y]) \tag{2.5.4}$$

(ii) ι_R is an isomorphism of Lie algebras between $Lie(G_1)$ and $\mathfrak{X}_R(G_1)$.

STEP1. $\iota_L(Lie(G_1)) \subset \mathfrak{X}_L(G_1)$. By analiticity of multiple operation of G_1 and the product rule in calculas, $\iota_L(Lie(G_1)) \subset \mathfrak{X}_L(G_1)$. For any $g \in G_1$ and $f \in C^\infty(G_1)$ and $x \in G_1$,

$$\begin{aligned}
& \pi_L(g)\iota_L(X)(f)(x) \\
= & \ \iota_L(X)(f)(g^{-1}x) \\
= & \ \frac{d}{dt} f((g^{-1}x)\eta(exp(tX)))|_{t=0} \\
= & \ \frac{d}{dt} f(g^{-1}(x\eta(exp(tX))))|_{t=0} \\
= & \ \frac{d}{dt} \pi_L(g) f(x\eta(exp(tX)))|_{t=0} \\
= & \ \iota_L(X)\pi_L(g)f(x)
\end{aligned} \tag{2.5.5}$$

So $\iota_L(X)$ is left invariant. □

STEP2. $\iota_R(Lie(G_1)) \subset \mathfrak{X}_R(G_1)$. It is easy to show this by the similar method to STEP1. □

STEP3. ι_L and ι_R are $\mathbb{R}$-linear and injective. It is easy to show this by the similar method to Proposition2.39. □

STEP4. ι_L and ι_R are surjective. Let us fix any $F \in \mathfrak{X}_L(G_1)$. By Proposition2.39, there is $X \in Lie(G_1)$ such that

$$F(f)(e) = \iota(X)(f)\ (\forall f \in C^\infty(G_1),\ \forall x \in G_1) \tag{2.5.6}$$

Because F is a left invariant vector field, for any $x \in G_1$,

$$\begin{aligned}
& F(f)(x) = \\
= & \ \pi_L(x^{-1})(F(f))(e) \\
= & \ F(\pi_L(x^{-1})(f))(e) \\
= & \ \frac{d}{dt} \pi_L(x^{-1})(f)(\eta(exp(tX)))|_{t=0} \\
= & \ \frac{d}{dt} f(x\eta(exp(tX)))|_{t=0} \\
= & \ \iota_L(X)(f)(x)
\end{aligned} \tag{2.5.7}$$

□

STEP5. Calculas of $\iota([X,Y])$. Let us fix any $f \in C^\infty(1_{G_1})$.

By Proposition2.18,

$$\begin{aligned} & \iota([X,Y])(f) \\ = & \frac{d}{dt} f(\eta(exp(t[X,Y])))|_{t=0} \\ = & \frac{d}{dt} f(\eta(exp(\sqrt{t}X)exp(\sqrt{t}Y)exp(-\sqrt{t}X)exp(-\sqrt{t}Y)))|_{t=0} \end{aligned} \tag{2.5.8}$$

□

STEP6. Taylor expansion of $f(\eta(exp(t_1X_1)exp(t_2X_2)exp(t_3X_3)exp(t_4X_4)))$. By the definition of ι_L, for any $i_4 \in \mathbb{Z} \cap [0,\infty)$,

$$\begin{aligned} & \iota_L(X_4)^{i_4}(f)(exp(t_1X_1)exp(t_2X_2)exp(t_3X_3)) \\ = & (\frac{\partial}{\partial t_4})^{i_4} f(\eta(exp(t_1X_1)exp(t_2X_2)exp(t_3X_3)exp(t_4X_4)|_{t_4=0} \end{aligned} \tag{2.5.9}$$

By repeating the above discussion in the same manner below, for any $i_1, i_2, i_3, i_4 \in \mathbb{Z} \cap [0,\infty)$,

$$\begin{aligned} & \iota_L(X_1)^{i_1}\iota_L(X_2)^{i_2}\iota_L(X_3)^{i_3}\iota_L(X_4)^{i_4}(f)(e) \\ = & (\frac{\partial}{\partial t_1})^{i_1}...(\frac{\partial}{\partial t_4})^{i_4} f(\eta(\Pi_{k=1}^4 exp(t_kX_k))|_{\boldsymbol{t}=\boldsymbol{0}} \end{aligned} \tag{2.5.10}$$

So,

$$\begin{aligned} & f(exp(t_1X_1)exp(t_2X_2)exp(t_3X_3)exp(t_4X_4)) \\ = & f(e) \\ + & \sum_{k=1}^{4} \iota_L(X_k)(f) \\ + & \sum_{t_1+...+t_4=2} \frac{1}{i_1!}\frac{1}{i_2!}\frac{1}{i_3!}\frac{1}{i_4!} \iota_L(X_1)^{i_1}...\iota_L(X_4)^{i_4} f(e) t^{i_1}...t^{i_4} \\ + & o(|\boldsymbol{t}|^2) \end{aligned} \tag{2.5.11}$$

□

STEP7. Showing $\iota_L([X,Y]) = [\iota_L(X), \iota_L(Y)]$. In we set $t_1 = t_2 = -t_3 = -t_4 = t$ and $X_1 = -X_3 = X$ and $X_2 = -X_4 = Y$ in (2.5.11),

$$\begin{aligned} & f(exp(\sqrt{t}X)exp(\sqrt{t}Y)exp(-\sqrt{t}X)exp(-\sqrt{t}Y)) \\ = & f(e) \\ + & [\iota(X), \iota(X)](f)t \\ + & o(|\boldsymbol{t}|) \end{aligned} \tag{2.5.12}$$

By (2.5.8),

$$\iota([X,Y])(f) = [\iota(X), \iota(X)](f) \tag{2.5.13}$$

□

STEP4 in the proof of Proposition2.42 implies the following Proposition.

Proposition 2.43. *Let G_1 be a Lie group which is locally isomorphic to a Lie subgroup G_2. Let us fix any $F_1, F_2 \in \mathfrak{X}_L(G_1)$ such that $F_1(f)(e) = F_2(f)(e)$ $(\forall f \in C^\infty(e))$. Then $F_1 = F_2$.*

2.5.4 Taylor expansion of C^ω-class function

STEP6 in the proof of Proposition2.42 implies the following Proposition.

Proposition 2.44. *Let*

(S1) G_1 be a Lie group which is locally isomorphic to a Lie subgroup G_2.

(S2) f be a C^∞-class function at a neighborhood of 1_{G_1}.

(S3) $X_1, ..., X_m \in Lie(G_1)$.

(S4) $g(\boldsymbol{t}) := f(\sum_{i=1}^m t_i X_i)$.

Then

$$(\frac{\partial}{\partial t_1})^{i_1}...(\frac{\partial}{\partial t_m})^{i_m} g(0) = \iota_L(X_1)^{i_1}...\iota_L(X_m)^{i_m} f \tag{2.5.14}$$

Theorem 2.5. *Let*

(S1) $G_{1,1}$ is a Lie group which is isomorphic to a Lie subgroup $G_{1,2}$ of $GL(n,\mathbb{C})$.

(S2) $G_{2,1}$ is a Lie group which is isomorphic to a Lie subgroup $G_{2,2}$ of $GL(n,\mathbb{C})$.

then the followings are equivalent.

(i) $Lie(G_{1,1})$ and $Lie(G_{2,1})$ are isomorphic.

(ii) $G_{1,1}$ and $G_{2,1}$ are locally isomorphic.

Proof of (ii) $\Longrightarrow$ (i). If (ii), by the same argument of the proof of Proposition2.2 and Lemma2.6 and von Neumann-Cartan's theorem, (ii) $\Longrightarrow$ (i). □

Proof of (i) $\Longrightarrow$ (ii). Let $\Phi : Lie(G_{1,1} \to Lie(G_{2,1}$ be an isomorphism. Let $X_{1,1}, ..., X_{1,m}$ be a basis of $Lie(G_{1,1}$. And let us set $X_{2,i} := \Phi(X_{1,i})$ $(i = 1, 2, ..., m)$. We set $e_j : (-\epsilon, \epsilon)^m \ni (t_1, ..., t_m) \to \Pi_{i=1}^m exp(t_i X_{j,i})$ $(j = 1, 2)$. There is $\epsilon > 0$ such that $e_j((-\epsilon, \epsilon)^m)$ is an open subset of G_j and $e_j((-\epsilon, \epsilon)^m) \subset V_j$ and e_j is homeomorphism$(j = 1, 2)$.

We set $\Psi : \eta_1(e_1((-\epsilon, \epsilon)^m)) \to \eta_2(e_2((-\epsilon, \epsilon)^m))$ by $\Psi(e_1(\boldsymbol{t})) := e_2(\boldsymbol{t})$. There is $\delta > 0$ such that $e_j((-\delta, \delta)^m)e_j((-\delta, \delta)^m) \subset e_j((-\epsilon, \epsilon)^m)$ $(j = 1, 2)$. We set $\phi_{j,i} : (-\delta, \delta)^{2m} \to (-\epsilon, \epsilon)$ by

$$e_j(\boldsymbol{x})e_j(\boldsymbol{y}) = e_j(\phi_{j,1}(\boldsymbol{x}, \boldsymbol{y}), ..., \phi_{j,m}(\boldsymbol{x}, \boldsymbol{y})) \tag{2.5.15}$$

$(j = 1, 2)$. We set $\psi_{j,i}(e_j(\boldsymbol{x})e_j(\boldsymbol{y})) := \phi_{j,i}(\boldsymbol{x}, \boldsymbol{y})$. By von Neumann-Cartan's theorem, $\phi_{\{j,i\}}$ are relal analytic functions. So, for each j, i there are $C_{j,i,I,J}$ $I, J \in \mathbb{Z}^m$

$$\phi_{j,i}(\boldsymbol{x}, \boldsymbol{y}) = \sum C_{j,i,I,J} t^{(I,J)} \tag{2.5.16}$$

We will show $\phi_{1,i} = \phi_{2,i}$ $(i = 1, 2, ..., m)$. By Proposition2.44,

$$C_{j,i,I,J} = \iota_L(X_1)^{i_1}...\iota_L(X_m)^{i_m}\iota_L(X_1)^{j_1}...\iota_L(X_m)^{j_m}\psi_{j,i}(0) \tag{2.5.17}$$

Let us fix $k, l \in \{1, 2, ..., m\}$. Because Φ is an isomorphism, there is $c_{k,l,1}, ..., c_{k,l,m} \in \mathbb{R}$ such that

$$[X_{j,k}, X_{j,l}] = \sum_{i=1}^m c_{k,l,i} X_{j,i} \tag{2.5.18}$$

So, by (2.5.4),

$$\iota_L(X_{j,k})\iota_L(X_{j,l}) = \iota_L(X_{j,l})\iota_L(X_{j,k}) + \sum_{i=1}^m c_{k,l,i}\iota_L(X_{j,i}) \tag{2.5.19}$$

By repeating apply of this equation to $\iota_L(X_1)^{i_1}...\iota_L(X_m)^{j_m}$, $C_{1,i,I,J} = C_{2,i,I,J}$. So $\phi_{1,i} = \phi_{2,i}$ $(i = 1, 2, ..., m)$.

We set $W_j := \eta_j(e_j((-\delta, \delta)^2 m))$ $j = 1, 2$. Because $\phi_{1,i} = \phi_{2,i}$ $(i = 1, 2, ..., m)$, for each $x, y \in W_1$

$$xy \in W_1 \iff \Psi(x)\Psi(y) \in W_2 \tag{2.5.20}$$

and if $xy \in W_1$

$$\Psi(xy) = \Psi(x)\Psi(y) \tag{2.5.21}$$

Consequently, $G_{1,1}$ and $G_{2,1}$ are locally isomorphic. □

2.5.5 Differential representation

Clearly the following holds.

Proposition 2.45 (Definition of differential representation of a continuous representation of Lie group)**.** *Let*

(S1) G_1 *is a Lie group which is locally isomorphic to a Lie subgroup of* $GL(n, \mathbb{C})$*.* G_2 *has at most countable connected components.*

(S2) (π, V) *is a finite dimensional continuous representation of* G_1*.*

(S3) $P := \{v_1, v_2, ..., v_r\}$ *is a basis of* V*.*

(S4) For each $f \in End_{\mathbb{C}}(V)$*, denote the representation matrix with respect to* P *by* $\Phi(f)$*.*

(S5) By $\Phi|GL(V) : GL(V) \to GL(n, \mathbb{C})$*, introduces a topology of* $GL(V)$*.*

Then

(i) $\Phi|GL(V) : GL(V) \to GL(n, \mathbb{C})$ *is an isomorphism of topological groups. So,* $GL(V)$ *is a Lie group.*

(ii) $\pi : G_1 \to GL(V)$ *is an homomorphism of Lie groups.*

(iii) $Lie(GL(V)) = M(n\mathbb{C})$*. By Proposition2.2,* π *introduces the homomorphism from* $Lie(G_1)$ *to* $M(n\mathbb{C})$*. we denote this homomorphism by* $d\pi_e$*. We call* $d\pi_e$ *the differential representation of* π*.*

(iv) $d\pi$ *is continuous.*

Proof of (iv). Because $d\pi$ is a linear mapping from $Lie(G_1)$ to $M(n\mathbb{C})$, $d\pi$ is continuous. □

Proposition 2.46 (Adjoint representation of a Lie group)**.** *Let*

(S1) G_1 *is a Lie group which is locally isomorphic to a Lie subgroup of* $GL(n, \mathbb{C})$*.* G_2 *has at most countable connected components.*

(S2) For each $g \in G_1$*, we define* $\sigma(g) \in Auto(G)$ *by* $\sigma(g)(x) := gxg^{-1}$ $(x \in G_1)$*.*

Then

(i) For any $g \in G_1$*,* $\sigma(g)$ *is an automorhism of a Lie group. By Proposition2.2, we denote the endmorphism of* $Lie(G_1)$ *by* $Ad(g)$*.*

(ii) $Ad(G_1) \subset GL(Lie(G_1))$

(iii) $(Ad, GL(Lie(G_1)))$ *is a continuous representation of* G_1 *on* $\mathbb{R}$*.*

Proof of (i). Because $\sigma(g^{-1}) = \sigma(g)^{-1}$ and analyticity of the group operation on G_1, (i) holds. □

Proof of (ii). Because $\sigma(1_{G_1}) = id_{G_1}$, $Ad(1_{G_1}) = id_{Lie(G_1)}$. Let us fix any $g, h \in G_1$. Because $\sigma(gh) = \sigma(g)\sigma(h)$, $Ad(gh)$ is the homomorphism of a Lie algebra $Lie(G_1)$ derived from $\sigma(g)\sigma(h)$. By Proposition2.35, $Ad(gh) = Ad(g)Ad(h)$. So, $Ad(G_1) \subset GL(Lie(G_1))$. □

Proof of (iii). Let us fix $v := (v_1, v_2, ..., v_r)$ which is a basis of $Lie(G_1)$. We denote the representation matrix of $Ad(g)$ respect to v by $R(g)$. Let us fix $\epsilon > 0$ such that $exp(B(O, \epsilon) \cap Lie(G_1)) \subset V$. Let us fix $\delta > 0$ such that $\{vY|Y \in B(0, 2\delta) \cap \mathbb{C}^r\} \subset B(O, \epsilon) \cap Lie(G_1)$. For any $Y \in B(0, 1) \cap \mathbb{C}^r$, $exp(\delta Ad(g)vY) = \tau(g\eta(exp(\delta Y))g^{-1})$. So,

$$vR(g)Y = \frac{1}{\delta}log(\tau(g\eta(exp(\delta Y))g^{-1})) \tag{2.5.22}$$

By setting $Y = e_1$, ..., $Y = e_r$, $vR(\cdot)$ is continuous. Because v is $N \times r$-matrix and $rank(v) = r$, $R(\cdot)$ is continuous. So, $(Ad, Lie(G_1))$ is a continuous representation of G_1. □

Proposition 2.47. *Here are the settings and assumptions.*

(S1) G_1 *is a Lie group which is locally isomorphic to a Lie subgroup of* $GL(n, \mathbb{C})$*.*

Then

(i) $dAd = ad$*.*

(ii) $Ad(Exp(X)) = Exp(ad(X))$ $(\forall X \in Lie(G_1)$*.*

Proof of (i). Let us assume $i : Lie(G_1) \to T_e(G_1)$ be an isomorphism of vector spaces in Proposition2.39. Let us fix any $X, Y \in Lie(G_1)$ and $s, t \in \mathbb{R}$ such that $|s| \ll 1, |t| \ll 1$ and $f \in C^\infty(e)$. Then

$$f(Exp(sAd(Exp(tX))Y)) = f(Exp(tX)Exp(sY)Exp(-tX))$$

And, by Proposition2.4,

$$Ad(Exp(tX)) = exp(tdAd(X))$$

Because

$$\begin{aligned}
& \frac{d}{dt}|_{t=0}\frac{d}{ds}|_{s=0} f(Exp(tX)Exp(sY)Exp(-tX)) \\
= & \frac{d}{ds}|_{s=0}\frac{d}{dt}|_{t=0} f(\eta(exp(sY) + st[X,Y] + O(t^2))) \\
= & \frac{d}{ds}|_{s=0} i(s[X,Y])(f) \\
= & i([X,Y])(f) = i(ad(X)Y)(f)
\end{aligned}$$

and

$$\begin{aligned}
& \frac{d}{dt}|_{t=0}\frac{d}{ds}|_{s=0} f(Exp(sAd(Exp(tx))Y)) \\
= & \frac{d}{dt}|_{t=0} i(Ad(Exp(tX))(Y))(f) \\
= & \frac{d}{dt}|_{t=0} i(exp(tdAd(X)(Y))(f) \\
= & \frac{d}{dt}|_{t=0} i(E + tdAd(X)(Y) + O(t^2))(f) \\
= & \frac{d}{dt}|_{t=0} i(E)(f) + ti(dAd(X)(Y))(f) + O(t^2) \\
= & i(dAd(X)(Y))(f)
\end{aligned}$$

$i(dAd(X)(Y))(f) = i(ad(X)Y)(f)$. So, $dAd = ad$. □

Proof of (ii). By (2.5.5) and (i),

$$Ad(Exp(X)) = Exp(dAd(X)) = Exp(ad(X))$$

□

Proposition 2.48. *Here are the settings and assumptions.*

(S1) G is a linear Lie group of $GL(n,\mathbb{C})$.

Then for any $g \in G$

(i) The representation matrix of $Ag(g)$ is $g \otimes (g^T)^{-1}$ with basis $\{E_{i,j}\}_{i,j}$.

(ii) $det(Ag(g)) = 1$.

Proof of (i). We set $h := g^{-1}$. Let us fix any i_0, j_0 and i, j. Then

$$(gE_{i_0,j_0}g^{-1})_{i,j} = (gE_{i_0,j_0}g^{-1})_{i,j} = \sum_l (gE_{i_0,j_0})_{i,l}h_{l,j} = \sum_{k,l} g_{i,k}(E_{i_0,j_0})_{k,l}h_{l,j} = g_{i,i_0}h_{j_0,j} = g_{i,i_0}h^T_{j,j_0}$$

So, the representation matrix of $Ag(g)$ is $g \otimes (g^T)^{-1}$. □

Proof of (ii). By Proposition1.5 and (i), (ii) holds. □

2.5.6 Baker-Campbell-Hausdorff formula

Proposition 2.49. *Here are the settings and assumptions.*

(S1) $S, T \in M(n,\mathbb{C})$.

Then

$$\frac{d}{ds}|_{s=0} exp(-S)exp(S+sT) = \frac{E - exp(-ad(S))}{ad(S)}T = \sum_{p=0}(-1)^p\frac{ad(S)^p}{(p+1)!}T$$

STEP1. Simplifieing S. Clearly

$$\frac{d}{ds}|_{s=0} exp(-S)exp(S+sT)$$

and

$$\sum_{p=0}(-1)^p \frac{ad(S)^p}{(p+1)!}T$$

are continuous respects to S. For any $P \in GL(n, \mathbb{C})$

$$\begin{aligned} & P\frac{d}{ds}|_{s=0} exp(-S)exp(S+sT)P^{-1} \\ = & \frac{d}{ds}|_{s=0} exp(-PSP^{-1})exp(PSP^{-1}+sPTP^{-1}) \end{aligned}$$

and

$$\begin{aligned} & P\sum_{p=0}(-1)^p \frac{ad(S)^p}{(p+1)!}TP^{-1} \\ = & \sum_{p=0}(-1)^p \frac{ad(PSP^{-1})^p}{(p+1)!}PTP^{-1} \end{aligned}$$

So, we can assume S is a diagonal matrix. □

STEP2. Linearity respects to T. By Wierstrass's theorem,

$$\begin{aligned} & exp(-S)exp(S+sT) \\ = & exp(-S)\lim_{m\to\infty}\frac{d}{ds}|_{s=0}\sum_{i=0}^{m}\frac{(S+sT)^i}{i!} \end{aligned}$$

We set

$$L_m(T) := exp(-S)\frac{d}{ds}|_{s=0}\sum_{i=0}^{m}\frac{(S+sT)^i}{i!}$$

Because

$$\begin{aligned} & \frac{d}{ds}|_{s=0}(S+sT)^i \\ = & \frac{d}{ds}|_{s=0}\sum_{j=0} sS^jTS^{i-j-1} + o(s) \\ = & \sum_{j=0} S^jTS^{i-j-1} \end{aligned}$$

$L_m(\cdot)$ is linear for any $m \in \mathbb{N}$. Because $L_m(\cdot)$ normed converges to

$$\frac{d}{ds}|_{s=0} exp(-S)exp(S+s\cdot)$$

$\frac{d}{ds}|_{s=0} exp(-S)exp(S+s\cdot)$ is linear. □

STEP3. Simplifying T. By STEP2, we can assume $T = E_{i,j}$. □

STEP4. Showing this equation. If $[S,T]=0$, the both side equals to T. So, we can assume $[S,T] \neq 0$. We set $\lambda_1, ..., \lambda_n$ by

$$S = \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n \end{pmatrix}$$

We set $\lambda = \lambda_i - \lambda_j$. Then

$$ST = \lambda T$$

Because $[S,T] \neq 0$, $\lambda_i \neq \lambda_j$ and $i \neq j$. Because $\lambda_j T$ and T are commutative, by replacing S by $S - \lambda_j T$, we can assume $\lambda_j = 0$. Then

$$TS = T^2 = O$$

So

$$ad(S)T = \lambda T$$

$$\begin{aligned}
& \frac{d}{ds}|_{s=0} exp(-S)exp(S+sT) \\
= & \frac{d}{ds}|_{s=0} exp(-S)\{\sum_{i=1}^{m} s\frac{S^{i-1}}{i!}T + o(1)\} \\
= & exp(-S)\sum_{i=1}^{m}\frac{S^{i-1}}{i!}T \\
= & exp(-S)\sum_{i=1}^{m}\frac{\lambda^{i-1}}{i!}T \\
= & exp(-\lambda)\sum_{i=1}^{m}\frac{\lambda^{i-1}}{i!}T \\
= & exp(-\lambda)\frac{exp\lambda - 1}{\lambda}T \\
= & \frac{1 - exp(-\lambda)}{\lambda}T \\
= & \sum_{i=1}^{m}(-1)^{i+1}\frac{\lambda^{i-1}}{i!}T \\
= & \sum_{i=1}^{m}(-1)^{i+1}\frac{ad(S)^{i-1}}{i!}T
\end{aligned}$$

Consequently, this Proposition holds. □

Proposition 2.50. *Let*

(S1) $S, T \in M(n, \mathbb{C})$.

Then

(i) If $|t| < \dfrac{log2}{||X|| + ||Y||}$ *then* $Z(t) := log(exp(tX)exp(tY))$ *converges.*

(ii) We set $\{Z_m\}_{m=1}^{\infty}$ *by* $Z(t) = \sum_{m=1}^{\infty} Z_m t^m$ *then*

$$Z_1 = X + Y$$

and for any $m \in \mathbb{N} \cap [2, \infty)$

$$Z_m = \sum_{\epsilon \in \{0,1\}^{m-2}} C_\epsilon ad(W_{\epsilon_1})...ad(W_{\epsilon_{m-2}})ad(X)Y \tag{2.5.23}$$

Here $W_0 := X$ *and* $W_1 := Y$ *and* $C_\epsilon \in \mathbb{Q}$ *and* C_ϵ *does not* X, Y.

(iii) If $||X|| + ||Y|| < log2$ *then* $Z := \sum_{m=1}^{\infty} Z_m$ *exists and* $exp(X)exp(Y) = expZ$.

Proof of (i). If $|t| < \dfrac{log2}{||X|| + ||Y||}$ then

$$\begin{aligned}
& ||exp(tX)exp(tY) - E|| \\
\leq & \lim_{m\to\infty} ||\sum_{i=0}^{m}\frac{1}{i!}t^i X^i|\sum_{i=0}^{m}\frac{1}{i!}t^i Y^i - E|| \\
\leq & \lim_{m\to\infty} |\sum_{i=0}^{m}\frac{1}{i!}|t|^i ||X||^i \sum_{i=0}^{m}\frac{1}{i!}|t|^i ||Y||^i - 1| \\
\leq & |exp|t|||X|||exp|t|||Y|| - 1| \\
\leq & |exp|t|(||X|| + ||Y||) - 1| \\
< & 1
\end{aligned}$$

So, if $|t| < \dfrac{log2}{||X|| + ||Y||}$ then $log(exp(tX)exp(tY))$ converges. □

Proof of (ii). By Proposition2.47,

$$\begin{aligned} & \frac{d}{dt}exp(Z(t)) \\ = & \frac{d}{dt}exp(tX)exp(tY) \\ = & exp(tX)Xexp(tY) + exp(tX)exp(tY)Y \\ = & exp(tX)exp(tY)exp(-tY)Xexp(tY) + exp(tX)exp(tY)Y \\ = & exp(Z(t))(exp(-tY)Xexp(tY) + Y) \\ = & exp(Z(t))(exp(-tad(Y))X + Y) \end{aligned}$$

So

$$exp(-Z(t))\frac{d}{dt}exp(Z(t)) = exp(-tad(Y))X + Y$$

Because

$$\begin{aligned} & exp(-Z(t))\frac{d}{dt}exp(Z(t)) \\ = & exp(-Z(t))\frac{d}{ds}|_{s=0}exp(Z(t+s)) \\ = & exp(-Z(t))\frac{d}{ds}|_{s=0}exp(Z(t) + sZ'(t) + o(s)) \\ = & exp(-Z(t))\frac{d}{ds}|_{s=0}exp(Z(t) + sZ'(t)) + o(s) \\ = & exp(-Z(t))\frac{d}{ds}|_{s=0}exp(Z(t) + sZ'(t)) \end{aligned}$$

by Proposition2.49,

$$\sum_{p=0}(-1)^p\frac{ad(Z(t))^p}{(p+1)!}Z'(t) = exp(-tad(Y))X + Y$$

So,

$$Z'(t) = \sum_{p=1}(-1)^{p+1}\frac{ad(Z(t))^p}{(p+1)!}Z'(t) + exp(-tad(Y))X + Y$$

Because

$$\sum_{p=1}(-1)^{p+1}\frac{ad(Z(t))^p}{(p+1)!}Z'(t)$$

has no constant,

$$Z_1 = X + Y$$

We assume $Z_1, ..., Z_m$ satisfies the condition (2.5.23). Because

$$Z(t) = Z_1t + Z_2t^2 + ... + Z_mt^m + ...$$

and

$$Z'(t) = t + 2Z_2t + ... + mZ_mt^{m-1} + (m+1)Z_{m+1}t^m...$$

$$(m+1)Z_{m+1} = \sum_{k=1}^{m}\sum_{i_1+...+i_k+(l-1)=m-1} lZ_{i_1}...Z_{i_k}Z_l + \frac{(-1)^m}{m!}ad(Y)^mX$$

Because of (2.5.6) and the assumption of this mathematical induction,

$$\begin{aligned}&(m+1)Z_{m+1}\\ =&\sum_{\epsilon\in\{0,1\}^{m-1}} D_{1,\epsilon}ad(W_{\epsilon_1})...ad(W_{\epsilon_{m-1}})ad(X)X\\ &+\sum_{\epsilon\in\{0,1\}^{m-1}} D_{2,\epsilon}ad(W_{\epsilon_1})...ad(W_{\epsilon_{m-1}})ad(X)Y\\ &+\sum_{\epsilon\in\{0,1\}^{m-1}} D_{3,\epsilon}ad(W_{\epsilon_1})...ad(W_{\epsilon_{m-1}})ad(Y)X\\ &+\sum_{\epsilon\in\{0,1\}^{m-1}} D_{4,\epsilon}ad(W_{\epsilon_1})...ad(W_{\epsilon_{m-1}})ad(Y)Y\end{aligned}$$

Because $ad(X)X=0$ and $ad(Y)Y=0$ and $ad(Y)X=-ad(X)Y$,

$$\begin{aligned}&(m+1)Z_{m+1}\\ =&\sum_{\epsilon\in\{0,1\}^{m-1}}(D_{2,\epsilon}-D_{3,\epsilon})ad(W_{\epsilon_1})...ad(W_{\epsilon_{m-1}})ad(X)Y\end{aligned}$$

So Z_{m+1} satisfies the condition (2.5.23). □

2.5.7 Analytic subgroup

Theorem 2.6 (Analytic subgroup)**.** *Let*

(S1) G_1 is a Lie group which is locally isomorphic to a linear Lie subgroup G_2 of $GL(n,\mathbb{C})$.

(S2) $\mathfrak{h}$ be a Lie subalgebra of $Lig(G_1)$.

Then there is H such that H is a subgroup of G_1 and H is a Lie group and $Lie(H)=\mathfrak{h}$. We say H is a analytic subgroup of G whose Lie algebra is $\mathfrak{h}$.

STEP1. Construction of H. There are $X_1,...,X_k,...,X_m,...,X_N\in M(n,\mathbb{C})$ such that $N=n^2$ and $X_1,...,X_N$ is a basis of $M(n,\mathbb{C})$ $X_1,...,X_k,...,X_m$ is a basis of $Lie(G_1)$ and $X_1,...,X_k$ is a basis of $\mathfrak{h}$. By von Neumann-Cartan's theorem, there is $\epsilon>0$ such that

$$e:(-\epsilon,\epsilon)^m\ni t\mapsto Exp(\sum_{i=1}^m t_iX_i)\in G_1$$

is a C^ω-class homeomorphism to an open subset of U and

$$E:(-\epsilon,\epsilon)^N\ni t\mapsto Exp(\sum_{i=1}^N t_iX_i)\in GL(n\mathbb{C})$$

is a C^ω-class homeomorphism to an open subset of $GL(n\mathbb{C})$. We set

$$H:=\{Exp(X_1)...Exp(X_l)|X_1,...,X_l\in\mathfrak{h},\ l\in\mathbb{N}\}$$

Clearly H is subgroup of G_1. □

STEP2. Constructing the topology of H. We set the topology of H whose fundamental neighborhood system of H is $\{hExp(B_k(O,s\epsilon))|0\le s<1,\ h\in H\}$. We will show $\{hExp(B_k(O,s\epsilon))|0\le s<1,\ h\in H\}$ satisfies the aixoms of a fundamental neighborhood system.

Let us fix any $exp(\sum_{i=1}^k t_iX_i)$ such that $t\in(-s\epsilon,s\epsilon)^k$. We will show there is $\delta>0$ such that

$$exp(\sum_{i=1}^k t_iX_i)exp(\sum_{i=1}^k(-\delta,\delta)X_i)\subset exp(\sum_{i=1}^k(-s\epsilon,s\epsilon)X_i) \tag{2.5.24}$$

There is $\epsilon_1>0$ such that $t+(-\epsilon_1,\epsilon_1)^k\subset(-s\epsilon,s\epsilon)^k$. There is $\delta\in(0,\epsilon)$ such that

$$exp(\sum_{i=1}^k t_iX_i)exp(\sum_{i=1}^k(-\delta,\delta)X_i)\subset exp(\sum_{i=1}^k t_iX_i+\sum_{i=1}^N(-\epsilon_1,\epsilon_1)X_i)$$

By the continuity of exp and log, we can assume

$$log(exp(\sum_{i=1}^{k} t_i X_i))exp(\sum_{i=1}^{k}(-\delta,\delta)X_i)) \subset \sum_{i=1}^{N}(-\epsilon,\epsilon)X_i$$

By Baker-Campbell-Hausdorff formula,

$$log(exp(\sum_{i=1}^{k} t_i X_i)exp(\sum_{i=1}^{k}(-\delta,\delta)X_i)) \subset \sum_{i=1}^{N}(-\epsilon,\epsilon)X_i \cap \mathfrak{h}$$

Because $exp|(\sum_{i=1}^{N}(-\epsilon,\epsilon)X_i)$ is injective,

$$\begin{aligned}
& exp(\sum_{i=1}^{k} t_i X_i))exp(\sum_{i=1}^{k}(-\delta,\delta)X_i) \\
\subset \;& exp(\sum_{i=1}^{k}(-\epsilon,\epsilon)X_i \cap \sum_{i=1}^{k} t_i X_i + \sum_{i=1}^{k}(-\epsilon_1,\epsilon_1)X_i) \\
= \;& exp(\sum_{i=1}^{k} t_i X_i + \sum_{i=1}^{k}(-\epsilon_1,\epsilon_1)X_i) \\
\subset \;& exp(\sum_{i=1}^{k}(-s\epsilon,s\epsilon)X_i)
\end{aligned}$$

Let us fix any $h_1, h_2 \in H$ such that

$$h_1 Exp(B_k(O,s_1\epsilon)) \cap h_2 Exp(B_k(O,s_2\epsilon)) \neq \phi$$

Then there is $u_1 \in Exp(B_k(O,s_1\epsilon))$ and $u_2 \in Exp(B_k(O,s_2\epsilon))$ such that $h_1u_1 = h_2u_2$. By (2.5.24), there is $\delta > 0$ such that $u_1 Exp(B_k(O,\delta)) \subset Exp(B_k(O,s_1\epsilon))$ and $u_2 Exp(B_k(O,\delta)) \subset Exp(B_k(O,s_2\epsilon))$.

$$\begin{aligned}
& h_1 Exp(B_k(O,s_1\epsilon)) \supset h_1u_1 Exp(B_k(O,\delta)) \\
= \;& h_1u_2 Exp(B_k(O,\delta)) \subset h_2 Exp(B_k(O,s_2\epsilon))
\end{aligned}$$

Consequently, $\{hExp(B_k(O,s\epsilon)) | 0 \leq s < 1,\ h \in H\}$ satisfies the aixoms of a fundamental neighborhood system.

□

STEP3. Showing properties of H. Clearly $Exp : \mathfrak{h} \to H$ is continuous. Because $B_k(O,\epsilon)$ is connected and Exp is continuous, $Exp(B_k(O,\epsilon))$ is a connected. So H is connected. And clearly H is Housdorff space. □

STEP4. Showing H is a topological group. It is enough to show continuity of the multiple operation and the inverse operation of H. Let us fix any $g_1, g_2 \in H$ and $s \in [0,1)$. We set $g := g_1^{-1}g_2$. It is enouth to show for sufficient small $s_1, s_2 \in [0,1)$ $\{g_1Exp((B_k(O,s_1\epsilon))\}^{-1}g_2Exp((B_k(O,s_2\epsilon))$ is contained $gExp((B_k(O,s\epsilon))$. For sufficient small $X, Y \in \mathfrak{h}$,

$$\begin{aligned}
& \{g_1Exp(X)\}^{-1}g_2Exp(Y) \\
= \;& Exp(-X)gExp(Y) \\
= \;& gg^{-1}Exp(-X)gExp(Y) \\
= \;& gExp(-Ad(g^{-1})X)Exp(Y)
\end{aligned}$$

By the defitnition of H, there are $Z_1, ..., Z_k \in \mathfrak{h}$ such that

$$g^{-1} = exp(Z_1)...exp(Z_k)$$

So, by Proposition2.47,

$$\begin{aligned}
& Ad(g^{-1})X \\
= \;& Ad(exp(Z_1))...Ad(exp(Z_k))X \\
= \;& exp(ad(Z_1))...exp(ad(Z_k))X
\end{aligned}$$

By Proposition2.3, $\mathfrak{h}$ is a closed subset of $M(n,\mathbb{C})$. So, $Ad(g^{-1})X \in \mathfrak{h}$. By Baker-Campbell-Hausdorff's formula, for sufficient small $X, Y \in \mathfrak{h}$,

$$Exp(-Ad(g^{-1})X)Exp(Y) \in Exp((B_k(O, s\epsilon))$$

So, the multiple operation and the inverse operation of H are continuous.

□

STEP5. Showing H is a Lie group. We can assume $\tau(e((-\epsilon,\epsilon)^m)) \subset V$. By Baker-Campbell-Hausdorff's formula, there is $\epsilon_1 > 0$ such that

$$\tau(e([-\epsilon_1,\epsilon_1]^k \times \{0\}^{m-k}))\tau(e([-\epsilon_1,\epsilon_1]^k \times \{0\}^{m-k})) \subset \tau(e((-\epsilon,\epsilon)^k \times \{0\}^{m-k}))$$

We set $V_H := \tau(e([-\epsilon_1,\epsilon_1]^k \times \{0\}^{m-k}))$. Clearly V_H is a neighborhood of the unit element in H and $V_H \subset V$. Because $\tau(e([-\epsilon_1,\epsilon_1]^k \times \{0\}^{m-k}))$ is compact subset of $GL(n,\mathbb{C})$, V_H is closed subset of $GL(n,\mathbb{C})$. We will show the topology of V_H is equal to the relative topology of $GL(n,\mathbb{C})$. It is enough to show for any $t \in [-\epsilon_1,\epsilon_1]^k$ such that for any $\alpha < \epsilon$

$$V_H \cap exp(\sum_{i=1}^{k} t_i X_i)exp(\sum_{i=1}^{k}(-\alpha,\alpha)X_i) = V_H \cap exp(\sum_{i=1}^{k} t_i X_i)exp(\sum_{i=1}^{N}(-\alpha,\alpha)X_i)$$

Let us fix any $t \in [-\epsilon_1,\epsilon_1]^k$ and $\alpha < \epsilon$ and

$$exp(\sum_{i=1}^{k} t_i X_i)u \in exp(\sum_{i=1}^{k} t_i X_i)exp(\sum_{i=1}^{N}(-\alpha,\alpha)X_i) \cap V_H$$

Because $exp(\sum_{i=1}^{k} -t_i X_i)exp(\sum_{i=1}^{k}[-\epsilon_1,\epsilon_1]X_i) \subset exp(\sum_{i=1}^{k}(-\epsilon,\epsilon)X_i)$ and exp is injective in $\sum_{i=1}^{N}(-\epsilon,\epsilon)X_i$,

$$u \in exp(\sum_{i=1}^{k}(-\epsilon,\epsilon)X_i)$$

So,

$$exp(\sum_{i=1}^{k} t_i X_i)u \in exp(\sum_{i=1}^{k} t_i X_i)exp(\sum_{i=1}^{k}(-\alpha,\alpha)X_i)$$

Consequently, H is a Lie group. Clearly $Lie(H) = \mathfrak{h}$. □

Proposition 2.51. *Let G be a Lie group and H is a closed subgroup of G. Then H is a Lie group.*

STEP1. Showing that H has at most countable connected components. For any $h \in H$, the connected component of H which contains h(called H_h) is contained some connected component of G.So, H has at most countable connected components. □

STEP2. Showing that H is a Lie group. We set

$$\mathfrak{h} := \{X \in M(n,\mathbb{C}) | Exp(tX) \in U \cap H \ (|t| \ll 1)\}$$

Because $U \cap H$ is closed, by the argument which is similar to the proof of Proposition2.3.3, $\mathfrak{h}$ is a Lie algebra. And clearly $\mathfrak{h}$ is a Lie subalgebra of $Lie(G)$. Let us take $X_1, ..., X_k, ..., X_m, ..., X_N$ which is a basis of $M(n,\mathbb{C})$ such that $X_1, ..., X_k$ is a basis of $\mathfrak{h}$ and $X_1, ..., X_m$ is a basis of $Lie(G)$. Because $U \cap H$ is closed and H satisfies the second countable axiom, by the argument which is similar to the proof of Lemma2.7 and Baker-Campbell-Hausdorff formula,

$$Exp(\mathfrak{h} \cap \sum_{i=1}^{k}(-\epsilon,\epsilon)X_i) = Exp(\sum_{i=1}^{m}(-\epsilon,\epsilon)X_i) \cap H = Exp(\sum_{i=1}^{N}(-\epsilon,\epsilon)X_i) \cap H$$

We set

$$V_H := Exp(\mathfrak{h} \cap \sum_{i=1}^{k}[-\frac{1}{2}\epsilon, \frac{1}{2}\epsilon]X_i)$$

So, by the argument which is similar to the proof of Theorem2.6, V_H is closed neighborhood of e and the relative topology of V_H to G is equal to the relative topology of V_H to $GL(n,\mathbb{C})$. So, by Proposition2.32, H is a Lie group and $\mathfrak{h} = Lie(H)$. □

2.6 Invariant measure

2.6.1 Existence of Invariant measure

Definition 2.19 (Baire measure). *Let X be a locally compact Housdorff space. We say μ is a Baire measure on X if*

$$C_c(X) \subset L^1(X, \mu)$$

Definition 2.20 (Invariant measure). *Let G be a locally compact topological group. We say μ is a left invariant measure on G if for any $f \in C_c(G)$ and any $g_0 \in G$*

$$\int_G f(g_0 g) d\mu(g) = \int_G f(g) d\mu(g)$$

We say μ is a right invariant measure on G or a right Haar measure on G if for any $f \in C_c(G)$ and any $g_0 \in G$

$$\int_G f(g g_0) d\mu(g) = \int_G f(g) d\mu(g)$$

We say G is unimodular if there is a left and right Haar measure on G. We call a left and right Haar measure on G a Haar measure on G.

We say μ is a right invariant measure on G

Notation 2.1. *Let G be a Lie group and $g_0 \in G$. For each $g \in G$ and $x \in G$, $L_{g_0}(x) := g_0 x$.*

Definition 2.21 (Left invariant form). *Let*

(S1) G is a Lie group and $m := Lie(G)$.

(S2) ω is a m-form on G.

We say ω is left invariant if for any $g \in G$ $dL_g \omega = \omega$. Here, for eahc $v_1, ..., v_m \in T_x(G)$,

$$(dL_g \omega)_x(v_1, ..., v_m) := \omega_{gx}(dL_g v_1, ..., dL_g v_m)$$

Lemma 2.8. *Let G be a Lie group and $m := Lie(G)$. And let us ω_e a antisymmetric m-th tensor at 1_G and $\omega \neq 0$. For each $x \in G$ and $v_1, ...v_m \in T_x(G)$,*

$$\omega_x(v_1, ..., v_m) := \omega_e(dL_x^{-1} v_1, ..., dL_x^{-1} v_m)$$

Then ω is a C^ω-class left invariant form.

Proof. Let us fix any $g, x \in G$ and $v_1, ..., v_m \in T_x(G)$.

$$\begin{aligned} & (L_g \omega)_x(v_1, ..., v_m) \\ = \ & \omega_{gx}(dL_g v_1, ..., dL_g v_m) \\ = \ & \omega_e(dL_{gx}^{-1} dL_g v_1, ..., dL_{gx}^{-1} dL_g v_m) \\ = \ & \omega_e(dL_x^{-1} dL_g^{-1} dL_g v_1, ..., dL_x^{-1} dL_g^{-1} dL_g v_m) \\ = \ & \omega_e(dL_x^{-1} v_1, ..., dL_x^{-1} v_m) = \omega_x(v_1, ..., v_m) \end{aligned}$$

□

Lemma 2.9. *Let*

(S1) G be a Lie group.

(S2) ω be a C^ω-class left invariant form.

(S3) $g \in G$.

(S4) (U_α, ψ_α) and (U_β, ψ_β) are local coordinates on G and $gU_\beta \cap U_\alpha \neq \phi$.

(S5) For any $x \in U_\alpha$ and $y \in U_\beta$

$$\omega_x = \Phi_\alpha(x) d\phi_{\alpha,1} \wedge ... \wedge d\phi_{\alpha,m}, \ \omega_y = \Phi_\beta(y) d\phi_{\beta,1} \wedge ... \wedge d\phi_{\beta,m}$$

Then, for any $x \in U_\beta \cap L_g^{-1} U_\alpha$,

$$\Phi_\beta(x) = det(J(\psi_\alpha \circ L_g \circ \phi_\beta)(\psi_\beta(x)))\Phi_\alpha(gx)$$

Proof. Let us fix any $x \in U_\beta \cap L_g^{-1}U_\alpha$. Then

$$\omega_x = \Phi_\beta(x)(d\phi_{\beta,1} \wedge ... \wedge d\phi_{\beta,m})_x$$

and

$$\omega_{gx} = \Phi_\alpha(gx)(d\phi_{\alpha,1} \wedge ... \wedge d\phi_{\alpha,m})_{gx}$$

So,

$$\omega_x((\frac{\partial}{\partial\psi_{\beta,1}})_x, ..., (\frac{\partial}{\partial\psi_{\beta,m}})_x) = \omega_{gx}(dL_g((\frac{\partial}{\partial\psi_{\beta,1}})_x), ..., dL_g((\frac{\partial}{\partial\psi_{\beta,m}})_x))$$

and

$$\omega_{gx}(dL_g((\frac{\partial}{\partial\psi_{\beta,1}})_x), ..., dL_g((\frac{\partial}{\partial\psi_{\beta,m}})_x)) = detJ(\psi_\alpha \circ L_g \circ \phi_\beta)(\psi_\beta(x))$$

These implies that

$$\Phi_\beta(x) = \Phi_\alpha(gx)detJ(\psi_\alpha \circ L_g \circ \phi_\beta)(\psi_\beta(x))$$

□

By following the argument of the proof of Lemma2.9 in reverse, we can show the following proposition.

Lemma 2.10. *Here are settings and assumptions.*

(S1) G is a Lie group.

(S2) $\{U_\alpha, \psi_\alpha\}_{\alpha\in\Lambda}$ is a system of local corrdinates of G.

(S3) $\{\Phi_\alpha\}_{\alpha\in\Lambda}$ is a family such that $\Phi_\alpha \in C^\infty(U_\alpha, \mathbb{R})$ $(\forall\alpha \in \Lambda)$.

(A1) Then, for any $g \in G$ and $x \in U_\beta \cap L_g^{-1}U_\alpha$,

$$\Phi_\beta(x) = det(J(\psi_\alpha \circ L_g \circ \phi_\beta)(\psi_\beta(x)))\Phi_\alpha(gx)$$

(S4) We set

$$\omega_x = \Phi_\alpha(x)d\phi_{\alpha,1} \wedge ... \wedge d\phi_{\alpha,m} \ (x \in U_\alpha, \alpha \in \Lambda)$$

Then ω is well-defined and C^ω left-invariant form.

Proposition 2.52. *Here are settings and assumptions.*

(S1) G is a Lie group.

(S2) ω is a C^∞ class form on G such that $\omega_g \neq 0$ $(\forall g \in G)$

(S3) μ is the measure on G induced by ω.

(A1) μ is left invariant.

Then ω is a left invariant form.

Proof. By Lemma2.9, There is a $\{U_\alpha, \psi_\alpha\}_{\alpha\in\Lambda}$ is a system of local corrdinates of G preserving the orientation of G and $\Phi_\alpha > 0$ on U_α $(\forall\alpha \in \Lambda)$ and $det(\phi_\alpha^{-1} \circ L_g \circ \psi_\beta) > 0$. Let us fix any $g \in G$ and $U_\beta \cap g^{-1}U_\alpha \neq \phi$. Let us fix any $f \in C_c(gU_\beta \cap U_\alpha)$. Because μ is left invariant,

$$\int_{U_\beta\cap g^{-1}U_\alpha} f(gx)d\mu(x) = \int_G f(gx)d\mu(x) = \int_G f(x)d\mu(x) = \int_{gU_\beta\cap U_\alpha} f(x)d\mu(x) = \int_{\psi_\alpha^{-1}(gU_\beta\cap U_\alpha)} f(\psi_\alpha(x))\Phi_\alpha(\psi_\alpha(x))dx$$

By change-of-variables formula for integral

$$\int_{U_\beta\cap g^{-1}U_\alpha} f(gx)d\mu(x) = \int_{\psi_\beta^{-1}(U_\beta\cap g^{-1}U_\alpha)} f(g\psi_\beta(y))\Phi_\beta(\psi_\beta(y))dy$$

$$= \int_{\psi_\alpha^{-1}(gU_\beta\cap U_\alpha)} f(\psi_\alpha(x))\Phi_\beta(g^{-1}\psi_\alpha(x))|det(\phi_\beta \circ L_g \circ \psi_\alpha)|^{-1}dx$$

So, for any $g \in G$ and $x \in U_\beta \cap L_g^{-1}U_\alpha$,

$$\Phi_\beta(x) = |det(J(\psi_\alpha \circ L_g \circ \phi_\beta)(\psi_\beta(x)))|\Phi_\alpha(gx)$$

Because $det(J(\psi_\alpha \circ L_g \circ \phi_\beta)(\psi_\beta(x))) > 0$,

$$\Phi_\beta(x) = det(J(\psi_\alpha \circ L_g \circ \phi_\beta)(\psi_\beta(x)))\Phi_\alpha(gx)$$

So, ω is left invariant form. □

Lemma2.9 implies the following.

Lemma 2.11. *Let G be a Lie group in which there is a left invariant form ω. Then G is orientable and ω is C^{ω}-class .*

Proof. By replacing two variables if necessary, there is a local coordinate system $\{U_\alpha, \psi_\alpha\}_{\alpha\in\Lambda}$ such that $\Phi_\alpha > 0$ $(\forall\alpha \in \Lambda)$. By Lemma2.9, $\{U_\alpha, \psi_\alpha\}_{\alpha\in\Lambda}$ preserves the orientation of G. □

Lemma 2.12. *Let*

- *(S1) M is a paracompact C^∞-class manifold.*
- *(S2) $H : M \to M$ is a C^∞-class homeomorphism.*
- *(S3) $\{U_\alpha\}_{\alpha\in\Lambda}$ is a open covering of M.*
- *(S4) f is a C^∞-class function on M.*
- *(A1) supp(f) is compact and there is $\alpha \in \Lambda$ such that supp$(f) \subset U_\alpha$.*

Then there are $\{U_{\beta_i}\}_{i=1}^N$ and $\{f_i\}_{i=1}^N \subset C^\infty(M)$ such that $\{H(U_{\beta_i})\}_{i=1}^N$ is a covering of supp(f) and

$$f = \sum_{i=1}^{N} f_i$$

and

$$supp(f_i) \subset U_\alpha,\ supp(f_i \circ H) \subset U_{\beta_i}\ (i = 1, 2, ..., N)$$

Proof. Because *supp*(f) is compact, there are $\{U_{\beta_i}\}_{i=1}^N$ such that $\{H(U_{\beta_i})\}_{i=1}^N$ is a covering of *supp*(f). Because *supp*(f) is paracompact and $\{H(U_{\beta_i})\}_{i=1}^N$ is a open covering of *supp*(f), there is $\{h_i\}_{i=1}^N \subset C^\infty(M)$ such that $\{h_i\}_{i=1}^N$ is a partition of unity which is subordinate to $\{H(U_{\beta_i})\}_{i=1}^N$. We set $f_i := h_i$ $(i = 1, 2, ..., N)$. Clearly $\{f_i\}_{i=1}^N$ satisfies the conditions in this Proposition. □

By Riesz-Markov-Kakutani representation theorem[8], any left invariant measure induces a measure.

Theorem 2.7. *Let*

- *(S1) G be a Lie group.*

Then

- *(i) There is C^∞-class left invariant form ω on G.*
- *(ii) G is orientable by ω.*
- *(iii) The measure induced from ω is left invariant. Specially, G has a left invariant measure.*

Proof. (i) is from Lemma2.8. (ii) is from Lemma2.11. We will show (iii). We set $m := Lie(G)$. Let us fix $f \in C_c^\infty(G)$ and $g_0 \in G$. For $x \in G$,

$$(L_{g_0} f)(x) := f(g_0 x)$$

By (ii) and the second contable axiom, there is $\{U_i, \psi_i, V_i, \Phi_i, \rho_i\}_{i=1}^\infty$ such that $\{U_i, \psi_i\}_{i=1}^\infty$ is a local coordinate system of G and $\{U_i, \psi_i\}_{i=1}^\infty$ is local finite and for each i $V_i \in \mathcal{O}(\mathbb{R}^m)$

$$\psi_i : U_i \to V_i$$

is an homeomorphism and $\{U_i, \psi_i\}_{i=1}^\infty$ preserves a orientation of G and for each i and $x \in U_i$

$$\omega_x = \Phi_i(x)(d\psi_{i,1} \wedge ... \wedge d\psi_{i,m})_x$$

and $\Phi_i > 0$ and $\{\rho_i\}_{i=1}^\infty$ is a partition of unity subordinating $\{U_i\}_{i=1}^\infty$. We set for each i, $f_i := f\rho_i$. By Lebesgue's convergence theorem,

$$\int_G f\omega = \sum_{i=1}^{\infty} \int_G f_i\omega,\ \int_G L_{g_0} f\omega = \sum_{i=1}^{\infty} \int_G L_{g_0} f_i\omega$$

So, it is enough to show for each i

$$\int_G f_i\omega = \int_G L_{g_0} f_i\omega$$

By Lemma 2.12, we can assume that for each i, there is j such that $supp(L_{g_0} f_i) \subset U_j$. Because $supp(f_i)$ is compact, there is an open set U_i' such that

$$supp(f_i) \subset U_i' \subset U_i$$

and

$$supp(L_{g_0} f_i) = L_{g_0}^{-1} supp(f_i) \subset L_{g_0}^{-1} U_i' \subset U_j$$

We set $\phi_i := \psi_i^{-1}$ and $V_i := \psi_i(U_i)$ and $\phi_j := \psi_j^{-1}$ and $V_j := \psi_j(U_j)$. By change-of-variables formula for integral and Lemma2.9,

$$\begin{aligned}
&\int_G L_{g_0} f_i \omega = \int_{\psi_j(L_{g_0}^{-1} U_i')} f_i(g_0 \phi_j(x)) \Phi_j(x) dx \\
= &\int_{\psi_j(L_{g_0}^{-1} U_i')} f_i(\phi_i(\psi_i(g_0 \phi_j(x)))) \Phi_j(x) dx \\
= &\int_{\psi_j(L_{g_0}^{-1} U_i')} f_i(\phi_i(\psi_i \circ L_{g_0} \circ \phi_j(x))) \\
&\times det(J(\psi_i \circ L_{g_0} \circ \phi_j))(\psi_j \circ L_{g_0}^{-1} \phi_i \circ \psi_i \circ L_{g_0} \circ \phi_j(x))))^{-1} \\
&\times \Phi_j(\psi_j \circ L_{g_0}^{-1} \phi_i \circ \psi_i \circ L_{g_0} \circ \phi_j(x)))) \\
= &\int_{V_i'} f_i(\phi_i(y)) det(J(\psi_i \circ L_{g_0} \circ \phi_j))(\psi_j \circ L_{g_0}^{-1} \circ \phi_i(y))^{-1} \\
&\times \Phi_j(\psi_j \circ L_{g_0}^{-1} \phi_i(y)) dy \\
= &\int_{V_i'} f_i(\phi_i(y)) \Phi_i(y) dy \\
= &\int_G f_i \omega
\end{aligned}$$

□

2.6.2 Haar measure

Theorem 2.8. *Let*

(S1) G be a Lie group with $m := dimLie(G)$.

(S2) ω^L is a left invariant m-form and ω^R is a right m-form on G.

(A1) $\omega_e^L = \omega_e^R$.

(S3) dg_L is the left invariant measure induced from ω^L. dg_R is the right invariant measure induced from ω^R.

Then

(i) $\omega^R = det(Ad(\cdot))\omega^L$.

(ii) $dg_R = |det(Ad(\cdot))| dg_L$. We set $\Delta_L(\cdot) := |det(Ad(\cdot))|$ and $\Delta_R(\cdot) := |det(Ad(\cdot))|^{-1}$.

Proof. It is enough to show (i). Let us fix any $g \in G$. and $v \in T_g(G)$ and $u := dL_g^{-1} v$. Then

$$\begin{aligned}
&\omega_g^R(v) = \omega_g^R(dL_g u) = \omega_e(dR_g dL_g u) = \omega_e(\iota(Ad(g)\iota^{-1}(u))) = det(Ad(g))\omega_e(u) \\
= \ &det(Ad(g))\omega_e(dL_g^{-1} v) = det(Ad(g))\omega^L(v)
\end{aligned}$$

This implies (i). □

Proposition 2.53. *Any compact Lie group is unimodular.*

Proof. Let us fix any G be a compact Lie group. Clearly, $|det(Ad(G))|$ is compact subgroup of $\mathbb{R}_{>0}^{\times}$. So, $|det(Ad(G))| = \{1\}$. □

2.6.3 Integral on all inverse elements

Proposition 2.54. *Let*

- *(S1) G is a Lie group.*
- *(S2) $I : G \ni g \mapsto g^{-1} \in G$.*
- *(S3) $f \in C_c(G)$.*
- *(S4) ω be a left invariant and right invariant form on G.*

then

$$\int_G f(g^{-1})\omega = \int_G f(g)\omega$$

STEP1. Construction of a left invariant form. We set $m := dim(Lie(G))$. Let us fix $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$ a system of local coordinates which preserves the orientation of G. Let us fix $\{a_\alpha\}_{\alpha\in\Lambda}$ such that for any $\alpha \in \Lambda$ $a_\alpha \in C^\infty(U_\alpha)$ and

$$\omega|U_\alpha = a_\alpha d\psi_\alpha^1 \wedge ... \wedge d\psi_\alpha^m$$

Then $\{(I(U_\alpha), \psi_\alpha \circ I^{-1})\}_{\alpha\in\Lambda}$ a system of local coordinates of G. For any $\alpha, \beta \in \Lambda$ such that $(I(U_\alpha) \cap (I(U_\beta) \neq \phi$,

$$\psi_\alpha \circ I^{-1} \circ (\psi_\beta \circ I^{-1})^{-1} = \psi_\alpha \circ \psi_\beta^{-1}$$

So, $\{(I(U_\alpha), \psi_\alpha \circ I^{-1})\}_{\alpha\in\Lambda}$ preserves the orientation of G.

We set ω' by

$$\omega'_g(u_1, u_2, ..., u_m) := \omega_{I^{-1}(g)}((dI)^{-1}_{I^{-1}(g)}u_1, ..., (dI)^{-1}_{I^{-1}(g)}u_m)$$

We will show ω' is left invarinat. Because ω is right invariant,

$$\begin{aligned}
&\omega'_{(L_x)(g)}((dL_x)_g v_1, ...(dL_x)_g v_m) = \omega'_{xg}((dL_x)_g v_1, ...(dL_x)_g v_m) = \omega_{I(xg)}((dI)^{-1}_{I(xy)}(dL_x)_g v_1, ..., (dI)^{-1}_{I(xy)}(dL_x)_g v_m) \\
= \;& \omega_{I(xg)}((dI)^{-1}_{I(xy)}(dL_I(x))^{-1}_g v_1, ..., (dI)^{-1}_{I(xy)}(dL_I(x))^{-1}_g v_m) = \omega_{I(xg)}(d(L_{I(x)} \circ I))^{-1}_{I(xy)} v_1, ..., d(L_{I(x)} \circ I))^{-1}_{I(xy)} v_m) \\
= \;& \omega_{I(xg)}(d(I \circ R_x)^{-1}_{I(xy)} v_1, ..., d(I \circ R_x)^{-1}_{I(xy)} v_m) = \omega_{R_{I(x)(I(g))}}(d(I \circ R_x)^{-1}_{I(xy)} v_1, ..., d(I \circ R_x)^{-1}_{I(xy)} v_m) \\
= \;& \omega_{R_{I(x)(I(g))}}((dR_x)^{-1}_{R_{I(x)(I(g))}}(dI)^{-1}_{I(g)} v_1, ..., (dR_x)^{-1}_{R_{I(x)(I(g))}}(dI)^{-1}_{I(g)} v_m) \\
= \;& \omega_{R_{I(x)(I(g))}}((dR_{I(x)})_{I(g)}(dI)^{-1}_{I(g)} v_1, ..., (dR_{I(x)})^{-1}_{I(g)}(dI)^{-1}_{I(g)} v_m) \\
= \;& \omega_{I(g)}(dI)^{-1}_{I(g)} v_1, ..., (dI)^{-1}_{I(g)} v_m) = \omega_{I^{-1}(g)}(dI)^{-1}_{I^{-1}(g)} v_1, ..., (dI)^{-1}_{I^{-1}(g)} v_m) = \omega'_g(v_1, ..., v_m)
\end{aligned}$$

So, ω' is left invariant. So, there is $C \in \mathbb{R}$ such that $\omega' = C\omega$. □

STEP2. Display of X using local coordinates.

$$\begin{aligned}
&\omega'_g(u_1, u_2, ..., u_m) = \omega_{I^{-1}(g)}((dI)^{-1}_{I^{-1}(g)}u_1, ..., (dI)^{-1}_{I^{-1}(g)}u_m) = \omega_e(d(L_{I^{-1}(g)})^{-1}_e(dI)^{-1}_{I^{-1}(g)}u_1, ..., d(L_{I^{-1}(g)})^{-1}_e(dI)^{-1}_{I^{-1}(g)}u_m) \\
= \;& \omega_e(d(I \circ L_{I^{-1}(g)})^{-1}_e u_1, ..., d(I \circ L_{I^{-1}(g)})^{-1}_e u_m) = \omega_e(d(L_g)^{-1}_e u_1, ..., d(L_g)^{-1}_e u_m)
\end{aligned}$$

For any $u_1, ..., u_m \in T_g(G)$,

$$\begin{aligned}
&\omega'_g(u_1, u_2, ..., u_m) = \omega_{I^{-1}(g)}((dI)^{-1}_{I^{-1}(g)}u_1, ..., (dI)^{-1}_{I^{-1}(g)}u_m) = \omega_{I^{-1}(g)}((dI)^{-1}_{I^{-1}(g)}u_1, ..., (dI)^{-1}_{I^{-1}(g)}u_m) \\
= \;& a_\alpha(I^{-1}(g))d\psi_\alpha^1 \wedge ... \wedge d\psi_\alpha^m((dI)^{-1}_{I^{-1}(g)}u_1, ..., (dI)^{-1}_{I^{-1}(g)}u_m) \\
= \;& a_\alpha(I^{-1}(g))d\psi_\alpha^1 \circ (dI)^{-1}_{I^{-1}(g)} \wedge ... \wedge d\psi_\alpha^1 \circ (dI)^{-1}_{I^{-1}(g)}(v_1, ..., v_m) \\
= \;& a_\alpha(I^{-1}(g))d(\psi_\alpha \circ I^{-1})^1_{I^{-1}(g)} \wedge ... \wedge d(\psi_\alpha \circ I^{-1})^m_{I^{-1}(g)}(v_1, ..., v_m)
\end{aligned}$$

this proposition holds. So,

$$\int_G f(g^{-1})\omega = \int_G f(g)\omega'$$

By setting $f = 1$, $\omega' = \omega$. So,

$$\int_G f(g^{-1})\omega = \int_G f(g)\omega$$

□

By the proof of Proposition2.54, the following holds.

Proposition 2.55. *Let*

- *(S1) G is a Lie group.*
- *(S2) $I : G \ni g \mapsto g^{-1} \in G$.*
- *(S3) $f \in C_c(G)$.*
- *(S4) ω be a left invariant on G.*

then

$$\int_G f(g^{-1})\omega = \int_G f(g)\Delta_R(g)\omega$$

2.6.4 $L^p(G)$

Proposition 2.56. *Let G be a Lie group. Then $L^p(G)$ is separable for any $p \in \mathbb{N} \cap [1, \infty)$.*

Proof. By Proposition2.31 there is $\{U_i\}_{i=1}^{\infty}$ which is a local finite open covering of G_1 and $\{\varphi_i\}_{i=1}^{\infty}$ is a partition of unity with respect to $\{U_i\}_{i=1}^{\infty}$ and for any i U_i is C^{∞}-class homeomorphic to $(0,1)^m$. For each i, $L^2(U_i)$ is separable. So, there is $\{f_{i,k}\}_{i,k} \subset C^{\infty}(G)$ such that $supp(f_{i,k}) \subset U_i$ $(\forall i, \forall k)$ and $\{f_{i,k}|U_i\}_k$ is dense in $L^p(U_i)$ $(\forall i)$. We set $A := \{\sum_{i=1}^{N} f_{i,k_i} | k_i \in \mathbb{N}\ (i = 1, 2, ..., N),\ N \in \mathbb{N}\}$. Clearly A is separable.

Let us fix any $f \in L^p(G)$. Let us fix any $\epsilon > 0$. Because $\lim_{N\to\infty} f * \chi_{\cup_{i=1}^{N} U_i} = f$ and $f \in L^p(G)$, by Lebesgue's convergence theorem, there is $N \in \mathbb{N}$ such that

$$||f - f * \chi_{\cup_{i=1}^{N} U_i}|| < \frac{\epsilon}{2}$$

We set $f_1 := f * \chi_{U_1}$ and $f_i := f * \chi_{U_i \setminus \cup_{k=i-1}^{N} U_k}$ $(i = 1, 2, ..., N)$. Then $f * \chi_{\cup_{i=1}^{N} U_i} = \sum_{i=1}^{N} f_i$. There are $f_{i,k_1}, ..., f_{i,k_N}$ such that $||f_i - f_{i,k_i}|| < \dfrac{\epsilon}{2N}$ $(i = 1, 2, ..., N)$. Clearly

$$||f * \chi_{\cup_{i=1}^{N} U_i} - \sum_{i=1}^{N} f_{i,k_i}|| < \frac{\epsilon}{2}$$

So, $||f - \sum_{i=1}^{N} f_{i,k_i}|| < \epsilon$. Consequently, $L^p(G_1)$ is separable. □

By the proof of Proposition2.56, the following holds.

Proposition 2.57. *Let G be a Lie group. Then there is at most countable subset of $C_c(G)$ which is dense in $L^p(G)$.*

2.6.5 Convolution

Definition 2.22 (Convolution of function and linear functional)**.** *Let*

- *(S1) G be a Lie group.*
- *(S2) $f \in C_c(G)$.*
- *(S3) T is a $\mathbb{C}$-linear functional on $C_c(G)$.*

Then

$$T * f(x) := T(\tau_x(f))\ (x \in G)$$

Here,

$$\tau_x(f)(y) = f(xy^{-1})\ (x, y \in G)$$

Notation 2.2 (Dirac delta function δ_x)**.** *Let G be a topological group and $x \in G$. We set δ_x by*

$$\delta_x(f) := f(x)\ (f \in C(G))$$

Definition 2.23 (Convolution of functions)**.** *Let G be a Lie group. Let us fix dg_r which is a right invariant measure on G. Let us fix $f, g \in C(G)$ and assume $supp(f)$ or $supp(g)$ is compact. We set*

$$f * g(x) := \int_G f(xy^{-1})g(y)dg_r(y)\ (x \in G)$$

Proposition 2.58. *We succeed notations in Definition2.23. Then* *(i)* $f * g \in C(G)$

(ii) *If* $f_1, f_2 \in C_c(G)$ *then* $f_1 * f_2 \in C_c(G)$ *and* $supp(f_1 * f_2) \subset supp(f_1)supp(f_2)$

(iii) *If* $f_3, f_3 \in C_c(G)$ *then* $(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3)$.

Proof of (i). Firstly let us assume $g \in C_c(G)$. Let us fix any $x \in G$ and $\epsilon > 0$.

$$f * g(x) = \int_G f(xy^{-1})g(y)dg_r(y) = \int_{supp(g)} f(xy^{-1})g(y)dg_r(y)$$

We set $K := dg_r(supp(g))$. Becase $f, g \in C(G)$, for each $y \in supp(g)$, there is $U_{x,y}$ and V_y such that $U_{x,y}$ is an open neighborhood of x and V_y is an open neighborhood of y and

$$|f(zw^{-1})g(w) - f(xw^{-1})g(w)| < \frac{\epsilon}{K+1} \ (\forall z \in U_{x,y}, \forall w \in V_y)$$

Because $supp(g)$ is compact, there are $V_{y_1}, ..., V_{y_n}$ such that $supp(g) \subset \cup_{i=1}^n V_{y_i}$. We set $U_x := \cap_{i=1}^n U_{x,y_i}$. Then clearly

$$|f(zw^{-1})g(w) - f(xw^{-1})g(w)| < \frac{\epsilon}{K+1} \ (\forall z \in U_x, \forall w \in V_y)$$

So,

$$|f * g(z) - f * g(x)| < \epsilon \ (\forall z \in U_x)$$

This means $f * g$ is continuous.

Firstly let us assume $f \in C_c(G)$. Let us fix any $x \in G$.

$$f * g(x) = \int_G f(xy^{-1})g(y)dg_r(y) = \int_G f((yx^{-1})^{-1})g(yx^{-1}x)dg_r(y) = \int_G f(y^{-1})g(yx)dg_r(y)$$
$$= \int_{supp(f)^{-1}} f(y^{-1})g(yx)dg_r(y)$$

So, we can prove continuity of $f * g$ by the argument which is similar to the proof in case $g \in C_c(G)$. □

Proof of (iii). Let us fix any $x \in G$.

$$(f_1 * f_2) * f_3(x) = \int_G f_1 * f_2(xy^{-1})f_3(y)dg_r(y) = \int_G \int_G f_1(xy^{-1}z^{-1})f_2(z)dg_r(z)f_3(y)dg_r(y)$$
$$= \int_G \int_G f_1(x(zy)^{-1})f_2(zyy^{-1})dg_r(z)f_3(y)dg_r(y) = \int_G \int_G f_1(xz^{-1})f_2(zy^{-1})dg_r(z)f_3(y)dg_r(y)$$

by Fubini Theorem

$$= \int_G f_1(xz^{-1}) \int_G f_2(zy^{-1})f_3(y)dg_r(y)dg_r(z) = \int_G f_1(xz^{-1})f_2 * f_3(z)dg_r(z) = f_1 * (f_2 * f_3)(x)$$

□

2.7 Various types of Lie group

2.7.1 Connected component of Lie group

Proposition 2.59. *Let*

(S1) G_1 *is a Lie group which is locally isomorphic to a linear Lie subgroup of* $GL(n, \mathbb{C})$ *and* G_1 *be connected.*

(A1) There is open neighborhood of 1_{G_1} U *such that for any* $x, y \in U$ $xy = yx$.

Then G_1 *is commutative.*

Proof. By Proposition2.31, we can assume that for any $g \in G_1$ there are $g_1, ..., g_M \in U$ such that $g = g_1 \cdot g_2 ... g_M$. Let us fix any $g = g_1 \cdot g_2 ... g_M$ and $h = h_1 \cdot h_2 ... h_N$ such that $g_1, ..., g_M, h_1, ..., h_N \in U$.

$$\begin{aligned} gh &= g_1 \cdot g_2 ... g_M \cdot h_1 \cdot h_2 ... h_N \\ &= h_1 \cdot h_2 ... h_N \cdot g_1 \cdot g_2 ... g_M \\ &= hg \end{aligned} \tag{2.7.1}$$

□

Proposition 2.60. *Let*

(S1) G_1 be a Lie group which is locally isomorphic to a linear Lie subgroup of $GL(n,\mathbb{C})$.

(S2) $G_{1,0}$ be the connected component of G_1.

Then $G_{1,0}$ is path-connected.

Proof. For sufficient small $\epsilon > 0$, $N(\epsilon) := Exp(B(O,\epsilon))$ is path-connected. Clearly, finite multiple of $N(\epsilon)$ is path-connected. So, by Proposition2.31, $G_{1,0}$ is path-connected. □

2.7.2 Reductive Lie group

Definition 2.24 (Reductive Lie group)**.** *Let $G \subset GL(n,\mathbb{C})$ be a linear Lie group. We say G is a reductive Lie group if for any $g \in G$ $\bar{g}^T \in G$. Let G be a Lie group. We say G is reductive if G is locally isomorphic to a reductive linear Lie group and G has finite connected components.*

The followings clearly hold.

Proposition 2.61. *Let $G \subset GL(n,\mathbb{C})$ be a linear Lie group and G be reductive. Then*

(i)

$$G = \{\bar{g}^T | g \in G\}$$

(ii)

$$Lie(G) = \{\bar{X}^T | X \in Lie(G)\}$$

Proof of (i). For any $g \in G$, $g = \overline{\bar{g}^T}^T$. So the above equation holds. □

Proof of (ii). For any $X \in Lie(G)$, $exp(t\bar{X}^T) = \overline{exp(tX)}^T$. So $Lie(G) = \{\bar{X}^T | X \in Lie(G)\}$. □

Proposition 2.62. *Let $\mathfrak{g}$ be a Lie algebra. We set*

$$(X,Y) := ReTr(X^T\bar{Y}) \ (X,Y \in \mathfrak{g})$$

then

(i) $(\cdot,\cdot)$ is an inner product on $\mathfrak{g}$.

(ii) $(ad(X)Y,Z) = (Y, ad(\bar{X}^T)Z)$ for any $X,Y,Z \in \mathfrak{g}$.

Proof of (i). For any $X,Y \in \mathfrak{g}$,

$$\begin{aligned}(Y,X) &= ReTr(Y^T\bar{X}) = ReTr(\bar{X}^TY) \\ &= ReTr(X^T\bar{Y}) = (X,Y) = \overline{(X,Y)}\end{aligned}$$

Also,

$$(X,X) = \sum_{i,j} |x_{i,j}|^2$$

So, (i) holds. □

Proof of (ii). Because $Tr(X^TY^T\bar{Z}) = Tr(\bar{Z}X^TY^T)$,

$$\begin{aligned}&(ad(X)Y,Z) = ReTr((XY-YX)^T\bar{Z}) \\ =\ & ReTr((Y^TX^T - X^TY^T)\bar{Z}) = ReTr(Y^TX^T\bar{Z} - Y^T\bar{Z}X^T) \\ =\ & ReTr(Y^T\overline{ad(\bar{X}^T)Z}) = (Y, ad(\bar{X}^T)Z)\end{aligned} \tag{2.7.2}$$

So, (ii) holds. □

Lemma 2.13. *Let $\mathfrak{g}$ be a Lie algebra and $\bar{\mathfrak{g}}^T = \mathfrak{g}$. For any $\mathfrak{h}$ which is an ideal of $\mathfrak{g}$, $\mathfrak{h}^\perp$ is also ideal. Here, we assume the inner product of $\mathfrak{g}$ is $(\cdot,\cdot)$.*

Proof. Let us fix any $X \in \mathfrak{g}$, $Y \in \mathfrak{h}^\perp$, $Z \in \mathfrak{h}$. By the assumption, $ad(\bar{X}^T)Z \in \mathfrak{h}$. By Proposition2.62,

$$(ad(X)Y,Z) = (Y, ad(\bar{X}^T)Z) = 0 \tag{2.7.3}$$

So $(ad(X)Y \in \mathfrak{h}^\perp$. □

Proposition 2.63. *Let G_1 is a reductive Lie group such that G_1 is locally isomorphic to G_2 which is linear Lie group of $GL(n,\mathbb{C})$. Then $Lie(G_1)$ is a reductive Lie algebra. And we denote the center of $Lie(G_1)$ by $\mathfrak{z}$ and denote $\langle[Lie(G_1), Lie(G_1)]\rangle$ by $\mathfrak{g}_1$. Then*

$$Lie(G_1) = \mathfrak{z} \oplus \mathfrak{g}_1 \tag{2.7.4}$$

and $\mathfrak{g}_1$ is a semisimple Lie algebra or $\{0\}$.

Proof. We set $\mathfrak{g} := Lie(G_1)$. If $Lie(G_1)$ has no trivial ideal, then $Lie(G_1)$ is reductive. Otherwise, $Lie(G_1)$ has a trivial ideal $\mathfrak{h}$. By Proposition2.13, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$. We set $\mathfrak{h}_1 := \mathfrak{h}$ and $\mathfrak{h}_2 := \mathfrak{h}^{\perp}$. If $\mathfrak{h}_1$ has a subset which is a not trivial ideal of $\mathfrak{h}_1$, by Proposition2.13, the subset is a not trivial ideal of $\mathfrak{g}$. By repeating the above argument, there are $\mathfrak{g}_1, ..., \mathfrak{g}_r, \mathfrak{g}_{r+1}, .., .\mathfrak{g}_m$ such that $\mathfrak{g}_1, ..., \mathfrak{g}_r, \mathfrak{g}_{r+1}, .., .\mathfrak{g}_m$ are ideals of $\mathfrak{g}$ and $\mathfrak{g}_1, ..., \mathfrak{g}_r$ are one-dimensional abelian Lie algebras and $\mathfrak{g}_{r+1}, ..., \mathfrak{g}_m$ are simple Lie algebras. So $\mathfrak{g}$ is reductive. Clearly $\mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_r$ is the center of $\mathfrak{g}$. Clearly $\langle[\mathfrak{g}, \mathfrak{g}]\rangle \subset \langle[\mathfrak{g}_{r+1}, \mathfrak{g}_{r+1}]\rangle \oplus ... \oplus \langle[\mathfrak{g}_m, \mathfrak{g}_m]\rangle$. So $\langle[\mathfrak{g}, \mathfrak{g}]\rangle \subset \mathfrak{g}_{r+1} \oplus ... \oplus \mathfrak{g}_m$. Because for each $j \in \{r+1, ..., m\}$ $\mathfrak{g}_j$ is simple Lie algebra, $\langle[\mathfrak{g}_j, \mathfrak{g}_j]\rangle = \mathfrak{g}_j$. So $\mathfrak{g}_{r+1} \oplus ... \oplus \mathfrak{g}_m \subset \langle[\mathfrak{g}, \mathfrak{g}]\rangle$. ☐

Proposition 2.64. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_m = \mathfrak{h}_1 \oplus ... \oplus \mathfrak{h}_n$ and $\mathfrak{g}_i$ and $\mathfrak{h}_j$ are ideal of $\mathfrak{g}$ and simple Lie algebras. Then $m = n$ and there is $\sigma : \{1, 2, ..., m\} \to \{1, 2, ..., m\}$ such that σ is bijective and $\mathfrak{g}_{\sigma(i)} = \mathfrak{h}_i$ ($\forall i \in \{1, 2, ..., m\}$).*

Proof. For each i, $\mathfrak{g}_1 \supset \langle[\mathfrak{g}_1, \mathfrak{g}_1]\rangle = \langle[\mathfrak{g}_1, \mathfrak{h}_1]\rangle \oplus ... \oplus \langle[\mathfrak{g}_1, \mathfrak{h}_n]\rangle$. Because $\langle[\mathfrak{g}_1, \mathfrak{g}_1]\rangle$ is not zero, there is $\sigma(1)$ such that $\left\langle[\mathfrak{g}_1, \mathfrak{h}_{\sigma(1)}]\right\rangle$ is not zero. Because $\left\langle[\mathfrak{g}_1, \mathfrak{h}_{\sigma(1)}]\right\rangle \subset \mathfrak{h}_1$ and $\mathfrak{h}_{\sigma(1)}$ is simple and $\mathfrak{g}_1$ is simple, $\mathfrak{g}_1 = \left\langle[\mathfrak{g}_1, \mathfrak{h}_{\sigma(1)}]\right\rangle = \mathfrak{h}_{\sigma(1)}$. By repeating the above argument, ☐

2.7.3 Discrete subgroup and Abelian Lie group

Definition 2.25 (Discrete subgroup)**.** *Let G is a topological group. We call $H \subset G$ a discrete subgroup of G if H is a subgroup of G and the relative of H to G is equal to the discrete topology.*

Proposition 2.65. *Let*

- *(S1) G_2 is a Lie group which is locally isomorphic to a linear Lie subgroup of $GL(n,\mathbb{C})$.*
- *(S2) H is a subgroup of G_1.*

then the followings equivalent.

- *(i) H is a discrete subgroup of G_1.*
- *(ii) There is an open neighborhood of 1_{G_1} U such that $U \cap H = \{1_{G_1}\}$.*
- *(iii) H is a closed subgroup of G_1 and H is a Lie group which is locally isomorphic to $\{1_{G_2}\}$. And $Lie(H) = \{0\}$.*

Proof of that (i) $\Longrightarrow$ (ii): Because $\{1_{G_1}\}$ is an open set of relative topology, there is an oen set U such that $\{1_{G_1}\} = U \cap H$. ☐

Proof of that (ii) $\Longrightarrow$ that H is closed set: There is U_1 such that U_1 is open neighborhood of 1_{G_1} and $U_1^{-1}U_1 \subset U$. There is U_2 such that U_2 is open neighborhood of 1_{G_1} and $U_2^{-1} \subset U_1$ and $U_2 \subset U_1$. Let us assume there is $g \in \bar{H} \setminus H$. There is $u \in U_2$ and $h \in H$ such that $gu = h$. So $g \in hU_1$. Because G_1 is a Housdorff space, there is U_3 such that U_3 is an open neighborhood of 1_{G_1} and $U_3 \subset U_2$ and $h^{-1}g \notin U_3^{-1}$. So $h \notin gU_3$. Because $g \in \bar{H}$, there is $h_2 \neq h$ such that $h_2 \in gU_3$. So there is $u_3 \in U_3$ such that $h_2 = gu_3$. So $h_2u_3^{-1} = hu^{-1}$. Because $h^{-1}h_2 \in U_2^{-1}U_3 \subset U$. So $h^{-1}h_2 \in U \cap H = \{1_{G_1}\}$. This implies $h = h_2$. This is contradiction. ☐

Proof of that H is a Lie group: Because of (ii), H is locally isomorphic to $\{1_{G_2}\}$. Because $\{1_{G_2}\}$ is a linear Lie group of $GL(n,\mathbb{C})$, H is a Lie group. ☐

Proof of that (ii) $\Longrightarrow$ that $Lie(H) = \{0\}$: By von-Neumann-Cartan's theorem, exp is locally injective. So $Lie(H) = \{0\}$. ☐

Proof of that (iii) $\Longrightarrow$ (ii): By von Neumann-Cartan's theorem, there is $\epsilon > 0$ such that

$$exp(B(O,\epsilon)) \cap \tau(H \cap U) = exp(Lie(H) \cap B(O,\epsilon)) = \{1_{G_2}\} \tag{2.7.5}$$

So

$$\begin{aligned} & \eta(exp(B(O,\epsilon) \cap V) \cap H \\ = \; & \eta(exp(B(O,\epsilon)) \cap \tau(H \cap U)) \\ = \; & exp(Lie(H) \cap B(O,\epsilon)) = \{1_{G_1}\} \end{aligned} \tag{2.7.6}$$

This means (ii). ☐

Proof of that (ii) $\implies$ (i): For any $h \in H$, $\{h\} = hU \cap H$. This means (i). $\square$

Proposition 2.66. *Let us fix any H which is a discrete subgroup of $\mathbb{R}^n$. Then there are linearly independent subset $X_1, ..., X_r \subset \mathbb{R}^n$ such that $H = \sum_{i=1}^{r} \mathbb{Z}X_i$. $r = 0$ means $H = \{0\}$.*

Proof of that $n = 1$. We can assume $H \neq \{0\}$. There is $Y \in H \setminus \{0\}$. We set $t_0 := inf\{t > 0 | tY \in H\}$. We assume $t_0 = 0$. There is $\{t_i\} \subset (0, \infty)$ such that $\lim_{i\to\infty} t_i = 0$ and $t_iY \in H$ $(\forall i)$. Let us fix any $t > 0$. $tY = \lim_{i\to\infty} \lceil \frac{t}{t_i} \rceil t_iY$. Because H is closed, $tY \in H$. This implies $\mathbb{R}Y \subset H$ and $Y \neq \mathbf{0}$. This contradicts with H is a discrete subgroup.

So $t_0 > 0$. We set $X_1 := t_0Y$. We assume there is $X \in H \setminus \mathbb{Z}X_1$. There is $t \in H \setminus \mathbb{Z}$ such that $X = tX_1$. $(t - \lceil t \rceil)t_0Y = (t - \lceil t \rceil)X_1 \in H$. This contradicts with the definition of t_0. $\square$

Proof of that $n > 1$. We assume the Proposition is true if $n < N$ and $N \leq 1$. Let us take $X_1 \in H$ as in the $N = 1$ case. $(0, 1)X_1 \cap H = \phi$.

There is $X_2, ..., X_N \in \mathbb{R}^N$ such that $X_1, X_2, ..., X_N$ is a basis of $\mathbb{R}^N$. We set $H' := \{\boldsymbol{t'} \in \mathbb{R}^{N-1} | \exists s \in \mathbb{R}$ *such that* $sX_1 + \sum_{i=2}^{N} tX_i \in H\}$. Clearly H' is a subgroup of $\mathbb{R}^{N-1}$.

We assume H' is a not discrete subgroup of $\mathbb{R}^{N-1}$. By the same argument as above, there is a sequence $\{\boldsymbol{t_i}'\}_{i=1}^{\infty} \subset H'$ such that $\lim_{i\to\infty} \boldsymbol{t_i}' = 0$. Because $X_1 \in H$, there is a sequence $\{s_i\}_{i=1}^{\infty} \subset [-\frac{1}{2}, \frac{1}{2}]$ such that $s_iX_1 + \sum_{i=2}^{N} t_iX_i \in H$ $(\forall i)$. We can assume there is $s_0 \in [-\frac{1}{2}, \frac{1}{2}]$ such that $\lim_{i\to\infty} s_i = s_0$. Because H is closed, $s_0X_1 \in H$. By the definition of X_1, $s_0 = 0$. Because $s_iX_1 + \sum_{j=2}^{N} t_{i,j}X_j \in H \setminus \{0\}$ $(\forall i)$ and $\lim_{i\to\infty} s_iX_1 + \sum_{j=2}^{N} t_{i,j}X_j = 0$. This means H is a not discrete subgroup. This is contradiction. So H' is a discrete subgroup.

By the assumption of the mathematical induction, there is $Z_1, ..., Z_r \in \mathbb{R}^{N-1}$ such that $Z_1, ..., Z_r$ are linear independent and $H' = \sum_{i=1}^{r} \mathbb{Z}Z_i$. There are $s_1, ..., s_r \in \mathbb{R}$ such that $X'_{i+1} := s_iX_1 + \sum_{j=1}^{r} Z_{i,j}X_j \in H$ $(\forall i)$. Because

$$(X_1, X'_2, ..., X'_{r+1}) = (X_1, ..., X_N) \begin{pmatrix} 1 & s_1 & ... & s_r \\ 0 & z_{1,1} & ... & z_{r,1} \\ ... & ... & ... & ... \\ 0 & z_{1,N-1} & ... & z_{r,N-1} \end{pmatrix} \tag{2.7.7}$$

and the rank of $\begin{pmatrix} 1 & s_1 & ... & s_r \\ 0 & z_{1,1} & ... & z_{r,1} \\ ... & ... & ... & ... \\ 0 & z_{1,N-1} & ... & z_{r,N-1} \end{pmatrix}$ is $(r + 1)$, $X_1, X'_2, ..., X'_{r+1}$ are linear independent.

Let us fix any $X \in H$. Because $X_1, X_2, ..., X_N$ is a basis of $\mathbb{R}^N$, there are s and $t_2, ..., t_N$ such that $X = sX_1 + t_2X_2 + ... + t_NX_N$. Because $(t_2, ..., t_N) \in H'$, there are $m_2, ..., m_N \in \mathbb{Z}$ such that $(t_2, ..., t_N)^T = m_2Z_2 + ... + m_NZ_N$.

Because $X - \sum_{i=1}^{r} X'_i \in \mathbb{R}X_1 \cap H = \mathbb{Z}X_1$, $X \in \mathbb{Z}X_1 + \sum_{i=1}^{r} \mathbb{Z}X'_i$. Consequently, $H = \mathbb{Z}X_1 + \sum_{i=1}^{r} \mathbb{Z}X'_i$. $\square$

Proposition 2.67. *Let*

- *(S1) G_1 is a Le group which is locally isomorphic to a Lie subgroup of $GL(n, \mathbb{C})$.*
- *(A1) G_1 is connected.*

Then the followings are equivalent.

- *(i) G_1 is abelian.*
- *(ii) $Lie(G_1)$ is abelian.*

STEP1. Showing (i) $\implies$ (ii). Let us fix any $X, Y \in Lie(G_1)$. Because

$$\begin{aligned} & exp(t(X + Y) + t^2[X, Y] + O(t^3)) \\ = \;& exp(tX)exp(tY) \\ = \;& exp(t(X + Y) + t^2[Y, X] + O(t^3)) \end{aligned} \tag{2.7.8}$$

, $[X, Y] = [Y, X]$. So $Lie(G_1)$ is abelian. $\square$

STEP2. Showing (ii) $\Longrightarrow$ (i). There is $\epsilon > 0$ such that $exp(B(O,\epsilon))exp(B(O,\epsilon)) \subset V$. Let us fix any $g, h \in \eta(exp(B(O,\epsilon)))$. There is $X, Y \in B(O,\epsilon)$ such that $g = \eta(exp(X)),\ h = \eta(exp(Y)0$. Because X and Y are commutative,

$$\begin{aligned} gh &= \eta(exp(X))\eta(exp(Y)) \\ &= \eta(exp(X)exp(Y)) \\ &= \eta(exp(X+Y)) = \eta(exp(Y+X)) \\ &= \eta(exp(Y)exp(X)) = \eta(exp(Y))\eta(exp(X)) = hg \end{aligned} \tag{2.7.9}$$

By Proposition2.59, G_1 is abelian. □

Proposition 2.68. *Let*

(S1) G_1 is a Lie group.

(A1) G_1 is abelian.

(A2) G_1 is connected.

(S2) $N := dimLie(G_1)$.

Then there is $r \in \{1, 2, ..., n\}$ such that $\mathbb{T}^r \times \mathbb{R}^{N-r}$ is C^ω-class isomorhic as Lie group to G.

STEP1. Showing that $Exp : Lie(G_1) \to G_1$ is continuous and surjective. There is $\epsilon > 0$ such that for any $g \in G$ there are $exp(X_1), ..., exp(X_M) \in V_\epsilon := exp(B(O,\epsilon))$ which satisfies $g = exp(X_1)...exp(X_M)$. Because $Lie(G_1)$ and G_1 are commutative, $Exp : Lie(G) \to G_1$ is homomorphism of topological group.

Because Exp is a locally isomorphism from $Lie(G_1) \cap B(O,\epsilon) \to \eta(exp(B(O,\epsilon))) \cap V^\circ$, by Proposition2.31, Exp is surjective. □

STEP2. Showing that $Exp^{-1}(\{1_G\})$ is a discrete subgroup of $\mathbb{R}^N$. By von-Neumann-Cartan's theorem, there is $\epsilon > 0$ such that $exp^{-1}(\{1_G\}) \cap B(O,\epsilon) = O$. So $exp^{-1}(\{1_G\})$ is a discrete subgroup of $\mathbb{R}^N$. □

STEP3. exp is an open map. Because G is abelian, for any $X \in Lie(G)$ $exp(B(X,\epsilon)) = exp(X)exp(B(O,\epsilon))$. Because $exp(B(O,\epsilon))$ is open, exp is an open map. □

STEP4. Construction of a isomorphism of Lie groups. By Proposition2.66, there are $X_1, ..., X_N \in Lie(G)$ and r such that $X_1, ..., X_N$ is a basis of $Lie(G)$ and

$$exp^{-1}(\{1_G\}) = \sum_{i=1}^{r} \mathbb{Z}X_i \tag{2.7.10}$$

We set $i : \mathbb{T}^r \times \mathbb{R}^{N-r} \to G$ by

$$i(exp(i2\pi\theta_1), ..., exp(i2\pi\theta_r), \boldsymbol{t}) := exp(\sum_{i=1}^{r} \theta_i X_i + \sum_{i=r+1}^{N} t_i X_i) \tag{2.7.11}$$

By STEP3, i is an open map. So i is homeomorphism and isomoriphism of topological groups. By Proposition2.2, i is a C^ω-class isomorphism of Lie groups. □

2.7.4 Nilpotent Lie group

Definition 2.26 (Nilpotent Lie algebra, Lie group)**.** *Let G be a Lie group and $\mathfrak{g} := Lie(G)$. We set*

$$\mathfrak{g}_0 := \mathfrak{g},\ \mathfrak{g}_i := [\mathfrak{g}_{i-1}, \mathfrak{g},]\ (i = 1, 2, ...) \tag{2.7.12}$$

We call $\mathfrak{g}$ is a Nilpotent Lie algebra if there is $n \in \mathbb{N}$ such that $\mathfrak{g}_n = \{0\}$. We call G is a Nilpotent Lie group if G is connected and $Lie(G)$ is a Nilpotent Lie algebra.

Proposition 2.69. *Let G be a Lie subgroup of $GL(n\mathbb{C})$ and G be a Nilpotent Lie group. Then $Exp : Lie(G) \to G$ is surjective.*

Proof. Let us fix any $g \in G$. By Proposition2.31, there are $X_1, ..., X_m \in Lie(G)$ such that $g = exp(X_1)exp(X_2)...exp(X_m)$.

Let us fix any $X, Y \in Lie(G)$. By Baker-Campbell-Hausdorff formula, there is a polynomial $Z(t)$ sucht that for $|t| \ll 1$

$$exp(tX)exp(tY) = exp(Z(t)) \tag{2.7.13}$$

Because $exp(\cdot X)exp(\cdot Y)$ is holomorphic, the power series of $exp(\cdot X)exp(\cdot Y)$ is equal to the power series of $exp(Z(t))$. The convergence radius of the power series of $exp(Z(t))$ is ∞. By identity theorem of holomorphic function(see [6]),

$$exp(X)exp(Y) = exp(Z(1))$$

So exp is surjective. □

2.8 Universal covering group of Lie group

Proposition 2.70 (Universal covering group)**.** *Let*

(S1) G_1 is a Lie group which is locally isomorphic to a Lie subgroup of $GL(n,\mathbb{C})$ G_2.

(A1) G_1 is path-connected.

Let

$$\hat{G_1} := [([0,1],\{0\}),(G_1,\{1_{G_1}\})]$$

and for each $c_1, c_2 \in \hat{G_1}$ $c_1 \sim c_2$ if there is a homotop Φ from c_1 to c_2 such that

$$\Phi(s,0) = e,\ \ \Phi(s,1) = c_1(1) = c_2(2)\ (\forall s)$$

and

$$\tilde{G_1} := \hat{G_1}/\sim$$

and

$$p : \hat{G_1} \ni c \mapsto [c] \in \tilde{G_1}$$

and

$$q : \tilde{G_1} \ni [c] \mapsto c(1) \in G_1$$

and

$$[c_1]\cdot[c_2] := [c_1c_2]\ (for\ c_1, c_2 \in \hat{G_1})$$

Then

(i) There is a Lie group structure of $\tilde{G_1}$ such that $p : \tilde{G_1} \to G_1$ is locally isomorphism of Lie groups.

(ii) $Lie(G_1) = Lie(\tilde{G_1})$

STEP1. Showing $\sim$ is equivalent relationship on $\hat{G}_1$. It is easy to show by the fact homotop is equivalent relationship. □

STEP2. Showing the multiple operation of $\tilde{G}$ is well-defined. Let us fix any $c_1, d_1, c_2, d_2 \in \hat{G}$ such that $c_1 \sim c_2$ and $d_1 \sim d_2$. Then there is Φ_c, Φ_d such that Φ_c is a homotopy from c_1 to c_2 and Φ_d is a homotopy from d_1 to d_2. Because $\Phi_c \cdot \Phi_d$ is a homotopy from $c_1 \cdot d_1$ to $c_2 \cdot d_2$, $c_1 \cdot d_1 \sim c_2 \cdot d_2$. So, the multiple operation of $\tilde{G}$ is well-defined.

□

STEP3. Showing q is surjective. This is from (A1). □

STEP4. Showing $\tilde{G}_1$ is group. This is from the group structure on G_1. □

STEP5. Constructing the topology of $\tilde{G}_1$. There is $\epsilon > 0$ such that

$$Exp : Lie(G_1) \cap B(O,\epsilon) \to Exp(B(O,\epsilon)) \cap G_1$$

is C^ω-class homeomorphism and

$$\sup_{X \in B(O,\epsilon)} ||exp(X) - E|| < 1$$

For each $s \in [0,1]$, we set

$$W_{e,s} := \{[[0,1] \ni t \to Exp(tsX)] | X \in Lie(G_1) \cap sB(O,\epsilon)\}$$

and for each $\tilde{g} \in \tilde{G}_1$

$$W_{\tilde{g},s} := \tilde{g}W_{e,s}$$

We will show $\{W_{\tilde{g},s}\}_{\tilde{g}\in\tilde{G}_1, s\in[0,1]}$ satisfies the axiom of system of fundamental neighborhoods.

Let us fix any $[c][d] \in [c]W_{e,s}$, $[d] \in W_{e,s}$. Clearly, there is $s_1 \in [0,1]$ such that for any $t \in [0,1]$

$$d(t)Exp(s_1B(O,\epsilon)) \subset Exp(sB(O,\epsilon))$$

Let us fix any $X \in s_1B(O,\epsilon)$. We set $Z := d(1)Exp(X)$. Because $Exp(sB(O,\epsilon))$ is simply connected, $d(\cdot)Exp(\cdot X) \sim Exp(\cdot Z)$. This implies that

$$c(\cdot)d(\cdot)Exp(\cdot X) \sim c(\cdot)Exp(\cdot Z)$$

So,

$$[cd]W_{e,s_1} \subset [c]W_{e,s}$$

Let us fix any $[c_1][d_1] = [c_2][d_2] \in [c_1]W_{e,s_1} \cap [c_2]W_{e,s_2}$, $[d_1] \in W_{e,s_1}$ and $[d_2] \in W_{e,s_2}$. By the argument in the previous paragraph, there is $s_3 \in [0,1]$ such that

$$[c_1 d_1]W_{e,s_3} \subset [c_1]W_{e,s_1},\ [c_2 d_2]W_{e,s_3} \subset [c_2]W_{e,s_2}$$

So,

$$[c_1 d_1]W_{e,s_3} \subset [c_1]W_{e,s_1} \cap [c_2]W_{e,s_2}$$

□

STEP6. Showing that $\tilde{G}$ is a topological group. Firstly, we will show $\tilde{G}$ is Housdorff space. Let $[c]\tilde{G} \setminus \{e\}$. Because G is Housdorff space, there is $s \in (0,1]$ such that

$$e \notin c(1)Exp(B_m(O, s\epsilon))$$

So,

$$[e] \notin [c]W_{e,s}$$

Consequently, $\tilde{G}$ is Housdorff space.

□

STEP7. Showing that q is a local isomorphism. Because $ExpB_m(O,\epsilon)$ is simply connected,

$$q|_{W_{e,1}} : W_{e,1} \ni [c] \to c(1) \in Exp(B_m(O,\epsilon))$$

is injective. And clearly $q|_{W_{e,1}}$ is surjective. Because $ExpB_m(O,\epsilon)$ is simply connected, for any $s \in [0,1]$ and $[c] \in W_{e,1}$ such that $[c]W_{e,s} \in W_{e,1}$,

$$q([c]W_{e,s}) = c(1)ExpB_m(O, s\epsilon)$$

So, $q|_{W_{e,1}}$ is continuous and open map. Because Exp is continuous, there is $s_0 \in [0,1]$ such that

$$Exp(B_m(O, s_0\epsilon)Exp(B_m(O, s_0\epsilon) \subset Exp(B_m(O, s_0\epsilon))$$

Because $ExpB_m(O,\epsilon)$ is simply connected,

$$[c_1][c_2] \in W_{e,s_0} \iff c_1(1)c_2(1) \in Exp(B_m(O, s_0\epsilon))$$

Consequently, q is a local isomorphism. □

Showing that $\tilde{G}$ is path-connected. Let us fix any $[c] \in \tilde{G}$. We set, for each $s \in [0,1]$,

$$C(s) := [c(s\cdot)]$$

Then, clearly, C is a continuous path from $[\{e\}]$ to c. □

Proposition 2.71. *Let G be a path-connected topological group and $\tilde{G}$ be a universal covering group of G. Let us assume $*$ be the operation of $\pi(G)$. Then for any $c_1 \in C([0,1], G)$ such that $c(0) = e$ and $c_2 \in \pi(G)$,*

$$[c_1] \cdot [c_2] = [c_1] * [c_2] = [c_2] \cdot [c_1]$$

Proof. We set

$$\Phi_1(s,t) := c_1(L(s(2t-1)) + (1-s)t)c_2(L(2st) + (1-s)t)$$

and

$$\Phi_2(s,t) := c_2(L(s(2t-1)) + (1-s)t)c_1(L(2st) + (1-s)t)$$

Here,

$$L(u) := \begin{cases} 0 & (u \leq 0) \\ u & (0 \leq u < 1) \\ 1 & (u \geq 1) \end{cases}$$

Clearly, Φ_1 is a homotop from $c_1 \cdot c_2$ to $c_1 * c_2$ and Φ_2 is a homotop from $c_2 \cdot c_1$ to $c_1 * c_2$. □

By Proposition2.71, the following holds. We will show another proof using adjoint representation of Lie group.

Proposition 2.72. *Let G be a path-connected Lie group and $\tilde{G}$ be a universal covering group of G. Then $q^{-1}(e)$ is contained in the center of $\tilde{G}$. In special, $\pi(G)$ is commutative group.*

STEP1. Showing that $Ad(g) = id$ ($\forall g \in q^{-1}(e)$). Let us fix any $g_0 \in q^{-1}(e)$ and $Y \in Lie(\tilde{G})$. By the definition of Ad,

$$g_0 Exp(tY) g_0^{-1} = Exp(tAd(g_0)Y) \ (|t| \ll 1)$$

So,

$$Exp(t\iota(Y)) = q(Exp(tY)) = q(g_0 Exp(tY) g_0^{-1}) = q(Exp(tAd(g_0)Y)) = Exp(t\iota(Ad(g_0)Y))$$

This implies

$$\iota(Y) = \iota(Ad(g_0)Y)$$

Because q is a local isomorphism, ι is an isomorphism. So, $Y = Ad(g_0)Y$. □

STEP2. Showing that $q^{-1}(e)$ is contained in the center of $\tilde{G}$. Because $\tilde{(G)}$ is path-connected, it is enough to show g_0 is commutative with $Exp(B(O, \epsilon))$ for sufficient small $\epsilon > 0$.

$$g_0 Exp(Y) = g_0 Exp(Y) g_0^{-1} g_0 = Exp(Ad(g_0)Y) g_0 = Exp(Y) g_0$$

□

Theorem 2.9. *Let*

- *(S1) $G_{i,1}$ is a Lie group which is locally isomorphic to a Lie subgroup of $GL(n, \mathbb{C})$ $G_{i,2}$ $(i = 1, 2)$.*
- *(A1) $Lie(G_{1,1})$ and $Lie(G_{2,1})$ are isomorphic as Lie algebras.*

then $G_{1,1}$ and $G_{2,1}$ are isomorphic as Lie groups.

2.9 Compact Lie group

Definition 2.27 (Killing form)**.** *Let $\mathfrak{g}$ be a Lie algebra. We set*

$$[X, Y] := Trace(ad(X)ad(Y))$$

3 Irreducible decomposition of unitary representation

3.1 Some facts admitted without proof

In this subsection, We will present some facts that we will use without proof in the pages that follow.

For the following Proposition, see [?].

Proposition 3.1 (Shur LemmaII)**.** *Let G be a topological group and (π, V) be an continuous irreducible representation of G and $A : V \to V$ be a continuous intertwining operator with respect to G such that $A \neq 0$. Then there is $\lambda \in \mathbb{C}$ such that $A = \lambda I$.*

Definition 3.1 (Extreme point)**.** *Let*

(S1) V is a vector space on $\mathbb{C}$.

(S2) A is a convex set of V.

(S3) $x \in A$.

We say x is an extreme point of A if for any $y, z \in A$ and $\lambda \in [0, 1]$ such that $x = \lambda y + (1 - \lambda)z$ $x = y = z$. We denote the set of all extreme points of A by $Ex(A)$.

Definition 3.2 (Extreme set)**.** *Let*

(S1) V is a vector space on $\mathbb{C}$.

(S2) A is a convex set of V.

(S3) $B \in A$.

We say B is an extreme set of A if for any $y, z \in A$ and $\lambda \in [0, 1]$ such that $x = \lambda y + (1 - \lambda)z \in B$ then $y, z \in B$.

For the following three Propositions, see [5].

Theorem 3.1 (S.Mazur Theorem)**.** *Let*

(S1) $(V, \{p_n\}_{n\in\mathbb{N}})$ is a semi-normed space on $\mathbb{R}$.

(S2) $x_0 \in V$.

(S3) $A \subset V$ is a closed convex subset with $x_0 \notin A$.

Then there is real-valued continuous linear function f such that $f(x_0) = 1$ and $|f(x)| < 1$ $(\forall x \in A)$.

Proposition 3.2. *Let*

(S1) $(V, \{p_n\}_{n\in\mathbb{N}})$ is a semi-normed space.

(S2) f is a real-valued continuous linear functional on V.

(S3) K is a compact convex subset of V.

Then $\{x \in K | f(x) = \max\{f(x)|x \in K\}\}$ is an extreme set of K.

Proposition 3.3 (Krein-Millman Theorem)**.** *Let*

(S1) $(V, \{p_n\}_{n\in\mathbb{N}})$ is a semi-normed space.

(S2) K is a compact convex subset of V.

(S3) $Ex(K)$ is the set of all extreme ompact convex subset of V.

Then

(i) $Ex(K)$ is not empty.

(ii) K is the closure of the convex full of $Ex(K)$.

Theorem 3.2 (Stone Weierstrass Theorem, lattice version)**.** *Let*

(S1) X is a compact metric space.

(S2) V is a $\mathbb{R}$-vector subspace of $C(X, \mathbb{R})$.

(A1) $\vee$ means the pointwise maximum. Then $f \vee g \in V$ $(\forall f, g \in V)$.

(A2) For any $x, y \in X$ such that $x \neq y$, there is $f \in V$ such that $f(x) \neq f(y)$.

Then V is dense in $C(X, \mathbb{R})$.

3.2 Continuity of representation

3.2.1 Baire Category Theorem

Theorem 3.3 (Baire Category Theorem)**.** *Let*

(S1) X is a complete metric space.

(S2) $\{A_n\}_{n=1}^{\infty}$ is a sequence of closed sets of X such that $A_n \subset A_{n+1}$ $(\forall n \in \mathbb{N})$.

(A1) $X = \cup_{n=1}^{\infty} A_n$.

Then there is $n \in \mathbb{N}$ such that $A_n^{\circ} \neq \phi$.

Proof. Let us assume

$$A_n^{\circ} = \phi \ (\forall n \in \mathbb{N}) \tag{3.2.1}$$

Let us fix $x_0 \in A_1$. In this proof, for each $x \in X$ and $\epsilon > 0$ we denote $D(x, \epsilon) := \{y \in X | d(x, y) \leq \epsilon\}$. Then there is $x_1 \in B(x_0, 1) \setminus A_1$. Because A_1^c is an open set, there is $\varphi(1) \in \mathbb{N} > 1$ such that $D(x_1, \frac{1}{\varphi(1)}) \subset A_1^c \cap B(x_0, 1)$. If you repeat this procedure in the same way below, there is $\varphi : \mathbb{N} \to \mathbb{N}$ and $\{x_n\}_{n=1}^{\infty} \subset X$ such that φ is narrow sense monotonically increasing and $D(x_n, \frac{1}{\varphi(n)}) \subset A_n^c \cap B(x_{n-1}, \frac{1}{\varphi(n-1)})$ $(\forall n \in \mathbb{N})$. Because clearly $\{x_n\}_{n=1}^{\infty}$ is a cauchy sequence, $x_\infty := \lim_{n\to\infty} x_n$ exists. By (A1), there is $n \in \mathbb{N}$ such that $x_\infty \in A_n$. Because $x_m \in D(n, \frac{1}{\varphi(n)}) \subset A_n^c$ $(\forall m \geq n)$, $x_\infty \in D(n, \frac{1}{\varphi(n)}) \subset A_n^c$. This is contradiction. □

3.2.2 Uniform boundedness principle

Theorem 3.4 (Uniform boundedness principle)**.** *Let*

(S1) X is a banach space.

(S2) Y is a normed space.

(S3) $\{T_\lambda\}_{\lambda\in\Lambda} \subset B(X, Y)$.

(A1) For any $v \in X$, $\{||T_\lambda v||\}_{\lambda\in\Lambda}$ is bounded.

Then $\{||T_\lambda||\}_{\lambda\in\Lambda}$ is bounded.

Proof. We set $A_n := \{v \in X| \ ||T_\lambda v|| \leq n \ (\forall \lambda \in \Lambda)\}$ $(n \in \mathbb{N})$. $\{A_n\}_{n=1}^{\infty}$ satisfies the assumptions in Baire category thereom. By Baire category thereom, there is $n \in \mathbb{N}$ such that $A_n^{\circ} \neq \phi$. So there is $v_0 \in X$ and $\epsilon > 0$ such that $B(v_0, 2\epsilon) \subset A_n$. For any $\lambda \in \Lambda$ and $w \in X$ such that $||w|| = 1$,

$$\begin{aligned}
&||T_\lambda w|| = ||\frac{1}{\epsilon}T_\lambda(\epsilon w + v_0) - \frac{1}{\epsilon}T_\lambda v_0|| \\
&\text{because } v_0, w + v_0 \in B(v_0, \epsilon) \\
&= ||\frac{1}{\epsilon}T_\lambda(\epsilon w + v_0) - \frac{1}{\epsilon}T_\lambda v_0|| \leq ||\frac{1}{\epsilon}T_\lambda(\epsilon w + v_0)|| + ||\frac{1}{\epsilon}T_\lambda v_0|| \leq \frac{n}{\epsilon} + \frac{n}{\epsilon} = \frac{2n}{\epsilon}
\end{aligned}$$

So, $||T_\lambda|| \leq \frac{2n}{\epsilon}$ $(\forall \lambda \in \Lambda)$ □

3.2.3 Weakly continuity of representation

Theorem 3.5. *Let*

(S1) G is a local compact topological group.

(S2) (π, V) is a representation of G.

(A1) For any $u \in V$, $G \ni g \mapsto \pi(g)u \in \mathbb{C}$ is continuous.

Then (π, V) is continuous.

Proof. Let us fix U_0 which is a local compact neighborhood of e. By (A1) and uniform boundedness principle,

$$\sup_{g\in U_0} ||\pi(g)|| < \infty$$

Let us fix any $\epsilon > 0$ and $g_0 \in G$ and $u_0 \in V$. By (A1), there is U_1 which is an open neighborhood of e such that $U_1 \subset U_0$

$$||\pi(g_0U_1)u_0 - u_0|| < \frac{\epsilon}{2}$$

So, for any $x \in U_1$ and $u \in B(u_0, \frac{\epsilon}{2(\sup_{g\in U_0} ||\pi(g)|| + 1)})(||\pi(g_0)|| + 1)$,

$$||\pi(g_0x)u - \pi(g_0)u_0|| \leq ||\pi(g_0x)u - \pi(g_0x)u_0|| + ||\pi(g_0x)u_0 - \pi(g_0)u_0|| < ||\pi(g_0)||_{op}||\pi(x)||_{op}||u - u_0|| + \frac{\epsilon}{2} < \epsilon$$

□

In speciality, the following holds. However, this theorem can be proved without using Theorem3.5. The proof is given below.

Theorem 3.6. *Let*

- *(S1) G is a topological group.*
- *(S2) (π, V) is a unitary representation of G.*
- *(A1) For any $u, v \in V$, $G \ni g \mapsto (\pi(g)u, v) \in \mathbb{C}$ is continuous.*

Then (π, V) is continuous.

Proof. Let us fix any $u \in V$ and $g \in G$. Let us fix any $v \in B(u, \frac{\epsilon}{12(2||u|| + 1)})$. There is U which is an open neighborhood of e such that

$$|(\pi(g^{-1}h)u, u) - ||u||^2| \leq \frac{\epsilon}{2}$$

By (S2), for any $h \in gU$ and $v \in B(u, \frac{\epsilon}{2(||u|| + 1)})$,

$$||\pi(h)u - \pi(g)v||^2 = ||u||^2 - 2Re(\pi(g^{-1}h)u, v) + ||v||^2 = ||u||^2 - 2Re(u, v) + ||v||^2 + 2Re(u, v) - 2Re(\pi(g^{-1}h)u, v)$$
$$= ||u - v||^2 + 2Re(u - \pi(g^{-1}h)u, v) = ||u - v||^2 + 2Re(u - \pi(g^{-1}h)u, u) + 2Re(u - \pi(g^{-1}h)u, v - u)$$
$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + 2||u - \pi(g^{-1}h)u||||v - u|| \leq \frac{2\epsilon}{3} + 2(||u|| + ||\pi(g^{-1}h)u||)||u - v|| = \frac{2\epsilon}{3} + 2(||u|| + ||u||)||u - v||$$
$$= \frac{2\epsilon}{3} + 4||u||||u - v|| \leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

So, (π, V) is continuous. □

3.3 Cyclic representation and Unitary dual

Definition 3.3 (Cyclic representation)**.** *Let G be a topological group and (π, V) be a continuous representation of G. We say (π, V) is a cyclic representation of G if there is $v \in V$ such that*

$$\overline{\{\sum_{i=1}^{N} \pi(g_i)v | g_1, ..., g_N \in G\}} = V$$

Clearly the following holds.

Proposition 3.4. *Let G be a topological group. Any continuous irreducible representation of G is a cyclic representation.*

By Proposition2.31, the following holds.

Proposition 3.5. *Let G be a Lie group and (π, V) be a continuous cyclic representation of G. Then V is countable. In speaciality, if π is unitary representation and $dim\pi = \infty$, then $V \simeq l^2$ as Hilbert space.*

By Proposition3.5, we can set of all continuous irreducible unitary representations of a Lie group.

Notation 3.1. *Let G be a Lie group. We set*

$$\Omega_c := \{(\pi, V) |\ V \text{ is closed subspace of } l^2 \text{ and } (\pi, V) \text{ is a continuous cyclic representation of } G\}$$

Definition 3.4 (Unitary dual)**.** *Let G be a Lie group. We set*

$$\hat{G} := \{(\pi, V) |\ V \text{ is closed subspace of } l^2 \text{ and } (\pi, V) \text{ is a continuous irreducible representation of } G\}/\simeq$$

Here, $\simeq$ is the isomorphic relation as unitary representations. We call $\hat{G}$ the unitary dual of G.

Proposition 3.6. *Let*

(S1) G is a Lie group.

(S2) (π_i, V_i) is a continuous unitary cyclic representation of G with cyclic vector v_i such that $||v_i|| = 1$ $(i = 1, 2)$.

(A1) $(\pi_1(g)v_1, v_1) = (\pi_2(g)v_2, v_2)$ $(\forall g \in G)$.

Then (π_1, V_1) and (π_2, V_2) are isomorphic as continuous unitary representation of G.

STEP1. Construction of orthonormal basis of V_1. Let $\{g_i\}_{i=1}^{\infty}$ is a dense subset of G. We set $\{h_i\}_{i=1}^{\infty}$ is a subgroup of G generated by $\{g_i\}_{i=1}^{\infty}$. There is a $\{f_i\}_{i=1}^{\infty} \subset \{h_i\}_{i=1}^{\infty}$ such that $\{\pi_1(f_i)v_1\}_{i=1}^{\infty}$ is a basis of the vector space W_1 which is generated by $\{\pi_1(h_i)v_1\}_{i=1}^{\infty}$. We take $\{w_i\}_{i=1}^{\infty}$ which is the orthonormal basis of W_1 by Gram-Schmit orthogonalization.

At the end of this step, we will show $\{\pi_2(f_i)v_2\}_{i=1}^{\infty}$ is a basis of the vector space W_2 which is generated by $\{\pi_2(h_i)v_2\}_{i=1}^{\infty}$. For showing this proposition, it is enough to show for each $a_1, ..., a_N \in \mathbb{C}$

$$\sum_{i=1}^{N} a_i \pi_1(f_i)v_1 = 0 \iff \sum_{i=1}^{N} a_i \pi_2(f_i)v_2 = 0 \tag{3.3.1}$$

Because of (S_2) and (A_1),

$$\sum_{i=1}^{N} a_i \pi_1(f_i)v_1 = 0 \iff (\sum_{i=1}^{N} a_i \pi_1(f_i)v_1, \pi_1(g)v_1) = 0\ (\forall g \in G) \iff \sum_{i=1}^{N} a_i(\pi_1(g^{-1}f_i)v_1, v_1) = 0\ (\forall g \in G)$$
$$\iff \sum_{i=1}^{N} a_i(\pi_2(g^{-1}f_i)v_2, v_2) = 0\ (\forall g \in G) \iff (\sum_{i=1}^{N} a_i \pi_2(f_i)v_1, \pi_2(g)v_1) = 0\ (\forall g \in G) \iff \sum_{i=1}^{N} a_i \pi_2(f_i)v_2 = 0$$

So, (3.3.1) holds. □

STEP2. Construction of orthonormal basis of V_2. By (A1), clearly

$$||\sum_{i=1}^{N} a_i \pi_1(f_i)v_1||_{V_1} = ||\sum_{i=1}^{N} a_i \pi_2(f_i)v_2||_{V_2}\ (\forall a_1, ..., a_N \in \mathbb{C}) \tag{3.3.2}$$

We set, for each $w_i = \sum_{j=1}^{N_i} a_{i,j}\pi_1(f_j)v_1$,

$$w'_i := \sum_{j=1}^{N_i} a_{i,j}\pi_2(f_j)v_2$$

We will show $\{w'_i\}_{i=1}^{\infty}$ is an orthonormal basis of V_2. By (A_1), $\{w'_i\}_{i=1}^{\infty}$ is clearly orthonormal. Let us fix any $k \in \mathbb{N}$. Then there are $a_1, ..., a_N \in \mathbb{C}$ such that

$$\pi_1(f_k)v_1 = \sum_{i=1}^{N} a_i w_i$$

Because $w_i \in W_1$, by (3.3.1),

$$\pi_2(f_k)v_2 = \sum_{i=1}^{N} a_i w'_i$$

So, $\{w'_i\}_{i=1}^{\infty}$ is an orthonormal basis of V_2. □

STEP3. Construction of isomorphism. We set

$$\Phi(\sum_{i=1}^{N} a_i w_i) := \sum_{i=1}^{N} a_i w_i' \ (a_1, ..., a_N \in \mathbb{C})$$

Clearly Φ is an unitary isomorphism between Hilbert spaces. We will show Φ is G-linear. Because $w_1 = v_1$ and $w_1' = v_2$,

$$\Phi(v_1) = v_2$$

Let us fix any $i \in \mathbb{N}$. Then there are $a_1, ..., a_n \in \mathbb{N}$ such that

$$\pi_1(g_i)v_1 = \sum_{j=1}^{n} a_j w_j$$

Because $w_i \in W_1$, by (3.3.1),

$$\pi_2(g_i)v_2 = \sum_{j=1}^{n} a_j w_j'$$

So,

$$\Phi(\pi_1(g_i)v_1) = \pi_2(g_i)\Phi(v_1)$$

Because W_1 is dense in V_1 and Φ is unitary, Φ is G-linear. $\square$

Proposition 3.7. *Let (π, V) be a continuous unitary representation of a topological groupG. Then there is a subset of G-invariant cyclic subspaces D such that*

$$V = \overline{\bigoplus_{W \in D} W}$$

Proof. We denote the all of nonzero invariant closed cyclic subspaces by $\mathfrak{D}$. Clearly $\mathfrak{D} \neq \phi$. We set

$$\mathfrak{T} := \{D \subset \mathfrak{D} | v_i \in W_i (i = 1, 2, ..., N), \{W_i\}_{i=1}^{N} \text{ is a distinct subset of } D, \sum_{i=1}^{N} v_i = 0 \implies v_i = 0\ (\forall i)\}$$

Let us fix any every totally ordered subset of $\mathfrak{T}$, T. Clearly $\cup_{D \in T} D \in \mathfrak{T}$. So, by Zorn's lemma, $\mathfrak{T}$ has a maximum element D. We set $V_0 := \bigoplus_{W \in D} W$. Let us assume $V_0^{\perp}$ is nonzero. Then $V_0^{\perp}$ has a nonzero invariant closed cyclic subspace W. Clearly, $D \cup \{W\} \in \mathfrak{T}$. This contradicts that D is a maximum element. So, $V_0^{\perp} = \{0\}$ and $V = \bar{V_0}$. $\square$

3.4 $*$-weak topology of $L^1(G)$

Definition 3.5 ($*$-weak topology)**.** *Let V be a normed space. We denote the weakest topology in which for any $x \in V$ $V^* \ni f \mapsto f(x) \in \mathbb{C}$ is continuous by $\mathcal{O}_w(V^*)$. We call this topology $*$-weak topology of V^*.*

Clearly the following two propositions holds.

Proposition 3.8. *Let V be a normed space. $\mathcal{O}_w(V^*)$ is induced by the family of seminorms $\{\cdot(x)\}_{x \in V}$.*

Proposition 3.9. *Let V be a separable normed space and $\{x_n\}_{n \in \mathbb{N}}$ be a dense subset of V. Then*

$$d : V^* \times V^* \ni (f, g) \mapsto \sum_{n=1}^{\infty} \frac{|f(x_n) - g(x_n)|}{1 + |f(x_n) - g(x_n)|} \in [0, \infty)$$

is a metric on V^* and $\mathcal{O}_w(V^*)$ is induced by d.

Theorem 3.7 (Banach-Alaoglu theorem)**.** *Let V be a separable normed space and $\{x_n\}_{n \in \mathbb{N}}$ be a dense subset of V. Then $B := \{f \in V^* | ||f|| \leq 1\}$ is a compact subset in $\mathcal{O}_w(V^*)$.*

Proof. Because $(V^*, \mathcal{O}_w)$ is metrizable, it is enough to show $(V^*, \mathcal{O}_w)$ is sequencial compact. Let us fix any $\{f_n\}_{n \in \mathbb{N}} \subset B$. By the same argument as the proof of Proposition1.19, there is a subsequence $\{g_n\}_{n\mathbb{N}} = \{f_{\varphi}(n)\}_{n \in \mathbb{N}}$ such that for any $i \in \mathbb{N}$ $\lim_{n \to \infty} g_n(x_i)$ exists.

Let us fix $x \in V$ and $\epsilon > 0$. Let us fix x_i such that $||x - x_i|| < \frac{\epsilon}{3}$. Because $\{g_n(x_i)\}_{n \in \mathbb{N}}$ is a cauchy sequence, there is $n_0 \in \mathbb{N}$ such that $|g_n(x_i) - g_m(x_i)| < \frac{\epsilon}{3}$ $(\forall m, n \geq n_0)$. Then for any $m, n \geq n_0$

$$|g_m(x) - g_n(x)| \leq |g_m(x) - g_m(x_i)| + |g_m(x_i) - g_n(x_i)| + |g_n(x) - g_n(x_i)| \leq 2||x -_x i|| + \frac{\epsilon}{3} \leq \epsilon$$

So $\{g_n(x)\}_{n\in\mathbb{N}}$ is a cauchy sequence. This implies $\lim_{n\to\infty} g_n(x)$ exists. We set

$$g(x) := \lim_{n\to\infty} g_n(x) \ (x \in V)$$

Cearly $||g|| \leq 1$ and $w - \lim_{n\to\infty} g_n = g$. □

3.5 Positive definite function on a group

3.5.1 Definition and Basic properties

Definition 3.6 (Positive definite function on a group)**.** *Let G be a group and $\varphi \in C(G,\mathbb{C})$. We say φ is positive definite if for any $n \in \mathbb{C}$ and $g_1, g_2, ..., g_n \in G$ and $c_1, c_2, ..., c_n \in \mathbb{C}$*

$$\sum_{j,k} c_j \bar{c_k} \varphi(g_j^{-1} g_k) \geq 0 \tag{3.5.1}$$

Example 3.1. *Let G be a group and (π, V) be a unitary representation of G and $v \in V$. Then the following is a positive definite function.*

$$(\pi(\cdot)v, v) \tag{3.5.2}$$

Proof. For any $n \in \mathbb{C}$ and $g_1, g_2, ..., g_n \in G$ and $c_1, c_2, ..., c_n \in \mathbb{C}$

$$\sum_{j,k} c_j \bar{c_k} (\pi(g_j^{-1} g_k)v, v) = \sum_{j,k} c_j \bar{c_k} (\pi(g_k)v, \pi(g_j)v) = (\sum_k \bar{c_k} \pi(g_k)v, \sum_j \bar{c_j} \pi(g_j)v) = ||\sum_k \bar{c_k} \pi(g_k)v||^2 \geq 0$$

□

Proposition 3.10. *Let G be a group and φ is a positive definite function on G. Then*

(i) $\varphi(e) \geq 0$

(ii) $\varphi(g^{-1}) = \overline{\varphi(g)}$

(iii) $|\varphi(g)| \leq \varphi(e)$

(iv) $|\varphi(g_1) - \varphi(g_2)|^2 \leq \frac{1}{2}\varphi(e)|\varphi(e) - Re\varphi(g_1^{-1} g_2)|$

Proof of (i). We succeed in the notation of Definition3.1. By setting $n = 1$ and $g_1 = e$ and $c_1 =$, (i) holds. □

Proof of (ii). By setting $n = 2$ and $g_1 = e$ and $g_2 = g$ and $c_1 = 1$ and $c_2 = a$,

$$(1 + |a|^2)\varphi(e) + a\varphi(g) + \bar{a}\varphi(g^{-1}) \geq 0$$

By setting $a = 1$,

$$Im\varphi(g) = -Im\varphi(g^{-1})$$

By setting $a = i$,

$$Re\varphi(g) = Re\varphi(g^{-1})$$

So, (ii) holds. □

Proof of (iii). By the above proof of (ii),

$$(1 + |a|^2)\varphi(e) \geq -2Re(a\varphi(g))$$

By setting $a = -exp(-iarg(a))$,

$$2\varphi(e) \geq 2|\varphi(g)|$$

So, (iii) holds. □

Proof of (iv). We set $n = 3$, $c_3 = 1$, $g_3 = 3$ in (). Then we get

$$0 \leq c_1\bar{c_2}\varphi(g_1 g_2^{-1}) + c_2\bar{c_1}\varphi(g_2 g_1^{-1}) + c_1\varphi(g_1) + c_2\varphi(g_2) + \bar{c_1}\varphi(g_1^{-1}) + \bar{c_2}\varphi(g_2^{-1}) + \varphi(e) + |c_1|^2\varphi(e) + |c_2|^2\varphi(e)$$

By (ii),

$$0 \leq 2Re(c_1\bar{c_2}\varphi(g_1 g_2^{-1})) + 2Re(c_1\varphi(g_1) + c_2\varphi(g_2)) + \varphi(e) + |c_1|^2\varphi(e) + |c_2|^2\varphi(e)$$

Moreover, we set $c_1 = -c_2 = \alpha$. Then

$$\begin{aligned} 0 &\leq -2|\alpha|^2 Re(\varphi(g_1 g_2^{-1})) + 2Re(\alpha(\varphi(g_1) - \varphi(g_2)) + \varphi(e) + 2|\alpha|^2\varphi(e) \\ &= 2|\alpha|^2(\varphi(e) - Re(\varphi(g_1 g_2^{-1}))) + 2Re(\alpha(\varphi(g_1) - \varphi(g_2)) + \varphi(e) \end{aligned}$$

We can assume $\varphi(g_1) \neq \varphi(g_2)$. We set $\alpha = -\varphi(e)\frac{\overline{\varphi(g_1) - \varphi(g_2)}}{2|\varphi(g_1) - \varphi(g_2)|^2}$. Then $2Re(\alpha(\varphi(g_1) - \varphi(g_2)) + \varphi(e) = 0$ and $2|\alpha|^2(\varphi(e) - Re(\varphi(g_1 g_2^{-1}))) = \frac{\varphi(e)(\varphi(e) - Re(\varphi(g_1 g_2^{-1})))}{2|\varphi(g_1) - \varphi(g_2)|^2}$. So, we get (iv). □

The following is clear.

Proposition 3.11. *Let G be a group and φ is a positive definite function on G. Then*

(i) φ_1, φ_2 are positive definite functions on G and α_1, α_2 are positive numbers. Then $\alpha_1\varphi_1 + \alpha_2\varphi_2$ is a positive definite function on G.

(ii) We set

$$\mathbb{P}_1 := \{\varphi | \varphi \text{ is a continuous positive definite function on } G \text{ such that } \varphi(e) = 1\}$$

and

$$\mathbb{P}_0 := \{\varphi | \varphi \text{ is a continuous positive definite function on } G \text{ such that } \varphi(e) \leq 1\}$$

and

$$\mathbb{P} := \{\varphi | \varphi \text{ is a continuous positive definite function on } G\ \}$$

Then $\mathbb{P}_1$ and $\mathbb{P}_2$ and $\mathbb{P}$ are convex.

Theorem 3.8 (Schur product theorem)**.** *Let $M := \{m_{i,j}\}_{i,j}$ and $N := \{n_{i,j}\}_{i,j}$ be nonnegative definite m-th Hermitian matrices. Then $M \circ N := \{m_{i,j} n_{i,j}\}_{i,j}$ is nonnegative definite. We call $M \circ N$ the Hadamard product of M and N.*

Proof. There are $A := \{a_{i,j}\}_{i,j}$ and $A := \{b_{i,j}\}_{i,j}$ such that

$$M = A^* A,\ N = B^* B$$

This means

$$m_{i,j} = \sum_{i=1}^{m} \bar{a_{i,j}} a_{i,k}, n_{i,j} = \sum_{l=1}^{m} \bar{b_{l,j}} b_{l,k}$$

So,

$$m_{i,j} n_{i,j} = \sum_{i,l=1}^{m} a_{i,k} b_{l,k} \bar{a_{i,j}} \bar{b_{l,j}}$$

For each i, l, we set the $(m,1)$-matrix $v_{i,l}$ by

$$v_{i,l} = {}^t(a_{i,1} b_{i,1}, ..., a_{i,m} b_{i,m})$$

Then $v_{i,l} v_{i,l}^*$ is a m-th nonnegative definite Hermite matrix and

$$M \circ N = \sum_{i,l} v_{i,l} v_{i,l}^*$$

So, $M \circ N$ is nonnegative definite. □

Proposition 3.12. *Let φ_1, φ_2 are positive definite functions on a group G. Then $\varphi_1\varphi_2$ is a positive definite function on a group G.*

Proof. Let us fix any $g_1, ..., g_m \in G$. By Proposition3.10, $\{(\varphi_1\varphi_2)(g_i^{-1} g_j)\}_{i,j}$ is an Hermite matrix. By Theorem3.8, $\{(\varphi_1\varphi_2)(g_i^{-1} g_j)\}_{i,j}$ is nonnegative definite. So, $\varphi_1\varphi_2$ is a positive definite function on a group G. □

3.5.2 GNS construction for unitary representation

We introduce the following notation.

Notation 3.2. *Let G be a Lie group and $f \in C(G)$. Then*

$$f^*(x) := \Delta_R(x)\overline{f(x^{-1})} \ (x \in G)$$

Clearly the following holds.

Proposition 3.13. *Let G be a Lie group and $f \in C(G)$.*

(i) $f^* \in C(G)$.

(ii) $f^{**} = f$.

Theorem 3.9 (GNS construction)**.** *Let G is a Lie group.*

(S1) G *is a Lie group.*

(S2) φ *is a continuous positive definite function on* G.

(S3) We set $(f, g) := \varphi * f * g^*(e)$ $f, g \in C_c(G)$.

(S4) We set $\mathcal{H}_0 := C_c(G) \setminus N$. *Here,* $N := \{f \in C_c(G) | ||f|| = 0\}$.

(S5) $T_g[f] := [f(\cdot g)]$ $([f] \in \mathcal{H}_0, g \in G)$

Then

(i) $(f, g) = \int_G \varphi(x^{-1}y)\overline{f^*(y)}g^*(x)dx_R dy_R = \int_G \varphi(xy^{-1})f(y)\overline{g(x)}dx_R dy_R$

(ii) $\mathcal{H}_0$ *is a pre-Hilbert space.*

(iii) T *is well-defined continuous unitary representation on* $\mathcal{H}_0$ *of* G.

(iv) We set $\mathcal{H}$ *be the completion of* $\mathcal{H}_0$. *Then* T *is well-defined continuous unitary representation on* $\mathcal{H}$ *of* G.

(iv) $\mathcal{H}$ *is separable.*

(v) Let us assume $\{f_n\}_{n\in\mathbb{N}} \subset C_c(G)$ *and* $f \in C_c(G)$ *and* $sup_{n\in\mathbb{N}} ||f_n||_\infty < \infty$ *and* $\lim_{n\to\infty} f_n = f$ *(pointwise convergense). Then* $\lim_{n\to\infty} ||f_n - f|| = 0$.

(vi) $||f|| \leq \sup_{x,y\in supp(f)} |\varphi(xy^{-1})|^{\frac{1}{2}} ||f||_{L^1(G)}$ $(\forall f \in C_c(G))$

(vii) $(\mathcal{H}, T)$ *is cyclic.*

(viii) $\varphi(g) = (T_g v, v)$ $(\forall g \in G)$.

(ix) If $\varphi(\cdot) = (\pi(\cdot)u, u)$ *for* (π, V) *which is a continuous cyclic unitary representation of* G *with cyclic vector* u*. Then* (π, V) *and* $(T, \mathcal{H})$ *are isomorphic as continuous unitary representations.*

STEP1. Proof of (i).

$$(f, g) = (\varphi * f^{**}) * g^*(e) = \int_G \varphi * f^{**}(x^{-1})g^*(x)dx_R = \int_G \int_G \varphi(x^{-1}y^{-1})f^{**}(y)dy_R g^*(x)dx_R$$
$$= \int_G \int_G \varphi(x^{-1}y^{-1})\overline{f^*(y^{-1})}\Delta(y)dy_R g^*(x)dx_R$$

By Proposition2.55,

$$= \int_G \int_G \varphi(x^{-1}y)\overline{f^*(y)}g^*(x)dy_R dx_R = \int_G \int_G \varphi(x^{-1}y)f(y^{-1})\overline{g(x^{-1})}\Delta(y)\Delta(x)dy_R dx_R$$
$$= \int_G \int_G \varphi(xy^{-1})f(y)\overline{g(x)}dy_R dx_R$$

□

STEP2. Proof of $(f, f) \leq 0$ $(\forall f \in C_c(G))$*.* By the same argument as in the proof of Proposition4.2, there is $\{E_{n,i}\}_{n\in\mathbb{N},1\leq i\leq\varphi(n)}$ and $\{x_{n,i}\}_{n\in\mathbb{N},1\leq i\leq\varphi(n)}$ such that

$$\{E_{n,i}\}_{n\in\mathbb{N},1\leq i\leq\varphi(n)} \subset \mathcal{B}(G)\text{:disjoint } (\forall n \in \mathbb{N})$$

and

$$x_{n,i} \in E_{n,i} \ (\forall n \in \mathbb{N}, 1 \leq \forall i \leq \varphi(n))$$

and

$$||f(x)-f(x_{n,i})|| \leq \frac{1}{n} \ (\forall x \in E_{n,i}, \forall n \in \mathbb{N}, , 1 \leq \forall i \leq \varphi(n))$$

and

$$||\varphi(x^{-1}y)-\varphi(x_{n,i}^{-1}x_{n,y})|| \leq \frac{1}{n} \ (\forall x \in E_{n,i}, \forall y \in E_{n,j}, \forall n \in \mathbb{N}, , 1 \leq \forall i \leq \varphi(n))$$

We set

$$F_n(x,y) := \sum_{i,j} \varphi(x_{n,i}^{-1}x_{n,y})f(x_{n,i})\overline{f(x_{n,j})}\chi_{E_n,i}(x)\chi_{E_n,i}(y) \ (x, y \in G, n \in \mathbb{N})$$

and

$$F(x,y) := \varphi(x^{-1}y)f(x)\overline{f(y)} \ (x, y \in G)$$

Then clearly

$$\lim_{n\to\infty} F_n(x,y) = F(x,y) \ (\forall x, y \in G)$$

and

$$||F||_\infty \leq ||\varphi||_\infty ||f||_\infty^2$$

So, by Lebesugue convergence theorem,

$$\lim_{n\to\infty} \int_G \int_G F_n(x,y)dx_R dy_R = \int_G \int_G F(x,y)dx_R dy_R = ||f||^2$$

Because φ is positive definite,

$$\int_G \int_G F_n(x,y)dx_R dy_R = \sum_{i,j} \varphi(x_{n,i}^{-1}x_{n,j})f(x_{n,i})\overline{f(x_{n,j})} \geq 0$$

□

STEP3. Proof of $(g,f) = \overline{(f,g)}$ $(\forall f \in C_c(G))$. By Proposition3.10, $\varphi(yx^{-1}) = \overline{\varphi(xy^{-1})}$ $(\forall x, y \in G)$. So, by (i), $(g,f) = \overline{(f,g)}$ $(\forall f \in C_c(G))$ □

STEP4. Proof of (ii). By STEP2,

$$|(f,g)| \leq ||f||||g|| \ (\forall f, g \in C_c(G))$$

So, $(\cdot_1, \cdot_2)$ is well-defined on $\mathcal{H}_0$ by this inequality. Consequently, (ii) holds. □

STEP5. Proof of that $(T_z f, T_z g) = (f,g)$ $(\forall f, g \in C_c(G), \forall z \in G)$.

$$(T_z f, T_z g) = \int_G \int_G \varphi(xy^{-1})T_z f(x)\overline{T_z f(y)}dx_R dy_R = \int_G \int_G \varphi(xy^{-1})f(xz)\overline{f(yz)}dx_R dy_R$$
$$= \int_G \int_G \varphi(xz(yz)^{-1})f(xz)\overline{f(yz)}dx_R dy_R = \int_G \int_G \varphi(xy^{-1})f(x)\overline{f(y)}dx_R dy_R = (f,g)$$

□

STEP6. Proof of that T is well-defined and unitary. It is clear from STEP5. □

STEP7. Proof of (iii). By STEP6, it is enough to show T is continuous. Let us fix any $f, g \in C_c(G)$. By Theorem3.6, it is enough to show $G \ni z \to (T_z f, g) \in \mathbb{C}$ is continuous. Let us fix any $\epsilon > 0$ and fix any $z \in G$. Let us fix U such that U is a compact neighborhood of e and $U^{-1} = U$. For $x \in supp(f)U$, there is V_x and U_x such that V_x is an open neighborhood of x and U_x is a compact neighborhood of e and $U_x \subset U$ and $U_x^{-1} = U_x$

$$|f(yz) - f(y)| \leq \frac{\epsilon}{(\int_G \int_{supp(f)U_0} |\varphi(xy^{-1})T_{z^{-1}}g(x)|dx_R dy_R + 1)} \ (\forall y \in V_x, \forall z \in U_x)$$

Because $supp(f)U$ is compact, there is $V_{x_1}, ..., V_{x_n}$ which is a covering of $supp(f)U$. $U_0 := U_{x_1} \cap ... \cap U_{x_n}$. For any $w \in zU_0$,

$$|(T_w f, g) - (T_z f, g)| = |(T_{z^{-1}w} f, T_{z^{-1}} g) - (f, T_{z^{-1}} g)| \leq \int_G \int_{supp(f)U_0} |\varphi(x^{-1}y)g(x)||f(yz) - f(y)|dy_R dx_R \leq \epsilon$$

□

STEP8. Proof of (iv). By Proposition5.7, $\mathcal{H}\prime$ is clearly separable. Because $\mathcal{H}\prime$ is dense in $\mathcal{H}$, $\mathcal{H}$ is separable. □

STEP9. Proof of (v). (v) is proved by Lebesgue convergence theorem. □

STEP10. Proof of (vi). This is followed by

$$||f||^2 \leq \sup_{x,y \in supp(f)} |\varphi(xy^{-1})| (\int_G |f(g)|dg)^2 \ (\forall f \in C_c(G))$$

□

STEP11. Constructing a cyclic vector. There is $\{f_n\}_{n=1}^{\infty} \subset C_c(G)$ such that $supp(f_n) \subset exp(B(O, \frac{1}{n})$ and $f_n \geq 0$ and $\int_G f_n dg = 1$ $(\forall n \in \mathbb{N})$. Then for any $n \in \mathbb{N}$

$$||f_n||^2 \leq ||\varphi||_\infty \int_G f(x)f(y)dxdy = ||\varphi||_\infty$$

So, there is subsequence $\{f_{\alpha(n)}\}_{n=1}^{\infty}$ and $v \in \mathcal{H}$ such that

$$w - \lim_{n\to\infty} f_{\alpha(n)} = v$$

Then for any $f \in C_c(G)$

$$(f, v) = \lim_{n\to\infty} (f, f_n) = \lim_{n\to\infty} \int_{supp(f)} \int_{supp(f_n)} \varphi(xy^{-1}) f(y) f_n(x) dxdy$$

By the same argument as in the proof of STEP7,

$$\begin{aligned}
&\lim_{n\to\infty} |\int_{supp(f)} \int_{supp(f_n)} \varphi(xy^{-1}) f(y) f_n(x) dxdy - \int_{supp(f)} \varphi(y^{-1}) f(y) dy| \\
&= \lim_{n\to\infty} |\int_{supp(f)} \int_{supp(f_n)} \varphi(y^{-1}) f(yx) f_n(x) dxdy - \int_{supp(f)} \varphi(y^{-1}) f(y) dy| \\
&= \lim_{n\to\infty} \int_{supp(f)} \int_{supp(f_n)} \varphi(e) |f(yx) - f(y)| f_n(x) dxdy \\
&\leq \int_{supp(f)} \sup_{z \in supp(f_n)} \varphi(e) |f(yz) - f(y)| dy = 0
\end{aligned}$$

So,

$$(f, v) = \varphi * f(e)$$

□

STEP12. Calculas of $f * k^*$. Let us fix any $f, k \in C_c(G)$. By Proposition4.2, $\int_G T_{y^{-1}} f k^*(y) dy$ exists. By the same argument as in the proof of STEP2 and STEP7, there is $\{E_{n,i}\}_{n\in\mathbb{N}, 1\leq i\leq\varphi(n)}$ and $\{x_{n,i}\}_{n\in\mathbb{N}, 1\leq i\leq\varphi(n)}$ such that

$$\{E_{n,i}\}_{n\in\mathbb{N}, 1\leq i\leq\varphi(n)} \subset \mathcal{B}(G)\text{:disjoint } (\forall n \in \mathbb{N})$$

and

$$y_{n,i} \in E_{n,i} \ (\forall n \in \mathbb{N}, 1 \leq \forall i \leq \alpha(n))$$

and

$$||k^*(y) - k^*(y_{n,i})|| \leq \frac{1}{n} \ (\forall y \in E_{n,i}, \forall n \in \mathbb{N}, , 1 \leq \forall i \leq \alpha(n))$$

and

$$||f(xy^{-1}) - f(xy_{n,i}^{-1})|| \leq \frac{1}{n} \ (\forall x \in supp(f)supp(k), \forall y \in E_{n,j}, \forall n \in \mathbb{N}, , 1 \leq \forall i \leq \alpha(n))$$

We set for $n \in \mathbb{N}$

$$F_n(x) := \int_G \sum_{i=1}^{\alpha(n)} f(xy_{n,i}^{-1}) k^*(y_{n,i}) \chi_{E_{n,i}}(y) dy \ (x \in G)$$

Then

$$\lim_{n\to\infty} F_n = f * k^* \text{ (pointwise convergence)}$$

and

$$||F_n||_\infty \leq ||f||_\infty ||k^*||_\infty dg(supp(f)supp(k))dg(supp(k^*)) \ (\forall n \in \mathbb{N})$$

So, by (v),

$$\lim_{n\to\infty} F_n = f * k^* \text{ (in } \mathcal{H})$$

Also,

$$F_n = \sum_{i=1}^{\alpha(n)} T_{y_{n,i}^{-1}} f k^*(y_{n,i})$$

By Proposition4.2 and (vi),

$$\lim_{n\to\infty} F_n = \int_G T_{y^{-1}} f k^*(y) dy \text{ (in } \mathcal{H})$$

So,

$$\int_G T_{y^{-1}} f k^*(y) dy = f * k^*$$

□

STEP13. Proof of (vii). Let us fix any $f, k \in C_c(G)$.

$$(f,k) = \varphi * (f * k^*)(e) = (f * k^*, v) = (\int_G T_{y^{-1}} f k^*(y) dy, v) = \int_G (T_{y^{-1}} f k^*(y), v) dy = \int_G (f k^*(y), T_y v) dy$$
$$= \int_G (f, k(y^{-1}) T_y v) \delta_R(y) dy = (f, \int_G k(y^{-1}) T_y v \delta_R(y) dy)$$

So,

$$k = \int_G k(y^{-1}) T_y v \Delta_R(y) dy$$

By the same argument as in the proof of Proposition4.2, $k \in \overline{\{\sum_{i=1}^m c_i \pi(g_i) v | c_i \in \mathbb{C}, g_i \in G, i = 1,2,...,m, m \in \mathbb{N}\}}$ So, v is a cyclic vector of $\mathcal{H}$. □

STEP14 Proof of (viii). For any $f \in C_c(G)$,

$$\int_G \varphi(g^{-1} f(g) dg = \varphi * f(e) = (f, v) = (\int_G f(y^{-1}) T_y v \Delta_R(y) dy, v) = \int_G f(y^{-1})(T_y v, v) \Delta_R(y) dy$$
$$= \int_G f(y)(T_{y^{-1}} v, v) dy$$

So, for any $y \in G$,

$$\varphi(g^{-1}) = (T_{y^{-1}} v, v)$$

□

STEP15 Proof of (ix). This is clearly followed by Proposition3.6. □

By the proof of Theorem3.9, the following holds.

Proposition 3.14. *Let G is a Lie group. We will succeed in notations of Theorem3.9.*

(S1) G is a Lie group.

(S2) φ is a bounded borel measurable function on G.

(A1) $(f,f) := \varphi * f * f^*(e) \geq 0 \ (\forall f \in C_c(G))$.

Then by the same method to Theorem3.9, we can construct a cyclic continuous unitary representation $(T, \mathcal{H})$ with a cyclic vector v and $\varphi(g) = (T_g v, v)$ (a.e. $g \in G$).

3.5.3 The topology of positive definite functions

Definition 3.7 (The topology of $\mathbb{P}_1$). *Let G be a Lie group. We denote the minimal topology of $\mathbb{P}_1$ in which*

$$\mathbb{P}_1 \ni \varphi \mapsto \int_G \varphi(g)f(g)dg_r \in \mathbb{C} \text{ is continuous for every } f \in L^1(G) \tag{3.5.3}$$

by τ_1.

By Proposition2.31, there are $\{U_n\}_{n=1}^\infty \subset \mathcal{O}(G)$ such that U_n is relative compact and $U_n \subset U_{n+1}$ ($\forall n \in \mathbb{N}$) and $G = \cup_{n=1}^\infty U_n$.

$$d(f_1, f_2) := \sum_{i=1}^\infty \frac{||f_1 - f_2||_{L^\infty(\bar{U}_i)}}{2^i(1 + ||f_1 - f_2||_{L^\infty(\bar{U}_i)})} \ (f_1, f_2 \in \mathbb{P}_1)$$

By Proposition3.10, d is a metric on $\mathbb{P}_1$. We call this topology the pontryagin topology of $\mathbb{P}_1$ and denote this by τ_2.

The following is clear.

Proposition 3.15. *Let G be a Lie group and $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathbb{P}$ and φ be a complex-value function on G and $\{\varphi_n\}_{n\in\mathbb{N}}$ compact converges to φ. Then $\varphi \in \mathbb{P}$.*

Proposition 3.16. *Let G be a Lie group. Then there is $\{f_n\}_{n\in\mathbb{N}} \subset C_c(G)$ such that for every $f \in C_c(G)$ and $\epsilon > 0$ there is $n \in \mathbb{N}$ such that $||f - f_n||_\infty < \epsilon$.*

Proof. By Proposition2.31, there is a sequence of compact subsets of G $\{K_n\}_{n\in\mathbb{N}}$ such that $K_n \subset K_{n+1}^\circ$ ($\forall n \in \mathbb{N}$) and $G = \cup_{n\in\mathbb{N}} K_n$. Then there is $\{g_n\}_{n\in\mathbb{N}} \subset C_c(G)$ such that

$$g_n|K_n \equiv 1 \text{ and } supp(g_n) \subset K_{n+1}^\circ \ (\forall n \in \mathbb{N})$$

Because $C(K_n)$ is separable for every $n \in \mathbb{N}$(see [14]), for each $n \in \mathbb{N}$ there is $\{h_{n,m}\}_{m\in\mathbb{N}}$ which is a dense subset of $C(K_n)$. We set $f_{n+1,m} := g_n h_{n+1,m}$ $(m, n \in \mathbb{N})$. Clearly $\{f_{n,m}\}_{n,m\in\mathbb{N}} \subset C_c(G)$.

Let us fix any $f \in C_c(G)$ and $\epsilon > 0$. Then there is $n\mathbb{N}$ such that $supp(f) \subset K_n$. Because $f \in C(K_{n+1})$, there is $m \in \mathbb{N}$ such that $||f|K_{n+1} - h_{n+1,m}|K_{n+1}||_\infty < \epsilon$. Because $g|K_n \equiv 1$ and $supp(f) \subset K_n$, $||f - f_{n+1,m}|K_{n+1}||_\infty = ||gf|K_{n+1} - gh_{n+1,m}|K_{n+1}||_\infty = ||f|K_{n+1} - h_{n+1,m}|K_{n+1}||_\infty < \epsilon$. □

Proposition 3.17. *Let G be a Lie group. Then τ_1 satisfies the first countable axiom.*

Proof. Let us assume $\{f_n\}_{n\in\mathbb{N}}$ be in Proposition. Let us fix any $\varphi_0 \in \mathbb{P}_1$. We set

$$V(\varphi_0, f_n, \frac{1}{m}) := \{\varphi \in \mathbb{P}_1 | |\int_G (\varphi - \varphi_0) f_n dg_r| < \frac{1}{m}\} \ (n, m \in \mathbb{N})$$

Let us fix any $\epsilon > 0$ and $f \in L^1(G)$. Because $C_c(G)$ is dense in $L^1(G)$(Proposition5.7), by Proposition, there is $n, l \in \mathbb{N}$ such that $||f - f_n||_{L^1(G)} < \frac{\epsilon}{4}$. Let us fix $m \in \mathbb{N}$ such that $\frac{1}{m} < \frac{\epsilon}{4}$. Let us fix any $\varphi \in V(\varphi_0, f_n, \frac{1}{m})$.

$$|\int_G (\varphi(g) - \varphi_0(g))f(g)dg_r| \leq |\int_G (\varphi(g) - \varphi_0(g))f_n(g)dg_r| + \int_G |\varphi(g) - \varphi_0(g)||f(g) - f_n(g)|dg_r \leq \frac{\epsilon}{4} + 2\int_G |f(g) - f_n(g)|dg_r < \epsilon$$

So, $V(\varphi, f_n, \frac{1}{m}) \subset V(\varphi, f, \epsilon)$. Because $\{V(\varphi_0, f, \epsilon)\}_{f\in L^1(G), \epsilon>0}$ is a neighborhood basis at φ_0, $\{V(\varphi_0, f_n, \frac{1}{m})\}_{m,n\in\mathbb{N}}$ is also a neighborhood basis at φ_0. □

Proposition 3.18. *Let*

(i) X_1 and X_2 are topological spaces.

(ii) $f : X_1 \to X_2$ satisfies

$$\text{If } \{x_n\}_{n\in\mathbb{N}} \text{ converges } x \text{ in } X_1 \text{ then } \{f(x_n)\}_{n\in\mathbb{N}} \text{ converges } f(x) \text{ in } X_2$$

(iii) X_1 satisfies the first countable axiom.

then f is continuous.

Proof. Let us assume f is not continuous. Then there is an open set of X_2 O such that $f^{-1}(O)$ is not open set of X_1. Then there is $x \in f^{-1}(O)$ such that for any neighborhood of x N, $N \not\subseteq f^{-1}(O)$. By (iii), we can take $\{V_{x,n}\}_{n\in\mathbb{N}}$ which is a countable neighborhood basis at x. Then there is $\{x_n\}_{n\in\mathbb{N}} \subset X_1$ such that $x_n \in V_{x,n} \setminus f^{-1}(O)$ ($\forall n \in \mathbb{N}$). Because $\{x_n\}_{n\in\mathbb{N}}$ converges x, by (ii), $\{f(x_n)\}_{n\in\mathbb{N}}$ converges $f(x) \in O$. Because $f(x_n) \in O^c$ ($\forall n \in \mathbb{N}$), $f(x) \in \bar{O^c} = O^c$. This is contradiction. □

Notation 3.3. *Let G be a topological group. We denote the set of all continuous positive definite functions by $\mathbb{P}$. And we set*

$$\mathbb{P}_1 := \{\varphi \in \mathbb{P} | \varphi(e) = 1\}$$

Example 3.2. *Let G be a group and (π, V) is a unitary representation of G. Then $\Phi_\pi(v \otimes v)$ is a positive definite function.*

Proof. For any $n \in \mathbb{C}$ and $g_1, g_2, ..., g_n \in G$ and $c_1, c_2, ..., c_n \in \mathbb{C}$

$$\sum_{j,k} c_j \bar{c_k} \Phi_\pi(v \otimes v)(g_j^{-1} g_k) = \sum_{j,k} c_j \bar{c_k} (\pi(g_k)v, \pi(g_j)v) = (\sum_k c_k \pi(g_k)v, \sum_j c_j \pi(g_j)v) \geq 0$$

□

Lemma 3.1. *Let*

(i) G be a Lie group.
(ii) $f \in C_c(G)$.
(iii) $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathbb{P}_0$.
(iv) $\phi \in \mathbb{P}_0$.
(v) $\{\phi_n\}_{n\in\mathbb{N}}$ converges to ϕ in τ_1.

Then $\{\phi_n * f\}_{n\in\mathbb{N}}$ compact converges to $\phi * f$.

STEP1. Showing that $\{\phi_n * f\}_{n\in\mathbb{N}}$ pointwise converges to $\phi * f$. Let us fix any $g \in G$. Then

$$\phi_n * f(g) = \int_G \phi_n(gh^{-1}) f(h) dg_r(h) = \int_G \phi_n((hg^{-1})^{-1}) f((hg^{-1})g) dg_r(h) = \int_G \phi_n(h^{-1}) f(hg) dg_r(h)$$
$$= \int_G \phi_n(h) f(h^{-1}g) \Delta_r(h) dg_r(h)$$

by (v)

$$\to \int_G \phi(h) f(h^{-1}g) \Delta_r(h) dg_r(h) = \phi * f(g) \ (n \to \infty)$$

□

STEP2. Showing that $\{\phi_n * f\}_{n\in\mathbb{N}}$ are equicontinuous. We will show that for each $g_0 \in G$ and $\epsilon > 0$ there is a neighborhood of e V such that

$$|\phi_n * f(g) - \phi_n * f(g_0)| < \epsilon \ (\forall g \in g_0 V, \forall n \in \mathbb{N})$$

Let us fix any $g_0 \in G$ and $\epsilon > 0$. Because $f \in C_c(G)$, $f\Delta_r$ is uniformly continuous. So, there is a neighborhood of e V such that

$$|f(g) - f(h)| < \frac{\epsilon}{2(dg_r(supp(f)) + 1)(||\Delta_r(g)||_{L^\infty(supp(f))} + 1)} \ (\forall g, h \in G \ s.t \ g^{-1}h \in V)$$

Then, for any $g \in g_0 V$,

$$|\phi_n * f(g) - \phi_n * f(g_0)| = |\int_G \phi_n(h^{-1})(f(hg) - f(hg_0) dg_r(h)| \leq \int_G |f(hg) - f(hg_0)| dg_r(h) < \epsilon$$

□

STEP3. Showing that $\{\phi_n * f\}_{n\in\mathbb{N}}$ compact converges to φ. Let us fix any K is a compact subset of G and $\epsilon > 0$. Because φ is uniformly continuous on K, there is V which is a neighborhood of e such that

$$|\varphi(g_1) - \varphi(g_2)| < \frac{\epsilon}{3} \ (\forall g_1, g_2 \in K \ s.t \ g_1^{-1} g_2 \in V)$$

By STEP2, for each $g \in K$, there is $V_g \subset V$ which is a neighborhood of e such that

$$|\varphi_n(g) - \varphi_n(h)| < \frac{\epsilon}{3} \ (\forall h \in gV_g, n \in \mathbb{N})$$

Because $K \subset \cup_{g\in K} gV$ and K is compact, there is $g_1, g_2, ..., g_n$ such that $K \subset \cup_{i=1}^n g_i V_{g_i}$.

By STEP1, for each $i \in \{1, 2, ..., n\}$, there is k_i such that

$$|\varphi_m(g_i) - \varphi(g_i)| < \frac{\epsilon}{3} \ (\forall m \geq k_i)$$

We set $K := \max_{i\in\{1,2,\ldots,n\}} k_i$. Let us fix any $g \in G$ and $m \geq K$. There is i such that $g \in g_i V_{g_i}$.

$$|\varphi_m(g) - \varphi(g)| \leq |\varphi_m(g) - \varphi_m(g_i)| + |\varphi_m(g_i) - \varphi(g_i)| + |\varphi(g_i) - \varphi(g)| < \epsilon$$

□

Theorem 3.10 (D.A.Raikov-R.Godement-H.Yoshizawa Theorem)**.** *Let G be a Lie group and τ_1, τ_2 be topologies which are defined in Definition3.7. Then $\tau_1 = \tau_2$.*

Strategy for our proof. Clearly $\tau_1 \subset \tau_2$. Let us fix any $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathbb{P}_0$ and $\phi \in \mathbb{P}_0$ such that $\phi_n \to \phi$ in τ_1. By Proposition3.18, it is enough to show $\phi_n \to \phi$ in τ_2.

Let us fix any $\epsilon > 0$ and K which is a compact subset of G. By Proposition3.10, there is V which is a neighborhood of e such that

$$|\varphi(g_1) - \varphi(g_2)| < \frac{\epsilon}{3} \ (\forall g_1, g_2 \in K \ s.t \ g_1^{-1} g_2 \in V)$$

Then there is $f \in C_c(G)$ such that $supp(f) \subset V$ and $f \leq 0$ on G and $\int_G f dg_r = 1$. □

STEP1. Evaluation of $\varphi_n * f - f$. For any $n \in \mathbb{N}$ and $g \in G$

$$|\varphi_n * f(g) - \varphi_n(g)| \leq |\int_G (\varphi_n(gh^{-1}) - \varphi_n(g)) f(h) dg_r(h)| \leq \int_G |\varphi_n(gh^{-1}) - \varphi_n(g)| f(h) dg_r(h)$$

By Proposition3.18

$$\leq \int_G \frac{1}{\sqrt{2}} \int_G (\varphi_n(e) - Re\varphi_n(h))^{\frac{1}{2}} f(h)^{\frac{1}{2}} f(h)^{\frac{1}{2}} dg_r(h) \leq \frac{1}{\sqrt{2}} (\int_G (\varphi_n(e) - Re\varphi_n(h)) f(h) dg_r(h))^{\frac{1}{2}}$$
$$= \frac{1}{\sqrt{2}} (\int_G (Re\varphi(e) - Re\varphi_n(h)) f(h) dg_r(h))^{\frac{1}{2}}$$

Because $\phi_n \to \phi$ in τ_1, there is $n_0 \in \mathbb{N}$ such that

$$\int_G |Re\varphi_n(h) f(h) - Re\varphi(h) f(h)| dg_r(h) < \frac{\epsilon^2}{9}$$

So,

$$|\varphi_n * f(g) - \varphi_n(g)| \leq \frac{\epsilon}{3} \int_G |\varphi(e) - \varphi(h)| f(h) dg_r(h) < \frac{\epsilon}{3} \ (\forall g \in G, n \geq n_0)$$

Similarily,

$$|\varphi * f(g) - \varphi(g)| < \frac{\epsilon}{3} \ (\forall g \in G, n \geq n_0)$$

□

STEP2. Showing this theorem. By Lemma3.1, there is $n_1 \in \mathbb{N}$ such that

$$|\varphi_n * f(g) - \varphi * f(g)| < \frac{\epsilon}{3} \ (\forall g \in K, n \geq n_1)$$

So, by STEP1,

$$|\varphi_n(g) - \varphi(g)| < |\varphi_n(g) - \varphi_n * f(g)| + |\varphi_n * f(g) - \varphi * f(g)| + |\varphi * f(g) - f(g)| < \epsilon \ (\forall g \in K, n \geq \max n_0, n_1)$$

□

Proposition 3.19. *Let G be a Lie group. Then $\mathbb{P}_1$ is compact.*

Proof. Let us fix any $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathbb{P}_1$. By Banach-Alaoglu Theorem, there is a cauchy subsequence $\{\phi_{\alpha(n)}\}_{n\in\mathbb{N}}$ in $*$-weak topology. Because $L^1(G)^* = L^\infty(G)$ (see [8]), there is a bounded borel function φ such that $\{\phi_{\alpha(n)}\}_{n\in\mathbb{N}}$ converges to φ in weak-$*$ topology. So, φ satisfies assumptions in Proposition3.14. By Proposition3.14, we can assume φ is continuous. □

3.5.4 Extreme points

Proposition 3.20. *Let*

(S1) G is a Lie group.

(S2) φ_1, φ_2 are continuous functions on G.

(A1) $\varphi_1 * f = \varphi_2 * f$ $(\forall f \in C_c(G))$.

Then $\varphi_1 = \varphi_2$.

Proof. Let us fix any $g \in G$. There is a sequence $\{f_n\}_{n\in\mathbb{N}} \subset C_c^+(G)$ such that $\int_G f_n dg_r = 1$ $(\forall n \in \mathbb{N})$. By the same argument as the proof of Theorem3.9, $\varphi_1(g) = \varphi_2(g)$. □

Proposition 3.21. *We will succeed in notations of Theorem3.9. Let*

(S1) G is a Lie group.

(S2) φ_1, φ_2 are continuous positive definite functions on G.

(A1) $(\cdot_1, \cdot_2)_{\varphi_1} = (\cdot_1, \cdot_2)_{\varphi_2}$

Then $\varphi_1 = \varphi_2$.

Proof. By Theorem3.9, $\varphi_1 * f = \varphi_2 * f$ $(\forall f \in C_c(G))$. By Proposition3.20, $\varphi_1 = \varphi_2$. □

Proposition 3.22. *Let*

(S1) G is a Lie group.

Then $Ex(\mathbb{P}_0) \setminus 0 = Ex(\mathbb{P}_1)$.

Proof of $\subset$. Let us fix any $\varphi \in Ex(\mathbb{P}_0) \setminus 0$. If $\varphi(e) < 1$, then $\varphi = \varphi(e)\dfrac{\varphi}{\varphi(e)} + (1 - \varphi(e))0$. This means $\varphi \notin Ex(\mathbb{P}_0)$. So, $\varphi(e) = 1$. □

Proof of $\supset$. Let us fix any $\varphi \in Ex(\mathbb{P}_1)$. Let us fix any $\varphi_1, \varphi_2 \in Ex(\mathbb{P}_0)$ and $\alpha_1, \alpha_2 \in [0, 1]$ such that $\varphi = \alpha_1\varphi_1 + \alpha_2\varphi_2$. Then $1 = \varphi(e) = \alpha_1\varphi_1(e) + \alpha_2\varphi_2(e)$. Then $\varphi_1(e) = \varphi_2(e) = 1$. So, $\varphi = \varphi_1 = \varphi_2$. □

Proposition 3.23. *Let*

(S1) G is a Lie group.

(S2) By GNS construction we set

$$\Phi : \mathbb{P}_1 \ni \varphi \mapsto (T, \mathcal{H}_\varphi) \in \Omega_c$$

Then $Ex(\mathbb{P}_1) = \mathbb{P}_1 \cap \Phi^{-1}(\hat{G})$.

Proof of $\subset$. Let us fix any $\varphi \in Ex(\mathbb{P}_1)$. Let us fix any closed G-invariant subspaces of $\mathcal{H}_\varphi$ V_1, V_2 such that $\mathcal{H}_\varphi = V_1 + V_2$ and $V_1 \neq 0$. Let us set P_i be the orthogonal projection of V_i $(i = 1, 2)$. Let us fix $v \in \mathcal{H}_\varphi$ such that $\varphi(g) = (T_g v, v)$ $(\forall g \in G)$. Because $V_1 \perp V_2$ and P_i is commutative with T_g $(\forall i, g \in G)$ and $1 = ||v||^2 = ||P_1 v||^2 + ||P_2 v||^2$, $\varphi(g) = ||P_1 v||^2 \dfrac{(T_g P_1 v, P_1 v)}{||P_1 v||^2} + ||P_2 v||^2 \dfrac{(T_g P_2 v, P_2 v)}{||P_2 v||^2}$. Because $\varphi \in Ex(\mathbb{P}_1)$, $(T_g v, v) = (T_g P_1 v, P_1 v) = (T_g P_1 v, v)$ $(\forall g \in G)$. So, $(v, T_{g^{-1}} v) = (P_1 v, T_{g^{-1}} v)$ $(\forall g \in G)$. Because $(T, \mathcal{H}_\varphi)$ is cyclic, $v = P_1 v$. So, $V_1 = \mathcal{H}_\varphi$. □

Proof of $\supset$. Let us fix any $\varphi \in \mathbb{P}_1 \cap \Phi^{-1}(\hat{G})$. Let us fix $\varphi_1, \varphi_2 \in \mathbb{P}_1$ and $\alpha_1, \alpha_2 \in [0, 1]$ such that $\varphi = \alpha_1\varphi_1 + \alpha_2\varphi_2$. We set for $f + \{f \in C_c(G) | ||f||_\varphi = 0\}) \in C_c(G)/\{f \in C_c(G) | ||f||_\varphi = 0\}$

$$\pi_i(f + \{f \in C_c(G) | ||f||_\varphi = 0\}) := f + \{f \in C_c(G) | ||f||_{\varphi_i} = 0\} \ (i = 1, 2)$$

Because $\{f \in C_c(G) | ||f||_\varphi = 0\} \subset \{f \in C_c(G) | ||f||_{\varphi_i} = 0\}$ $(i = 1, 2)$, π_1, π_2 are well defined and surjective.

Let us fix any $w \in \mathcal{H}_{\varphi_1}$. Because $|(\pi_1(u), \pi_1(w))_{\mathcal{H}_{\varphi_1}}| \leq \dfrac{1}{\alpha_1}|(u, w)| \leq \dfrac{1}{\alpha_1}||u||||w||$. So, by Riez representation theorem, there is $Aw \in \mathcal{H}_\varphi$ such that $(\pi_1(u), \pi_1(w))_{\mathcal{H}_{\varphi_1}} = (u, Aw)$ $(\forall u \in \mathcal{H}_\varphi)$. Clearly A is continuous and linear. If $A = 0$, then $\varphi_1 = 0$. This is contradiction. So, $A \neq 0$. Because $(T, \mathcal{H}_\varphi)$ is irreducible, by Shur Lemma(see Proposition3.1), there is $\lambda_1 \in \mathbb{C}$ such that $T = \lambda_1 I$. There is $w_1 \in \mathcal{H}_{\varphi_1}$ such that $\pi_1(w_1) \neq 0$. Then $0 < ||\pi_1(w_1)||^2_{\varphi_1} = \bar{\lambda}||w_1||^2$. So, $\lambda_1 > 0$. And, $(\cdot_1, \cdot_2)_{\varphi_1} = \lambda_1(\cdot_1, \cdot_2)_\varphi$. By Proposition3.21, $\varphi_1 = \lambda_1\varphi$. $1 = \varphi_1(e) = \lambda_1\varphi(e) = \lambda_1$. So, $\varphi_1 = \lambda$. □

By Proposition3.23, Krein Millman Theorem(Theorem3.3), Raikov-Godement-Yoshizawa Theorem(Theorem3.10), the following hold.

Theorem 3.11 (I.M. Gelfand-D.A. Raikov Theorem). *Let*

(S1) G is a Lie group.

(S2) K is a compact subset of G.

(S3) $\epsilon > 0$.

(S4) φ is a continuous positive definite function on G.

Then $\alpha_1, ..., \alpha_m > 0$ and $\varphi_1, ..., \varphi_m \in Ex(\mathbb{P}_1)$ such that

$$||\varphi - \sum_{i=1}^{m} \alpha_i \varphi_i||_{L^\infty(K)} < \epsilon$$

Theorem 3.12 (I.M. Gelfand-D.A. Raikov Theorem). *Let*

(S1) G is a Lie group.

(S2) $g_1, g_2 \in G$.

(A1) $T_{g_1} = T_{g_2}$ $(\forall (T, V) \in \hat{G})$.

Then $g_1 = g_2$.

Proof. Let us fix $g_1, g_2 \in G$ such that $g_1 \neq g_2$. We set $g_0 := g_1 g_2^{-1}$. There is $f \in C_c^+(G)$ s.t $g_0 \notin supp(f)^{-1} supp(f)$ and $||f||_2 = 1$. We set

$$\varphi(g) := (R_g f, f) \ (g \in G)$$

Because the right regular representation R is continuous on $L^2(G)$, φ is continuous positive definite function on G.

$$\varphi(g_0) = \int_G f(gg_0) f(g) dg_r(g) = 0$$

Because $1 = \varphi(e) = \varphi(e) - \varphi(g_0)$, by Theorem3.11, Then $\alpha_1, ..., \alpha_m > 0$ and $\varphi_1, ..., \varphi_m \in Ex(\mathbb{P}_1)$ such that

$$\sum_{i=1}^{m} \alpha_i (\varphi_i(e) - \varphi_i(g_0)) \neq 0$$

So, there is i such that $\varphi_i(g_0) \neq 1$. Because $\varphi_i \in \mathbb{P}_1$, by Proposition3.23, $(T, \mathcal{H}_{\varphi_i}) \in \hat{G}$ and there is $v \in \mathcal{H}_{\varphi_i}$ such that $||v||_{\varphi_i} = 1$ and $\varphi_i(g_0) = (T_{g_0} v, v)_{\varphi_i}$. So, $T_{g_0} \neq I$. This implies that $T_{g_1} \neq T_{g_2}$. □

3.6 Topology of unitary dual

Definition 3.8 (Fell topology). *By GNS construction we set*

$$\Phi : \mathbb{P}_1 \ni \varphi \mapsto (T, \mathcal{H}_\varphi) \in \Omega_c$$

Here, we assume the topology of $\mathbb{P}_1$ is the pontryagin topology and Ω_c is the set of all separable cyclic unitary representation of G. We set the toplogy of Ω_c by $\{O \subset \Omega_c | \Phi^{-1}(O) \text{ is open set}\}$. We call this topology Fell topology of Ω_c.

3.7 Direct Integral of Hilbert spaces

Definition 3.9. *Let*

(S1) $(X, \mathfrak{B}, \mu)$ is a measurable space.

We say X is localizable if there is $N \subset X$ and $\{X_i\}_{i=1}^\infty \subset \mathfrak{B}$ such that

(i) $\{X_i\}_{i=1}^\infty$ is disjoint.

(ii) $N \cap \cup_{i=1}^\infty X_i = \phi$.

(iii) $X = N \cup \cup_{i=1}^\infty X_i$.

(iv) $\mu(X_i) < \infty$ $(\forall i \in \mathbb{N})$.

(v) $\mu(F) = \sum_{i=1}^{\infty} \mu(F \cap X_i)$ $\forall F \in \mathfrak{B}$.

Because Lie group is σ-compact, the following holds.

Proposition 3.24. *Let*

(S1) G *is a Lie group.*

(S2) μ *is a left invariant measure.*

Then $(G, \mathfrak{B}, \mu)$ *is localizable.*

Notation 3.4 (Locally almost everywhere)**.** *Let*

(S1) $(X, \mathfrak{B}, \mu)$ *is a meaurable space.*

(S2) For each $x \in X$*, the proposition* $P(x)$ *is given.*

We denote P *holds loc. a.e* $x \in X$ *if for any* $Y\mathfrak{B}$ *such that* $\mu(Y) < \infty$ P *holds loc. a.e* $x \in Y$*.*

Proposition 3.25 (Direct Integral of Hilbelt spaces)**.** *Let*

(S1) $(X, \mathfrak{B}, \mu)$ *is a meaurable space.*

(S2) $\{H(x)\}_{x \in X}$ *is a family of Hilbert spaces.*

(S3) $\Pi := \Pi_{x \in X} H(x)$.

(S4) $\mathfrak{G} \subset \Pi$.

(S5) $\mathfrak{R} := \{f \in \mathfrak{G} | f = 0 \text{ loc-a.e. } x \in X\}$

We say $\mathfrak{G}$ *is a Direct Integral of* $\{H(x)\}_{x \in X}$ *if*

(i) If $v_1, v_2 \in \mathfrak{G}$ *and* $a, b \in \mathbb{C}$ *then* $av_1 + bv_2 := \{av_1(x) + bv_2(x)\}_{x \in X} \in \mathfrak{G}$.

(ii) If $v \in \mathfrak{G}$ *then* $X \in x \mapsto ||v(x)||_{H(x)} \in \mathbb{R}$ *is measurable.*

(iii) If $v \in \mathfrak{G}$ *then* $\int_X ||v(x)||_{H(x)}^2 \mu(x) < \infty$.

(iv) Let us fix any $f \in \Pi$ *such that*

(a) There is $\varphi \in L^2(X)$ *such that* $||f||_{H(x)} \leq \varphi(x)$ $(\forall x \in X)$

(b) For any $g \in \mathfrak{G}$*,* $X \ni x \mapsto (f(x), g(x))_{H(x)} \in \mathbb{C}$ *is measurable.*

Then there is $h \in \mathfrak{G}$ *such that for any* $g \in \mathfrak{G}$

$$(f(x) - h(x), g(x)) = 0 \text{ for loc-a.e } x \in X \tag{3.7.1}$$

(v) Let us fix any $f \in \Pi$ *such that*

(a) $||f(\cdot)||_{H(\cdot)} \in L^2(X)$

(b) There is $h \in \mathfrak{G}$ *such that* $f(x) = h(x)$ *for loc-a.e* $x \in X$*.*

Then $f \in \mathfrak{G}$*.*

Then $\mathfrak{G}/\mathfrak{R}$ *is a Hilbert space. We call this Wils Direct Integral of* $(X, \mu, \{H(x)\}_{x \in X})$ *with respect to* $\mathfrak{G}$ *and denote this by* $\int_X^{\mathfrak{G}} H(x) d\mu(x)$

Proof. It is enough to show that any cauchy sequence of $\mathfrak{G}$ has a convergent subsequence. Let us fix any cauchy sequence of $\mathfrak{G}$, $\{v_n\}_{n=1}^{\infty}$. Then there is subsequence $\{v_{\varphi(i)}\}_{i=1}^{\infty}$ such that

$$\sum_{i=1}^{\infty} ||v_{\varphi(i+1)} - v_{\varphi(i)}||^2 < \infty$$

and

$$\sum_{i=1}^{\infty} ||v_{\varphi(i+1)} - v_{\varphi(i)}|| < \infty$$

So,

$$\int_X \sum_{i=1}^{\infty} ||v_{\varphi(i+1)}(x) - v_{\varphi(i)}(x)||_{H(x)}^2 d\mu(x) = \sum_{i=1}^{\infty} \int_X ||v_{\varphi(i+1)}(x) - v_{\varphi(i)}(x)||_{H(x)}^2 d\mu(x) = \sum_{i=1}^{\infty} ||v_{\varphi(i+1)} - v_{\varphi(i)}||^2 < \infty$$

So,

$$\sum_{i=1}^{\infty}||v_{\varphi(i+1)}(x)-v_{\varphi(i)}(x)||_{H(x)}^2<\infty \text{ loc-a.e } x\in X$$

So, $\{v_{\varphi(i)}(x)\}_{i=1}^{\infty}$ is cauchy sequence for loc-a.e $x\in X$. Because for any $x\in X$ $H(x)$ is Hilbert space, $\{v_{\varphi(i)}(x)\}_{i=1}^{\infty}$ converges to some $v(x)\in H(x)$ for loc-a.e $x\in X$. Because $||v(x)||_{H(x)}^2=\lim_{n\to\infty}(v_n(x),v_n(x))$ for loc-a.e $x\in X$, $||v(\cdot)||_{H(\cdot)}$ is measurable. For loc-a.e $x\in X$,

$$||v_n(x)||\leq||v_n(x)-v_1(x)||+||v_1(x)||\leq\sum_{i=2}^{n}||v_i(x)-v_{i-1}(x)||+||v_1(x)||$$

So, for loc-a.e $x\in X$,

$$||v(x)||\leq\sum_{i=2}^{\infty}||v_i(x)-v_{i-1}(x)||+||v_1(x)||$$

Here,

$$\int_X(\sum_{i=2}^{\infty}||v_i(x)-v_{i-1}(x)||+||v_1(x)||)^2d\mu(x)\leq\lim_{n\to\infty}\int_X(\sum_{i=2}^{n}||v_i(x)-v_{i-1}(x)||+||v_1(x)||)^2d\mu(x)$$
$$\leq\lim_{n\to\infty}(\sum_{i=1}^{n}||v_{i+1}-v_i||^2+||v_1||^2+||v_1||\sum_{i=1}^{n}||v_{i+1}-v_i||+(\sum_{i=1}^{n}||v_{i+1}-v_i||)^2)<\infty$$

So,

$$\sum_{i=2}^{\infty}||v_i(\cdot)-v_{i-1}(\cdot)||+||v_1(\cdot)||\in L^2(X,\mu)$$

Let us fix any $u\in\mathfrak{G}$ and $n\in\mathbb{N}$.

$$(v_n(x),u(x))=(\frac{1}{2}||v_n(x)+u(x)||^2-\frac{1}{2}||v_n(x)||^2-\frac{1}{2}||u(x)||^2)+i(\frac{1}{2}||v_n(x)+iu(x)||^2-\frac{1}{2}||v_n(x)||^2-\frac{1}{2}||iu(x)||^2)$$

So, $(v_n(\cdot),u(\cdot))$ is measurable. This implies that $(v(\cdot),u(\cdot))$ is measurable. By (iv), there is $v_0\in\mathfrak{G}$ such that for $u\in\mathfrak{G}$ and for loc-a.e $x\in X$

$$(v(x)-v_0(x),u(x))=0$$

So, for any $n\in\mathbb{N}$, $(v(x)-v_0(x),v_n(x)-v_0(x))=0$. This implies that for loc-a.e $x\in X$ $(v(x)-v_0(x),v(x)-v_0(x))=0$. So,

$$v(x)=v_0(x)\ \text{ loc-a.e } x\in X$$

By (v), $v\in\mathfrak{G}$.

For loc-a.e $x\in X$ and $n\mathbb{N}$,

$$||v(x)-v_n(x)||\leq2(\sum_{i=2}^{\infty}||v_i(x)-v_{i-1}(x)||+||v_1(x)||)$$

and $\sum_{i=2}^{\infty}||v_i(\cdot)-v_{i-1}(\cdot)||+||v_1(\cdot)||\in L^2(X)$. So, by Lebesgue convergence theorem,

$$\lim_{n\to\infty}||v-v_n||^2=\lim_{n\to\infty}\int_X||v(x)-v_n(x)||^2d\mu(x)=0$$

□

By Theorem3.6, the following holds.

Proposition 3.26 (Direct Integral of Unitary representations)**.** *Let*

(S1) $(X,\mathfrak{B},\mu)$ *is a meaurable space.*

(S2) $\{H(x)\}_{x\in X}$ *is a family of Hilbert spaces.*

(S3) $\Pi:=\Pi_{x\in X}H(x)$.

(S4) $\mathfrak{G}\subset\Pi$.

(S5) $\int_X^{\mathfrak{G}}H(x)d\mu(x)$ *is the direct integral of* $(X,\mu,\{H(x)\}_{x\in X})$ *with respects to* $\mathfrak{G}$.

(S6) G is a topological group.

(S7) π_x is a continuous unitary representation on $H(x)$ $(x \in X)$.

(A1) For any $g \in G$ and $v := \{v(x)\}_{x\in X} \in \mathfrak{G}$, $\pi(g)v := \{\pi_x(g)v(x)\}_{x\in X} \in \mathfrak{G}$

(A2) For any $v := \{v(x)\}_{x\in X} \in \mathfrak{G}$, $G \ni g \mapsto \pi(g)v\mathfrak{G}$ is continuous.

Then $(\pi, \int_X^{\mathfrak{G}} H(x)d\mu(x))$ is continuous unitary representation. We call this direct integral representation of $(X, \mu, \{\pi(x), H(x)\}_{x\in X})$ and denote this by $\int_X^{\mathfrak{G}} \pi(x)d\mu(x)$.

3.8 Decomposition of an affine type function

Definition 3.10 (Baire Set)**.** *Let X be a locally compact topological space. We denote the minimal borel family in which any element of $C_c(X)$ is measurable by $\mathfrak{B}_0$. We call the elamant of $\mathfrak{B}_0$ Baire set.*

Definition 3.11 (Support of measure)**.** *Let*

(S1) X is a locally compact topological space.

(S2) $\mathfrak{B}$ is the minimal borel set family containing all relative compact open sets.

(S3) μ is a nonnegative measure on $\mathfrak{B}$.

(S4) $F \subset X$.

We say F supports μ if for any $A\mathfrak{B}$ such that $A \cap F = \phi$, $\mu(A) = 0$.

Definition 3.12 (Regular borel measure)**.** *Let*

(S1) X is a locally compact hausdorff topological space.

(S2) $\mathfrak{B}$ is the minimal borel set family containing all relative compact open sets.

(S3) μ is a nonnegative measure on $\mathfrak{B}$.

(A1) For any compact set A, $\mu(A) < \infty$.

(A2) $\mu(A) = sup\{\mu(C)|C \in \mathfrak{B},\ C \subset A$ and C is compact.$\}$.

(A3) $\mu(B) = sup\{\mu(C)|C \in \mathfrak{B},\ A \subset C$ and C is an open set.$\}$.

Then we say μ is regular borel measure on X.

Definition 3.13 (Upper semicontinuous function)**.** *Let*

(S1) X is a topological space.

We say $f \in Map(X, \mathbb{R})$ is upper continuous for any $c \in \mathbb{R}$ $f^{-1}((-\infty, c))$ is an open set.

Definition 3.14 (Affine type function)**.** *Let $\mathcal{D}$ be a vector space and X be a convex subset of $\mathcal{D}$ and f be a real valued function on $\mathcal{D}$. We say f is affine type if*

$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y)\ (\forall \lambda \in [0,1], \forall x, y \in X)$$

We denote the set of all continuous affine type function on $\mathcal{D}$ by $A(X)$.

Notation 3.5. *Let*

(S1) $(\mathcal{D}, \{||\cdot||_n\}_{n\in\mathcal{N}}$ is a seminormed vector space.

(S2) X is a compact convex subset of $\mathcal{D}$.

We set

$$B(X) := \{f \in Map(X, \mathbb{R}|f \text{ is an upper semicontinuous and convex on } X\}$$

and

$$CB(X) := B(X) \cap C(X)$$

and

$$CB_0(X) := CB_0(X) - CB_0(X)$$

Definition 3.15 (Vector lattice)**.** *Let*

(S1) $(V, \leq)$ is a partialy ordered vector space.

(S2) $\vee$ is a binary operation on V.

We say $(V, \leq, \vee)$ is vector lattice if for any $x, y, z \in V$

(i) If $x \leq y$ then $x + z \leq y + z$.

(ii) If $x \leq y$ then $\alpha x \leq \alpha y$ $(\forall \alpha \geq 0)$.

(iii) $x \vee y$ is a least upper bound.

Proposition 3.27. *Let*

(S1) $(\mathcal{D}, \{||\cdot||_n\}_{n\in\mathcal{N}}$ is a seminormed vector space.

(S2) X is a compact convex subset of $\mathcal{D}$.

Then

(i) If $f, g \in CB(X)$ then $\max(f, g) \in CB(X)$.

(ii) $CB_0(X)$ is a vector lattice with the pointwise order and pointwise maximum.

(iii) $CB_0(X)$ is dense in $C(X)$.

Proof of (i). Let us fix any $x, y \in X$ and $\lambda \in [0, 1]$. Then

$$\begin{aligned}\max(f(\lambda x + (1-\lambda)y), g(\lambda x + (1-\lambda)y)) &\leq \max(\lambda f(x) + (1-\lambda)f(y), \lambda g(x) + (1-\lambda)g(y)) \\ &\leq \lambda \max(f(x), g(x)) + (1-\lambda) max(f(x), g(x))\end{aligned}$$

So, $\max(f, g) \in CB(X)$ □

Proof of (ii). Let us fix any $f_1, f_2, g_1, g_2 \in CB(X)$. For each $x \in X$

$$f_1(x) - g_1(x) \leq f_2(x) - g_2(x) \iff f_1(x) + g_2(x) \leq f_2(x) + g_1(x)$$

So,

$$\max(f_1 - g_1, f_2 - g_2) = \max(f_1 + g_2, f_2 + g_1) - (g_1 + g_2)$$

So, by (i), $\max(f_1 - g_1, f_2 - g_2) \in CB_0(X)$. □

Proof of (iii). By Hahn-Banach Theorem, for any $x, y \in X$ such that $x \neq y$, there is $h \in CB_0(X)$ such tat $h(x) \neq h(y)$. So, by Stone-Weierstrass Theorem in Vector Lattice(Theorem3.2), (iii) holds. □

Definition 3.16 (Order of Regular Borel measures)**.** *Let*

(S1) X is a locally compact hausdorff topological space.

(S2) $\mathfrak{B}$ is the minimal borel set family containing all relative compact open sets.

(S3) μ_1, μ_2 are regular borel measures on X.

We denote $\mu_1 \prec \mu_2$ if

$$\mu_1(f) \leq \mu_2(f) \ (\forall f \in CB(X))$$

Proposition 3.28. *Let*

(S1) $(\mathcal{D}, \{||\cdot||_n\}_{n\in\mathcal{N}}$ is a seminormed vector space.

(S2) X is a compact convex subset of $\mathcal{D}$.

(S3) μ_1, μ_2 are regular borel measure on X.

(A1) $\mu_1 \prec \mu_2$ and $\mu_2 \prec \mu_1$.

Then $\mu_1 = \mu_2$.

Proof. This is from Proposition3.27. □

Proposition 3.29. *Let*

(S1) $(\mathcal{D}, \{||\cdot||_n\}_{n\in\mathcal{N}}$ is a seminormed vector space.

(S2) X is a compact convex subset of $\mathcal{D}$.

(S3) μ_1, μ_2 are regular borel measure on X.

(A1) $\mu_1 \prec \mu_2$.

(S4) $f \in A(X)$.

Then $\mu_1(f) = \mu_2(f)$.

Proof. Because $f \in CB(X) \cap (-CB(X))$, $\mu_1(f) = \mu_2(f)$. $\square$

Definition 3.17 (Upper envelope function)**.** *Let*

(S1) $(\mathcal{D}, \{||\cdot||_n\}_{n\in\mathcal{N}}$ *is a seminormed vector space.*

(S2) X *is a compact convex subset of* $\mathcal{D}$.

(S3) $f \in C(X, \mathbb{R})$.

We set

$$\tilde{f}(x) := \inf\{h(x) | h \in A(X), h \geq f\} \ (x \in X)$$

Proof of $\{h \in A(X) | h \geq f\} \neq \phi$. Because X is compact and $f \in C(X, \mathbb{R})$, $||f||_{L^\infty(X)} < \infty$. Constant function with $||f||_{L^\infty(X)}$ is continuous affine type function. So, $\{h \in A(X) | h \geq f\} \neq \phi$. $\square$

Proposition 3.30. *Let*

(S1) $(\mathcal{D}, \{||\cdot||_n\}_{n\in\mathcal{N}}$ *is a seminormed vector space.*

(S2) X *is a compact convex subset of* $\mathcal{D}$.

Then

(i) For any $f \in C(X, \mathbb{R})$, $\tilde{f}$ *is bounded and upper semicontinuous.*

(ii) For any $f \in C(X, \mathbb{R})$, $f \leq \tilde{f}$.

(iii) For any $f \in CB(X)$, $f = \tilde{f}$.

(iv) For any $f, g \in CB(X)$, $\widetilde{f+g} \leq \tilde{f} + \tilde{g}$.

(v) For any $f, g \in CB(X)$, $|\tilde{f} - \tilde{g}| \leq ||f - g||_{L^\infty(X)}$.

(vi) For any $f \in CB(X)$ *and* $r \in (0, \infty)$, $\tilde{rf} = r\tilde{f}$.

Proof of (i). Because $\tilde{f} \leq ||f||_{L^\infty(X)}$, $\tilde{f}$ is bounded. Let us fix any $c \in \mathbb{R}$ and $x \in \tilde{f}^{-1}((-\infty, c))$. Then there is $h \in A(X)$ such that $h(x) < c$. Because h is continuous, there is V which is a neighborhood of 0 such that $h(x+y) < c$ $(\forall y \in V \cap X)$. So, $\tilde{f}(x+y) < c$ $(\forall y \in V \cap X)$. This means that $x + V \subset \tilde{f}^{-1}((-\infty, c))$. So, $\tilde{f}$ is upper semicontinuous. $\square$

Proof of (ii). (ii) is clear from the definition of upper envelope functions. $\square$

Proof of (iii). We set $K := \{(x, r) \in X \times \mathbb{R} | 0 \leq r \leq f(x)\}$. Because X is compact and f is continuous concave, K is compact convex subset of $X \times \mathbb{R}$. Aiming contradiction, let us assume $f(x_0) < \tilde{f}(x_0)$ for some $x_0 \in X$. $(x_0, \tilde{f}(x_0)) \notin K$. By Theorem3.1, there is L which is a continuous $\mathbb{R}$-linear functional on $\mathcal{D} \times \mathbb{R}$ such that

$$L(x_0, \tilde{f}(x_0)) > 1 > L(x, f(x)) \ (\forall x \in X)$$

This implies $(\tilde{f}(x_0) - f(x_0))L(0, 1) > 0$. So,

$$L(0, 1) > 0$$

We set

$$h(x) := \frac{1 - L(x, 0)}{L(0, 1} \ (x \in \mathfrak{D})$$

Then $h \in A(X)$ and

$$L(x, h(x)) = 1 \ (\forall x \in \mathfrak{D})$$

So,

$$L(x_0, \tilde{f}(x_0)) > L(x, h(x)) > L(x, f(x)) \ (\forall x \in X)$$

This implies

$$0 < L(x, h(x)) - L(x, f(x)) = L(0, h(x) - f(x)) = (h(x) - f(x))L(0, 1) \ (\forall x \in X)$$

So,

$$f(x) < h(x) \ (\forall x \in X)$$

Similarly,

$$h(x_0) < \tilde{f}(x_0)$$

These two equation contradict with each other. $\square$

Proof of (iv). Let us fix any $x \in X$ and $\epsilon > 0$. Then there is $h_1, h_2 \in A(X)$ such that $f \leq h_1$ and $g \leq h_2$ and $h_1(x) \leq \tilde{f}(x) + \epsilon$ and $h_2(x) \leq \tilde{g}(x) + \epsilon$. Because $h_1 + h_2 \in A(X)$ and $f + g \leq h_1 + h_2$. $\widetilde{f+g}(x) \leq h_1(x) + h_2(x)$. So, $\widetilde{f+g}(x) \leq \tilde{f}(x) + \tilde{g}(x) + 2\epsilon$. □

Proof of (v). By (iv), for any $x \in X$.

$$\tilde{f}(x) - \tilde{g}(x) \leq \widetilde{f-g+g}(x) - \tilde{g}(x) \leq \widetilde{f-g}(x)$$

Because $||f-g|| \in A(X)$, $\widetilde{f-g} \leq ||f-g||$. So, (v) holds. □

Proof of (vi). This is clear from the definition of upper envelope functions. □

Definition 3.18 (Convex cone)**.** *Let*

(S1) $\mathcal{D}$ *is a* $\mathbb{R}$*-vector space.*

(S2) $v_1, v_2, ..., v_m \in \mathcal{D}$.

Then

$$cc(v_1, v_2, ..., v_m) := \{\sum_{i=1} a_i v_i | a_i \geq 0 \ (\forall i)\}$$

Proposition 3.31. *Let*

(S1) $\mathcal{D}$ *is a* $\mathbb{R}$*-vector space.*

(S2) $v_1, v_2, ..., v_m \in \mathcal{D}$.

(A1) $0 \in ex(cc(v_1, v_2, ..., v_m))$.

Then there is $w_1, ..., w_n \in\in \mathcal{D}$ *such that* $w_1, ..., w_n$ *are linear independent and*

$$cc(v_1, v_2, ..., v_m) \subset cc(w_1, w_2, ..., w_n)$$

Proof. We set $n_0 := dim\{v_1, ..., v_m\}$. Using mathematical induction on $m - n_0$, we prove this proposition. Let us fix any $d \in \mathbb{N}$. Let us assume this proposition holds for $m - n_0 \leq d$ and $m - n_0 = d + 1$. Then we can assume

$$v_m = -\sum_{i=1}^{k} a_i v_i + \sum_{j=1}^{l} b_j v_{k+j}, k + l = m - 1$$

If $k = 0$ or $v_m \neq 0$, then $cc(v_1, v_2, ..., v_m) = cc(v_1, v_2, ..., v_{m-1})$. By the assumption of mathematical induction, this proposition holds. So, we can assume $k \neq 0$ and $v_m \neq 0$. If $l = 0$, $0 = \dfrac{1}{2}(v_m + \sum_{i=1}^{k} a_i v_i)$. This means $0 \notin ex(cc(v_1, ..., v_m))$. So, we can assume $l \neq 0$. Furthermore, we can assume

$$\begin{aligned} k := &\min\{K \in \mathbb{N} | \exists \sigma : \{1, ..., m\} \to \{1, ..., m\}\text{:bijective}, \exists c_1, ..., c_K > 0, \exists d_1, ..., d_L \geq 0 (L := m - K) \text{ s.t.} \\ &- \sum_{i=1}^{K} c_\sigma(i) v_{\sigma(i)} + \sum_{j=1}^{L} b_{\sigma(j)} v_{\sigma(k+j)} = 0 \ \} - 1 \end{aligned}$$

We set

$$v'_{k+j} = \frac{-1}{l} \sum_{i=1}^{k} a_i v_i + b_j v_{k+j} \ (j = 1, ..., l)$$

Because of the minimalism of k, $0 \in ex(cc(v_1, ...v_k, v'_{k+1}, ..., , v'_{k+l}))$. Because $v_{k+j} = \dfrac{1}{b_j}(\sum_{i=1}^{k} a_i v_i + v'_{k+j})$ $(\forall j)$ and $\sum_{j=1}^{l} v'_{k+j} = v_m$,

$$cc(v_1, v_2, ..., v_m) \subset cc(v_1, ..., v'_{k+l}), k + l = m - 1$$

By the assumption of mathematical induction, this proposition holds. □

Proposition 3.32. *Let*

(S1) $(\mathcal{D}, \{|| \cdot ||_n\}_{n \in \mathcal{N}}$ *is a seminormed vector space.*

(S2) X *is a compact convex subset of* $\mathcal{D}$.

(S3) $x \in X$.

(A1) $f(x) = \tilde{f}(x)$ $(\forall f \in C(X, \mathbb{R}))$.

Then $x \in ex(X)$.

Proof. Aiming contradiction, let us assume $x \notin ex(X)$. Then there is $y, z \in X$ such that $y \neq z$ and $x = \dfrac{y+z}{2}$. We set $f(\cdot) := d(x, \cdot)$. By Proposition3.30,

$$0 = f(x) = \tilde{f}(x) \geq \frac{1}{2}(\tilde{f}(y) + \tilde{f}(z)) = \frac{1}{2}(f(y) + f(z)) > 0$$

This is contradiction. □

Proposition 3.33. *Let*

(S1) X *is a locally compact hausdorff topological space.*

(S2) $\mathfrak{B}$ *is the minimal borel set family containing all relative compact open sets.*

(S3) $\mathfrak{M}$ *is the set of all regular borel measures on* X*.*

(S4) $\mu \in \mathfrak{M}$.

Then $M_\mu := \{\nu \in \mathfrak{M} | \mu \geq 0, \mu \prec \nu\}$ *has a maximal element.*

STEP1. We set

$$\Phi := \{T \subset M_\mu | T \text{ is totally ordered with } \prec\}$$

Let us fix any $\mathfrak{N}$ which is totally ordered subset of M_μ with inclusion relationship. Clearly $\cup_{T \in \mathfrak{N}} T$ is totally ordered with $\prec$. So, by Zorn Lemma, Φ has a maximal element F. Because F is totally ordered with $\prec$, for any finite elements $\tau_1, ..., \tau_m \in F$, $\cap_{i=1}^m M_{\tau_i} \neq \phi$. □

STEP2. We set

$$S := \{\mu \in \mathfrak{M} | \mu(1) = \nu(1)\}$$

Because $S \subset \{F \in C(X)^* | ||F|| \leq |\nu(1)|\}$ and S is closed subset in $*$-weak topology, by Banach-Alaogrou Theorem, S is compact subset in $*$-weak topology. For any $\tau \in F$,

$$M_\tau = \cap_{f \in CB(X)} \{\mu \in S | \mu(f) \geq \nu(f)\} \cap \cap_{f \in C_c^+(X)} \{\mu \in S | \mu(f) \geq 0\}$$

So, $M_\tau \subset S$ is closed subset in $*$-weak topology, □

STEP3. By STEP1 and STEP2, $\cap_{\tau \in F} M_\tau \neq \phi$. Let us take a $\mu_0 \in \cap_{\tau \in F} M_\tau$. For aiming contradiction, let us assume there is $\mu \in M_\nu$ such that $\mu_0 \prec \mu$ and $\mu \neq \mu_0$. By Proposition, $\mu \notin F$. But $F \cap \{\mu\}$ is totally ordered. This is contradiction. So, μ_0 is a maximal element of M_ν. □

Proposition 3.34. *Let*

(S1) $(\mathcal{D}, \{|| \cdot ||_n\}_{n \in \mathcal{N}}$ *is a seminormed vector space.*

(S2) X *is a compact convex subset of* $\mathcal{D}$.

(S3) μ *is a maximal element in* $\mathfrak{M}$.

Then

$$\mu(f) = \mu(\tilde{f}) \ (\forall f \in C(X, \mathbb{R}))$$

Proof. We set

$$\rho(g) := \mu(\tilde{g}) \ (g \in C(X, \mathbb{R}))$$

Clearly ρ is a seminorm on $C(X, \mathbb{R})$. Let us fix any $f \in C(X, \mathbb{R})$.

$$L(rf) := r\mu(\tilde{f}) \ (r \in \mathbb{R})$$

By Hahn Banach Theorem, L has an extension L' which is a $\mathbb{R}$-linear functional on $C(X, \mathbb{R})$ such that $L' \leq \rho$. Let us fix any $g \in C(X, \mathbb{R})^+$. Because $-g \leq 0$, $\tilde{-g} \leq 0$. So,

$$L(-g) \leq \rho(-g) = \mu(\tilde{-g}) \leq \mu(0) \leq 0$$

This implies $0 \leq L(g)$. So, by Riez representation theorem, L is a regular borel measure.

Let us fix any $h \in CB(X)$. Because $-h$ is continuous and concave, by Proposition,

$$L(-h) \leq \rho(-h) = \mu(\tilde{-h}) = \mu(-h)$$

So, $\mu \prec L$. This implies $\mu = L$. So,

$$\mu(\tilde{f}) = L(f) = \mu(f)$$

□

Proposition 3.35. *Let*

(S1) $(\mathcal{D}, \{||\cdot||_n\}_{n\in\mathcal{N}}$ *is a seminormed vector space.*

(S2) X *is a compact convex subset of* $\mathcal{D}$.

(S3) f *is continuous strictly convex function on* X.

(S4) $z \notin ex(X)$.

Then $f(z) < \tilde{f}(z)$

Proof. By there are $x, y \in X$ such that $x \neq y$ and $z = \frac{1}{2}(x+y)$ Let us fix any $h \in A(X)$ such that $f \leq h$. Then

$$f(z) < \frac{1}{2}(f(x) + f(y)) \leq \frac{1}{2}h(x) + h(y)) = h(z)$$

So,

$$f(z) < \frac{1}{2}(f(x) + f(y)) \leq \tilde{f}(z)$$

□

Theorem 3.13 (Choquet Theorem)**.** *Let*

(S1) $(\mathcal{D}, \{||\cdot||_n\}_{n\in\mathcal{N}}$ *is a seminormed vector space.*

(S2) X *is a compact convex subset of* $\mathcal{D}$.

(S3) $x_0 \in X$.

Then there are K *is a borel set and* μ *which is a regular borel probability measure on* X *such that* K *supports* μ *and* $X \setminus K \subset ex(X)$ *and*

$$\varphi(x_0) = \int_K \varphi(x) d\mu(x) \ (\forall \varphi \in A(X))$$

STEP1. Construction of continuous strictly convex function. We set $U := \{h \in A(X) | ||h||^{\infty} = 1\}$. Because X is compact metrizable, there is a countable set $\{h_n\}_{n\in\mathbb{N}} \subset U$ which is dense in U. We set

$$f := \sum_{n=1}^{\infty} \frac{h_n^2}{2^n}$$

We will show f is strictly convex. Let us fix any $x, y \in X$ such that $x \neq y$ and $\lambda \in (0,1)$. By Hahn-Banach Theorem, there is f which is a real-valued continuous linear functional on$\mathcal{D}$ and satisfies $f(x) > f(y)$. Because $\dfrac{f - \frac{f(x)+f(y)}{2}}{||f - \frac{f(x)+f(y)}{2}||_{L^{\infty}(\mathcal{D})}} \in U$, there is $n \in \mathbb{N}$ such that $h_n(x) > 0 > h_n(y)$.

$$\begin{aligned} &h_n(\lambda x + (1-\lambda)y)^2 = \lambda^2 h_n(x)^2 + (1-\lambda)^2 h_n(y)^2 + \lambda(1-\lambda)h_n(x)h_n(y) < \lambda^2 h_n(x)^2 + (1-\lambda)^2 h_n(y)^2 \\ &\leq \lambda h_n(x)^2 + (1-\lambda)h_n(y)^2 \end{aligned}$$

This implies that $f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y)$. So, f is strictly convex. □

STEP2. Construction of a regular borel measure. Because X is locally compact hausdorff space, by Riez-Markov-Kakutani Theorem, $\delta : C(X) \ni g \mapsto g(x) \in \mathbb{C}$ defines a regular borel measure. So, by Proposition3.33, there is a maximal element $\mu \in \mathfrak{M}$ such that $\delta \prec \mu$. By Proposition3.29, $\mu(g) = \delta(g)$ for any $g \in A(X)$. Because $1 \in A(X)$, $\mu(X) = 1$. □

STEP3. Construction of K. We set

$$K := \cup_{n\in\mathbb{N}} K_n, K_n := \{x \in X | \tilde{f}(x) - f(x) > \frac{1}{n}\}$$

Because $K_n = (\cap_{m\in\mathbb{N}}\{x \in X | \tilde{f}(x) - f(x) < \frac{1}{n} + \frac{1}{m}\})^c$ and $\tilde{f} - f$ is upper continuous, K_n is measurable for any $n \in \mathbb{N}$. So, K is borel measurable. By Proposition3.35, $X \setminus K \subset ex(X)$. By Proposition3.34, $\mu(f) = \mu(\tilde{f})$. So $\mu(K) = 0$. This implies $X \setminus K$ supports μ. □

3.9 Mautner-Teleman's theorem

Proposition 3.36. *Let*

(S1) G is a Lie group.

(S2) (π, V) is a continuous unitary cyclic representation of G with a cyclic vector ω.

Then there is a finite mesurable space $(X, \mathcal{M}, \mu)$ and a direct integral $\int_X^{\mathcal{G}} \omega(x)d\mu(x)$ which is isomorphic to (π, V) as continuous unitary representation.

STEP1. Decomposition of a matrix coefficient. We can assume

$$||\omega|| = 1$$

We set

$$\varphi(g) := (\pi(g)\omega, \omega) \ (g \in G)$$

Because $\mathbb{P}_1$ is a compact convex subset of $C(G)$ with compact convergence topology which is metrizable by countable seminorms. By Theorem3.13, there are μ which is a probability measure on $\mathbb{P}_1$ and X which is a borel mesurable set such that $X \subset ex(\mathbb{P}_1)$

$$F(\varphi) = \int_X F(\varphi_x)d\mu(x) \ (\forall F \in A(\mathbb{P}_1))$$

Here, $\varphi_x = x$. For any $g \in G$, $\mathbb{P}_1 \in \psi \mapsto Re\psi(g) \in \mathbb{R}$ and $\mathbb{P}_1 \in \psi \mapsto Im\psi(g) \in \mathbb{R}$ are continuous affine by Raikov-Godement-Yoshizawa Theorem(Theorem3.10). So,

$$\varphi(g) = \int_X \varphi_x(g)d\mu(x) \ (\forall g \in G)$$

□

STEP2. Construction of a family of irreducible representations. We set

$$(T(x), H(x)) : \text{The representation generated by the GNS construction } (x \in X)$$

and

$$\Pi := \Pi_{x \in X} H(x)$$

and

$$v(f, x) : \text{The projection of } f \text{ in } H(x) \ (f \in C_c(G), x \in X)$$

and

$$\mathfrak{D}_0 : \text{The vector space generated by } \{\lambda(\cdot)v(f, \cdot) | f \in C_c(G), \lambda \in L^\infty(X, \mu)\}$$

We set $\mathfrak{D}$ by the completion of $\mathfrak{D}_0$ with the inner product $(\cdot, \cdot) := \int_X (\cdot, \cdot)_{H(x)} d\mu(x)$. As we showed in the process of proving Proposition3.25, any cauchy sequence of $\mathfrak{D}_0$ has a subsequence which converges pointwise some element of Π. So, we can embedded $\mathfrak{D}$ in Π. Clearly $\mathfrak{D}$ is $\mathbb{C}$-linear subspace of Π. And, for each $\lambda \in L^\infty(X, \mu)$ and $f \in C_c(G)$, $X \ni x \mapsto ||\lambda(x)v(f, x)||_{H(x)}$ is measurable and L^2-integrable. So, forany $F \in \mathfrak{D}$, $X \ni x \mapsto ||F(x)||_{H(x)}$ is measurable and L^2-integrable. Clearly $\mathfrak{D}$ satisfies (v) in Proposition3.25. So, it is enough to show (iv) in Proposition3.25. Hereafter, let us fix any $u \in \Pi$ which satisfies (iv)(a) and (iv)(b) in Proposition3.25. There exists $\{v_n\}_{n \in \mathbb{N}} \subset \mathfrak{D}_0$ such that

$$\lim_{n \to \infty} ||v_n - u|| = \inf_{v \in \mathfrak{D}_0} ||v - u||$$

For each $u, v \in \Pi$,

$$P(u, v)(x) = \begin{cases} \dfrac{(u(x), v(x))}{||v(x)||^2} & (v(x) \neq 0) \\ 0 & (v(x) = 0) \end{cases} \quad (x \in X)$$

We will show

$$||u(x) - P(u, v)(x)|| \leq ||u(x) - v(x)|| \ (\forall v \in V, \forall x \in X) \tag{3.9.1}$$

Let us fix any $v \in V$ and $x \in X$. If $v(x) = 0$, (3.9.1) holds. So, we can assume $v(x) \neq 0$. Then

$$||u(x) - P(u, v)(x)||^2 = ||u(x)||^2 - \frac{|(u(x), v(x))|^2}{||v(x)||^2}$$

and

$$||u(x)-v(x)||^2 = ||u(x)||^2 - 2Re(u(x),v(x)) + ||v(x)||^2$$

So,

$$||v(x)||^2(||u(x)-v(x)||^2 - ||u(x)-P(u,v)(x)||^2) = |(u(x),v(x)) - ||v(x)||^2|^2 \geq 0$$

This implies (3.9.1). So, by (3.9.1) and Proposition1.14, $\{P(u,v_n)\}_{n\in\mathbb{N}}$ is a cauchy sequence. So, $u_0 := \lim_{n\to\infty} P(u,v_n) \in \mathfrak{D}$ exists. We will show u_0 $u \in \Pi$ which satisfies (iv)(3.7.1) in Proposition3.25. Aiming contradiction, let us assume that there are $u' \in \mathfrak{D}$ and a borel measurable set E such that $\mu(E) > 0$ and

$$(u(x)-u_0(x), u'(x)) \neq 0 \ (a.e x \in X)$$

As we showed in the process of proving Proposition3.25, any cauchy sequence of $\mathfrak{D}_0$ has a subsequence which converges pointwise some element of Π. So, we can assume $u' \in \mathfrak{D}_0$. We set

$$v := u' - P(u', u_0)$$

For any $x \in X$, we will show

$$(v(x), u_0(x)) = 0 \tag{3.9.2}$$

and

$$(u(x)-u_0(x), u_0(x)) = 0 \tag{3.9.3}$$

If $u_0(x) = 0$, the both clearly holds. So, we can assume $u_0(x) \neq 0$. Then,

$$(v(x), u_0(x)) = (u'(x), u_0(x)) - \frac{(u'(x), u_0(x))}{||u_0(x)||^2}|(u_0(x), u_0(x))|^2 = 0$$

This means (3.9.2) holds. Furthermore,

$$(u(x), u_0(x)) = (u(x), \frac{(u(x), v_\infty(x))}{||v_\infty(x)||^2} v_\infty(x)) = \frac{|(u(x), v_\infty(x))|^2}{||v_\infty(x)||^2} = (u_0(x), u_0(x))$$

This means (3.9.3) holds. For any $x \in E$,

$$(u(x)-u_0(x), u'(x)) = (u(x), u'(x)) - (u_0(x), u'(x)) = (u(x), v(x)) + (u(x), P(u', u_0)(x)) - (u_0(x), u'(x))$$
by (3.9.2)
$$= (u(x), v(x)) + (u(x), P(u', u_0)(x)) - (u_0(x), P(u', u_0)(x))$$
by (3.9.3)
$$= (u(x), v(x))$$

So,

$$(u(x), v(x)) \neq 0 \ (\forall x \in E) \tag{3.9.4}$$

We will show

$$P(u', u_0) \in \mathfrak{D} \tag{3.9.5}$$

Clearly,

$$\lambda \in L^\infty(X), w \in \mathfrak{D} \implies \lambda w \in \mathfrak{D}$$

For $n \in \mathbb{N}$, we set

$$\lambda_n(x) := \begin{cases} \frac{(u'(x), u_0(x))}{||u_0(x)||^2} & (||v(x)|| \geq \frac{1}{n} \text{ and } ||u'(x)|| \leq n) \\ 0 & (otherwise) \end{cases} \quad (x \in X)$$

Because $\lambda_n \in L^\infty(X,\mu)$, $\lambda_n u_0 \in \mathfrak{D}$. Let us fix any $n_0 \in \mathbb{N}$. If $m, n \geq n_0$,

$$||\lambda_m u_0 - \lambda_n u_0|| \leq \int_{||u_0(x)|| \leq \frac{1}{n_0}, ||u'(x)|| \geq n_0} ||u'(x)||^2 d\mu(x)$$

The right side of this equation converges to 0 when $n \to \infty$. So, $\{\lambda_m u_0\}_{m\in\mathbb{N}}$ is a cauchy sequence. So, $P(u', u_0) = \lim_{m\to\infty} \lambda_m u_0$(pointwise convergence) is in $\mathfrak{D}$. We set

$$u_1 := u_0 + P(u, v)$$

By the way which is similar to the proof of (3.9.5), $P(u,v) \in \mathfrak{D}$. This implies $u_1 \in \mathfrak{D}$.

$$||u-u_1||^2 = ||u-u_0||^2 - 2Re(u-u_0, P(u,v)) + \frac{|(u,v)|^2}{||v||^2}$$

by Proposition3.9.2

$$= ||u-u_0||^2 - 2Re(u, P(u,v)) + \frac{|(u,v)|^2}{||v||^2} = ||u-u_0||^2 - \frac{|(u,v)|^2}{||v||^2} < \inf_{v \in \mathfrak{D}_0} ||v-u||^2$$

This is a contradiction. So, $(X, \mathcal{B}(X), \mu, \Pi, \mathfrak{D})$ is a direct integral of Hilbert spaces. □

STEP3. Construction of continuous unitary representation. We set

$$T_g v(f,x) := v(R_g f, x) \ (f \in C_c(G), x \in X)$$

Because

$$(v(R_g f, x), v(R_g g, x)) = (v(f,x), v(g,x)) \ (\forall f, g \in C_c(G), \forall x \in X)$$

T_g is a unitary operator on $\mathfrak{D}_0$. Because $\mathfrak{D}_0$ is dense in $\mathfrak{D}$, T_g has the unique extension on $\mathfrak{D}$. For any $f \in C_c(G)$ and $g_1, g_2 \in G$, $||T_{g_1} v(f,\cdot) - T_{g_2} v(f,\cdot)|| \leq \mu(X) ||R_{g_1} f - R_{g_2} f||_{L^\infty}$. So,

$$G \ni g \mapsto T_g v(f,\cdot) \in \mathfrak{D}$$

is continuous. Because T is unitary and $\mathfrak{D}_0$ is dense in $\mathfrak{D}$, T is weak continuous. So, T is strong continuous. Let us take $\{f_n\}_{n\in\mathbb{N}} \subset C_c^+(G)$ such that $\int_G f_n dg_r = 1$ and $supp(f_n) \subset exp(\{X \in M(n,\mathbb{C}| \ ||X|| \leq \frac{1}{n}\}) \ (\forall n \in \mathbb{N})$. Then $\{v(f_n,\cdot)\}_{n\in\mathbb{N}}$ has a subsequence which converges some $v \in \mathfrak{D}$. By the same way as the proof of Theorem3.9, we can show the following.

$$(v(f,\cdot), v(g,\cdot)) = (v(f,\cdot), \int_G g(y^{-1}) T_y^{-1} v(g,\cdot)) \Delta_r(y) dg_r(y)) \ (\forall f, g \in C_c(G))$$

$$v(g,\cdot) = \int_G g(y^{-1}) T_y^{-1} v(g,\cdot)) \Delta_r(y) dg_r(y) \ (\forall g \in C_c(G))$$

By the same way as the proof of Theorem3.9, g is in the closed subspace generated by $T(G)v$. Because $\mathfrak{D}_0$ is dense in $\mathfrak{D}$, T is cyclic with cyclic vector v. Clearly the following holds.

$$(T_g v)(x) = T_g^x v(x) \ (\forall x \in X)$$

Here, T^x is the representation by GNS construction for $x \in X$. So,

$$\varphi(g) = \int_X \varphi_x(g) d\mu(x) = \int_X (T_g^x v(x), v(x)) d\mu(x) = \int_X (T_g v(x), v(x)) d\mu(x) = (T_g v, v) \ (\forall g \in G)$$

By Proposition3.6, (π, V) and $(T, \int_X^{\mathfrak{D}} H(x) d\mu(x))$ are isomorphic as continuous unitary representations. □

By Proposition3.7 and Proposition3.36, the following holds.

Theorem 3.14 (mautner-Teleman's theorem)**.** *Let*

(S1) G is a Lie group.

(S2) (π, V) is a continuous unitary representation of G.

Then there is a a family of direct integral of continuous unitary representations $\{\int_{X_\lambda}^{\mathcal{D}_\lambda} \omega(x) d\mu_\lambda(x)\}_{\lambda\in\Lambda}$ such that

(i) (X_λ, μ_λ) is a finite measurable space $(\forall \lambda \in \Lambda)$.

(ii) $\int_{X_\lambda}^{\mathcal{D}_\lambda} \omega_\lambda(x) d\mu_\lambda(x)$ is a continuous cyclic unitary representation of G.

(iii) (π, V) and $\bigoplus_{\lambda\in\Lambda} \int_{X_\lambda}^{\mathcal{D}_\lambda} \omega_\lambda(x) d\mu_\lambda(x)$ are isomorphic as continuous unitary representations of G.

3.10 Review

Please note that the statements in this subsection are generally inaccurate. In this chapter, the following mautner-Teleman theorem is the main theorem(Theorem3.14).

Theorem (mautner-Teleman theorem)**.** *Let*

(S1) G *is a Lie group.*

(S2) (π, V) *is a continuous unitary representation of* G*.*

Then there is a a family of direct integral of continuous unitary representations $\{\int_{X_\lambda}^{\mathcal{D}_\lambda} \omega(x) d\mu_\lambda(x)\}_{\lambda\in\Lambda}$ *such that*

(i) (X_λ, μ_λ) *is a finite measurable space* $(\forall\lambda \in \Lambda)$*.*

(ii) $\int_{X_\lambda}^{\mathcal{D}_\lambda} \omega_\lambda(x) d\mu_\lambda(x)$ *is a continuous cyclic unitary representation of* G*.*

(iii) (π, V) *and* $\bigoplus_{\lambda\in\Lambda} \int_{X_\lambda}^{\mathcal{D}_\lambda} \omega_\lambda(x) d\mu_\lambda(x)$ *are isomorphic as continuous unitary representations of* G*.*

This theorem states that any continuous unitary representation of Lie group is decomposed into irreducible continuous unitary representations. The direct integral of continuous unitary representations $\{X, \mathfrak{D}, \mu, T_x, H(x)\}$ is a subset of $\Pi := \Pi_{x\in X} H(x)$ which satisfies the following main conditions.

(i) For any $u, v \in \mathfrak{D}$, $(u(\cdot), v(\cdot))$ is measurable and integrable.

(ii) $\{T_x\}_{x\in X}$ defines T which is a continuous and unitary action on $\mathfrak{D}$.

(iii) If $v \in \Pi$ and $||v(\cdot)||$ is measurable and bounded by a L^2 function and $(v(\cdot), u(\cdot))$ is measurable, v can be seen as the element of $\mathfrak{D}$ in a sense.

In special, $(T, \mathfrak{D})$ is a continuous unitary representation of G.

I also think that the following Gelfand-Raikov Theorem(Theorem3.12) obtained in the process of showing mautner-Teleman theorem is also a very significant theorem. This theorem states that we can distinguish any two element of Lie group G by the unitary dual $\hat{G}$ of G. The definition of a unitary dual is the set of all continuous irreducible unitary representation of G.

Theorem (I.M.Gelfand-D.A.Raikov Theorem)**.** *Let*

(S1) G *is a Lie group.*

(S2) $g_1, g_2 \in G$*.*

(A1) $T_{g_1} = T_{g_2}$ $(\forall(T, V) \in \hat{G})$*.*

Then $g_1 = g_2$*.*

Below, I would like to review the process of obtaining these two theorems with my personal opinions and impressions. We begin by examining the cyclic representation rather than directly examining the irreducible representation. The definition of the cyclic representation (π, V) with a cyclic vector v is the representation space is spanned by $\pi(G)v$. The definition of the cyclic representation is the representation whose any vector is a cyclic vector. One of the reasons for focusing on cyclic representations is to investigate the Jordan normal form with respect to matrices that cannot be diagonalized in matrix decomposition theory. Supposing (π, V) is a representation of $\mathbb{Z}$, $\pi(1)$ is similar a jordan block if and only if (π, V) is cyclic[15].

By Zorn lemma and the same argument as the diagonalization of unitary matrices, we can show that any continuous unitary representation of Lie group is decomposed into cyclic continuous unitary representations (Proposition3.7). So, the proof of mautner-Teleman theorem is attributed to the case for cyclic representations.

We focus on matrix coefficients whose form is $\varphi := (\pi(\cdot)v, v)$ from a continuous cyclic representation (π, V) with a cyclic vector v. φ satisfies the following condition.

$$\sum_{i=1}^{N} a_i \pi(g_i) v = 0 \iff \sum_{i=1} a_i \varphi(g g_i) = 0 \ (\forall g \in G)$$

This implies if $(\pi_1(\cdot)v_1, v_1) = (\pi_2(\cdot)v_2, v_2)$ then π_1 and π_2 are isomorphic as continuous unitary representations (Proposition3.6). So, this is the kicker to investigate $\varphi := (\pi(\cdot)v, v)$. This function satisfies the following conditions.

(i) $\varphi(e) \geq 0$

(ii) $\varphi(g^{-1}) = \overline{\varphi(g)}$

(iii) $|\varphi(g)| \leq \varphi(e)$

(iv) $|\varphi(g_1) - \varphi(g_2)|^2 \leq \frac{1}{2}\varphi(e)|\varphi(e) - Re\varphi(g_1^{-1}g_2)|$

(v) If $(f,g)_\varphi := \int_G \varphi(xy^{-1})f(y)\bar{g(x)}dg_r(x)dg_r(y)$ $(f, g \in C_c(G))$, then $(\cdot,\cdot)_\varphi$ satisfies a nonnegative Hermitian semibilinear form.

We call functions which satisfies these conditions positive definite functions even if they don't have a form $(\pi(\cdot)v, v)$. The right regular action R preserves this nonnegative Hermitian semibilinear form and continuos. So, we construct continuous unitary representation (T, H_φ). Taking a sequence of $C_c^+(G)$ $\{f_n\}_{n\in\mathbb{N}}$ such that $||f_n||_{L^1(G)} = 1$ $(\forall n)$, by Banach-Alaogrou Theorem (Theorem3.7), $\{f_n\}_{n\in\mathbb{N}}$ has a convergent subsequence which converges to some $v \in H_\varphi$ in $*$-weak topology. Banach-Alaoglu Theorem states the unit sphere on of dual of a separable normed space is sequencial compact in $*$-weak topology. v likes a dirac delta function whose support $\{e\}$. For any $g \in G$, $T_g v$ likes a dirac delta function whose support $\{g^{-1}\}$. So, v is a cyclic vector of H_φ. Assigning $f = T_g v$ and $g = v$ in (v), we see $\varphi = (T.v, v)$. In special φ can been seen as a continuous positive definite function. This method of obtaining a continuous and cyclic unitary representation from a positive definite function is the GNS construction.

The GNS construction is a powerful technique that will be used with great success throughout this chapter. For example, if $g_1 \neq g_2$ in G, there is $f \in C_c^+(G)$ such that $g_1g_2^{-1} \notin supp(f)$ and $f(e) = 1$. So, the continuous cyclic unitary representation by GNS construction for $(R.f, f)$ separates e and $g_1g_2^{-1}$. So, by GNS construction, the claim is established with the 'irreducible' part in Gelfand-Raikov replaced by 'cyclic'.

We see GNS construction gives a map from the space of continuous positive definite functions to the set of all cyclic continuous unitary representations. So, we focus on $\mathbb{P}_1$ which is the set of all normalised continuous positive definite functions whose value at e is 1. There are two possible ways to set a topology in $\mathbb{P}_1$. One is the topology from compact convergence(Pontryagin topology). Another one is the $*$-weak topology. By the strong continuity (iv), these topology is the same. This is Raikov-Godement-Yoshizawa Theorem(Theorem3.10). A sketch of the proof of this theorem is shown below. Let us assume $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathbb{P}_1$ converges to $\varphi \in \mathbb{P}_1$ in $*$-weak topology. Then for any $f \in C_c(G)$, $\{f * \varphi_n\}_{n\in\mathbb{N}}$ converges to $f * \varphi \in \mathbb{P}_1$ pointwise. Because of (iv), $\{f * \varphi_n\}_{n\in\mathbb{N}}$ is equicontinuous on any compact subset. By the same argument of the proof of AscoliArzel theorem, $\{f * \varphi_n\}_{n\in\mathbb{N}}$ converges to $f * \varphi$. Because of (iv), taking f such that $supp(f)$ is sufficient small, $||\varphi_n - \{f * \varphi_n||_\infty\}_{n\in\mathbb{N}}$ and $||\varphi - \{f * \varphi||_\infty\}_{n\in\mathbb{N}}$ are uniformly small. So, $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathbb{P}_1$ compact converges to $\varphi \in \mathbb{P}_1$.

By this powerfull theorem, we can show important properties of the topology of $\mathbb{P}_1$. $*$-weak convergence preserves (iii) and (iv) and boundedness of positive definite functions. By GNS construnction, $*$-weak convergence preserves continuity of positive definite functions. So, $\mathbb{P}_1$ is closed subset of $*$-weak topology. By Banach-Alaoglu theorem and $L^1(G)^* = L^\infty(G)$, $\mathbb{P}_1$ is compact. Because $\mathbb{P}_1$ is convex, by Krein-Millman theorem, any $\varphi \in \mathbb{P}_1$ can be uniformly approximated by some convex combination of $\{\varphi_n\}_{n=1}^N \subset ex(\mathbb{P}_1)$ on any compact subset.

We see

$$ex(\mathbb{P}_1) = \mathbb{P}_1 \cap \Phi^{-1}(\hat{G})$$

Here, Φ is the map defined by GNS construction. Because by orthogonal projections we can get a convex combination decomposition of positive definite function from a decomposition of a representation space of GNS construction, the $\subset$ part is shown. By Shur Lemma, we can obtain a decomposition of a representation space of GNS construction from a decomposition of a element of $\mathbb{P}_1$. The above discussion show Gelfand-Raikov theorem.

Next step, we elaborate Krein-Millman theorem. I mean for each $\varphi \in \mathbb{P}_1$, there is a probability measure $\mu \in P(\mathbb{P}_1)$ such that there is $Y \subset ex(\mathbb{P}_1)$ which supports μ and

$$\varphi = \int_Y \varphi_x d\mu(x)$$

This is from Choquet Theorem(Theorem3.13).

I think our first step is to interpret the value $\varphi(g)$ in terms of inverted space and function. I mean for each $g \in G$, we interpret g as

$$f_g : \mathbb{P}_1 \ni \varphi \mapsto \varphi(g)$$

By Raikov-Godement-Yoshizawa Theorem, f_g is continuous. Because f_g is convex and concave, if we define

$$\mu_1 \prec \mu_2 :\iff \mu_1(f) \leq \mu_2(f) \text{ (for any } f \text{ which is a continuous convex function on } \mathbb{P}_1)$$

then

$$\varphi = \int_{\mathbb{P}_1} \varphi_x d\mu(x)$$

for any μ such that $\delta_\varphi \prec \mu$. As shown below, we find a mesurable subset of $\mathbb{P}_1$ which is defined by continuous strictly convex functions. If $f \in C(\mathbb{P}_1, \mathbb{R})$ is strictly convex, for any affine(convex and concave) function h which satisfies $f \leq h$,

$$\{x \in \mathbb{P}_1 | f(x) < h(x)\} \subset ex(\mathbb{P}_1)$$

It is rational to obtain the minimam function. So, we define the following upper envelope function $\tilde{f}$.

$$\tilde{f}(x) := \inf\{h(x)|f \leq h, h \in A(\mathbb{P}_1)\} \ (x \in \mathbb{P}_1)$$

Here, $A(\mathbb{P}_1)$ is the set of all continuous affine functions on $\mathbb{P}_1$. Becuase $\tilde{f}(x)$ is upper semicontinuous, $\{x \in \mathbb{P}_1 | f(x) < \tilde{f}(x)\}$ is measurable. Because convex combination of countable dense subset of $\{h \in A(\mathbb{P}_1) | ||h||_\infty = 1\}$ is continuous strictly convex by Hahn-Banach theorem, there is a continuous strictly convex function on $\mathbb{P}_1$. So, we find μ such that $\delta_\varphi \prec \mu$ and $\mu(f) = \mu(\tilde{f})$.

If $h \in C(\mathbb{P}_1, \mathbb{R})$ is convex, then $\tilde{-h} = -h$ by applying Hahn-Banach theorem to a convex set $\{(x,r) \in \mathbb{P} || 0 \leq r \leq h(x)\}$. This can be inferred by drawing a graph of h in the 1-dimensional case. By this fact and Hahn-Banach extension theorem and Riez-Markov-Kakutani theorem, for any μ such that $\delta_\varphi \prec \mu$, there is a regular borel measure L such that $\mu \prec L$ and $L(f) = \mu(\tilde{f})$. So, if we take μ which is a maximal element of $\{\mu | \delta_\varphi \prec \mu\}$ by Zorn Lemma, $\mu(f) = \mu(\tilde{f})$.

We set $X := \{x \in \mathbb{P}_1 | f(x) = \tilde{f}(x)\}$. By Theorem3.14, we can construct $\int_X^{\mathfrak{D}} H(x) d\mu(x)$ which is a direct integral unitary representations from $\{\Phi(x)\}_{x \in ex(X)}$. By the same way as GNS construction, we show $\int_X^{\mathfrak{D}} H(x) d\mu(x)$ is a continuous cyclic unitary with some cyclic vector v and $\varphi = (T.v, v)$. So, $\int_X^{\mathfrak{D}} H(x) d\mu(x)$ and π are isomorphic as continuous unitary repesentations.

4 Irreducible decomposition of unitary representation of compact group

4.1 Some facts admitted without proof

Theorem 4.1 (Stone Wierstrass Theorem). *Let*

- *(S1)* X *be a compact metric space.*
- *(S2)* $A \subset C(G)$.
- *(A1)* A *is a* $\mathbb{C}$*-vector subspace of* $C(G)$.
- *(A2)* $1 \in A$.
- *(A3) If* $f \in A$, *then* $\bar{f} \in A$.
- *(A4) If* $f, g \in A$, *then* $fg \in A$.
- *(A5) If* $x \neq y \in X$, *there is* $f \in C(G)$ *such that* $f(x) \neq f(y)$.

Then A *is dense subset of* $C(G)$ *in uniformly convergence topology.*

4.2 General topics on Bochner Integral

Definition 4.1 (Bochner Integral). *Let*

- *(S1)* $(X, \mathcal{B}, \mu)$ *is a measurable space.*
- *(S2)* Y *is a Banach space.*

Then

- *(i) We say* $F : X \to Y$ *is finite-value if there is* $S \in \mathcal{B}$ *such that* $F(S)$ *is a finite set and* $F(X \setminus S) = \{0\}$ *and* $\mu(S) < \infty$. *We set*
$$\int_X F(x)d\mu(x) = \sum_{\alpha \in F(S)} \alpha\mu(F^{-1}(\alpha))$$
- *(ii) We say* $F : X \to Y$ *is a strong measurable if there are* $\{F_n\}_{n=1}^{\infty}$ *such that for each* $n \in \mathbb{N}$ F_n *is a finite valued and* $\{F_n\}_{n=1}^{\infty}$ *almost everywhere pointwise converges to* F.
- *(iii) We say* $F : X \to Y$ *is Bochner integrable if* F *is strong measurable and there are* $\{F_n\}_{n=1}^{\infty}$ *such that for each* $n \in \mathbb{N}$ F_n *is a finite valued and* $\{F_n\}_{n=1}^{\infty}$ *almost everywhere pointwise converges to* F *and*
$$\int_X F(x)d\mu(x) := \lim_{n\to\infty} \int_X F_n(x)d\mu(x)$$
exists.

Because of the definition of Bochner integral, the following clearly holds.

Proposition 4.1. *Let*

- *(S1)* $(X, \mathcal{B}, \mu)$ *is a measurable space.*
- *(S2)* Y, Z *is a Banach space.*
- *(S3)* $F : X \to Y$ *is Bochner integrable.*
- *(S3)* $T : Y \to Z$ *is bounded linear.*

Then $T \circ F$ *is Bochner integrable and*
$$T \int_X F(x)d\mu(x) = \int_X T \circ F(x)d\mu(x)$$

Proposition 4.2. *Let*

- *(S1)* X *is a compact space.*
- *(S2)* B *is a banach space.*
- *(S3)* $F \in C(X, B)$.
- *(S4)* μ *is a finite borel measure on* X.

Then F is bochner integrable and

$$||\int_X F(x)d\mu(x)|| \leq \int_X ||F(x)||d\mu(x)$$

Proof. By (S1) and (S3), $F(X)$ is compact. So, for each $n \in \mathbb{N}$, there is a finite open covering of $F(X)$ $O(F(x_{n,i})$ $(n = 1, 2, ..\alpha(n))$ such that $O(F(x_{n,i})$ is an open neighborhood of $F(x_{n,i})$ and $O(F(x_{n,i}) \subset B(F(x_{n,i}), \frac{1}{n})$. We can assume that for each $n \in \mathbb{N}$ and each $i \in \{1, ..., \alpha(n+1)\}$ there is $j \in \{1, ..., \alpha(n)\}$ such that $O(F(x_{n+1,i})) \subset O(F(x_{n,j}))$.

$$F_n(x) := \begin{cases} F(x_{n,1}) & x \in F(X) \cap B(F(x_{n,1}), \frac{1}{2^n}) \\ F(x_{n,i+1}) & x \in F(X) \cap (B(F(x_{n,i+1}), \frac{1}{2^n}) \setminus \cup_{j=1}^{i} B(F(x_{n,j}), \frac{1}{2^n})) \end{cases}$$

Clearly, for any $n \in \mathbb{N}$, F_n is finite valued and

$$||F_n(x) - F(x)|| < \frac{1}{2^n}$$

and

$$||\int_X F_n(x)d\mu(x) - \int_X F_{n+1}(x)d\mu(x)|| < \frac{1}{2^n}\mu(X)$$

So,

$$\lim_{n\to\infty} F_n(x) = F(x) \ (\forall x \in X)$$

and by (S2)

$$\lim_{n\to\infty} \int_X F_n(x)d\mu(x)$$

exists. □

4.3 General topics on Compact self-adjoint Operator

Definition 4.2 (Compact operator)**.** *Let*

(S1) W is a normed linear space.

(S2) V is a Banach space.

We say $T : W \to V$ is a compact operator if T is linear and $T(B(0, 1))$ is a relative compact. We denote the set of all compact operator on V by $B_0(W, V)$.

Proposition 4.3. *Let*

(S1) W and V and U are normed linear space.

Then

(i) If V is a Banach space, then $B_0(W, V)$ is a closed subspace of $B(W, V)$.

(ii) If $T \in B_0(W, V)$ and W_0 which is a linear subspace of W , then T_{W_0} is a compact operator.

(iii) If $T \in B(W, V)$ and $dim(ImT) < \infty$, T is a compact operator.

(iv) If $T \in B_0(W, V)$ and $S \in B(V, U)$, then $S \circ T$ is a compact operator.

(v) If $T \in B_(W, V)$ and $S \in B_0(V, U)$, then $S \circ T$ is a compact operator.

Proof of (i). Let us fix any $\{F_n\}_{n=1}^{\infty} \subset B_0(W, V)$ such that $F := \lim_{n\to\infty} F_n$ exists. Let us fix any $\{x_n\}_{n=1}^{\infty} \subset B(0, 1)$. It is enough to show there is a subsequence $\{F(x_{\varphi(n)})\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} F(x_{\varphi(n)}$ exists. Because $\{F_n\}_{n=1}^{\infty} \subset B_0(W, V)$, there are subsequence $\{x_{\varphi_n(k)}\}_{k=1}^{\infty}$ $(n = 1, 2, ...)$ such that fo reach $n \in \mathbb{N}$ $\{x_{\varphi_{n+1}(k)}\}_{k=1}^{\infty}$ is a subsequence of $\{x_{\varphi_{n+1}(k)}\}_{k=1}^{\infty}$ and

$$||F_n(x_{\varphi_n(k)}) - F_n(x_{\varphi_n(l)})|| < \frac{1}{n} \ (\forall k, l \geq n)$$

We set

$$\psi(n) := \varphi_n(n) \ (n \in \mathbb{N})$$

Let us fix any $\epsilon > 0$. There is $n_0 \in \mathbb{N}$ such that

$$||F_k - F|| < \frac{\epsilon}{4} \ (\forall k \geq n_0)$$

and $\frac{1}{n_0} < \frac{\epsilon}{2}$. Let us fix any $k, l \geq n_0$. Then $\psi(k) = \varphi_k(k)$ and $\psi(l) = \varphi_l(l)$ and $k_0 \geq n_0$ and $l_0 \geq n_0$ and $\psi(k) = \varphi_{n_0}(k_0)$ and $\psi(l) = \varphi_{n_0}(l_0)$. So,

$$||F(x_{\psi(k)}) - F(x_{\psi(l)})|| \leq ||F_{n_0}(x_{\psi(k)}) - F_{n_0}(x_{\psi(l)})|| + \frac{\epsilon}{2} = ||F_{n_0}(x_{\varphi_{n_0}(k_0)}) - F_{n_0}(x_{\varphi_{n_0}(l_0)})|| + \frac{\epsilon}{2} \leq \epsilon$$

So, $\{F(x_{\psi(k)})\}_{k=1}^{\infty}$ is a cauchy sequence. Because V is Hilbert space, $\lim_{k\to\infty} F(x_{\psi(k)})$ exists. □

Proposition 4.4. *Let*

(S1) V is an inner product space.

(A1) $T \in B_0(V, V)$.

(A2) There is α which is a nonzero eigenvalue of T.

Any W which is eigenspace of α is finite dimensional.

Proof. Then there is a orthonormality $\{v_i\}_{i=1}^{\infty} \subset W$. Because $\frac{1}{\alpha}T$ is a compact operator, $\frac{1}{\alpha}TW = \{w \in W | ||w|| = 1\}$ is compact. By Proposition1.10, W has finite dimension. □

Lemma 4.1. *Let*

(S1) V is a Hilbert space.

(A1) T is a self adjoint operator from V to V.

(A2) $(Ku, u) = 0$ $(\forall u \in V)$.

Then $K = 0$

Proof. Let us fix any $v \in V$. We set $w := v + Kv$.

$$0 = (Kw, w) = (Kv + K^2 v, v + Kv) = 2||Kv||^2$$

So, $||Kv|| = 0$. This implies $Kv = 0$. □

Lemma 4.2. *Let*

(S1) V is a Hilbert space.

(A1) T is a self adjoint compact operator from V to V.

(A2) $\lambda_+ := \sup_{v \in V, ||v||=1} (Kv, v) > 0$.

Then there is a $u_0 \in V$ such that

$$\lambda_+ = (Ku_0, u_0), Ku_0 = \lambda_+ u_0$$

Proof. Then there is $\{v_i\}_{i=1}^{\infty}\{v \in V| \ ||v|| = 1\}$ such that

$$\lim_{i\to\infty} (Kv_i, v_i) = \lambda_+$$

By Proposition1.19, we can assume there is $v_0, u_0 \in V$ such that

$$w - \lim_{i\to\infty} v_i = v_0$$

and

$$\lim_{i\to\infty} Kv_i = u_0$$

We will show $(Kv_0, v_0) = \lambda_+$.

$$\begin{aligned} & (Kv_0, v_0) = (Kv_i, v_i) + (Kv_i - u_0, v_0 - v_i) + (u_0, v_0 - v_i) + (Kv_0, v_0) - (Kv_i, v_0) \\ = \ & (Kv_0, v_0) = (Kv_i, v_i) + (Kv_i - u_0, v_0 - v_i) + (u_0, v_0 - v_i) + (v_0, Kv_0) - (v_i, Kv_0) \\ \to \ & \lambda_+ \ (i \to \infty) \end{aligned}$$

Let us fix $v \in V$ such that $||v|| = 1$. We set

$$f(t) := (Kv(t), v(t)), \ v(t) := \frac{v_0 + tv}{||v_0 + tv||} \ (|t| \ll 1)$$

then

$$f(t) = \frac{(Kv_0, v_0) + 2tRe(Kv_0, v) + t^2(Kv, v)}{||v_0||^2 + 2tRe(v_0, v) + t^2||v||^2}$$

So,

$$f(t)(||v_0||^2 + 2tRe(v_0, v) + t^2||v||^2) = (Kv_0, v_0) + 2tRe(Kv_0, v) + t^2(Kv, v)$$

Because $f(0) = \lambda_+$ and $f'(0) = 0$,

$$\lambda_+ Re(v_0, v) = Re(Kv_0, v)$$

And

$$\lambda_+ Re(v_0, iv) = Re(Kv_0, iv)$$

These imply

$$\lambda_+(v_0, v) = (Kv_0, v)$$

This means

$$Kv_0 = \lambda_+ v_0$$

□

The following Proposition clealy holds.

Proposition 4.5. *Let*

(S1) T is a self-adjoint continuous linear operator of Hilbert space V.

Then

(i) Any eigenvalue of P is a real number.

(ii) If $\alpha_1, \alpha_2 \in \mathbb{R}$ are different eigenvalues of P, $V_{\alpha_1} \perp V_{\alpha_2}$. Here V_{α_i} is the eigenvalue space of α_i $(i = 1, 2)$.

(iii) If (π, V) is a continuous representation of a topological group G and W is a G-invariant subspace of V, then $W^\perp$ is a G-invariant.

Lemma 4.3. *Let*

(S1) V is a Hilbert space.

(A1) T is a compact self adjoint operator from V to V.

(S2) $\sigma_+(T)$ is the set of all positive eigenvalues of G.

Any assumulation point of $\sigma_+(T)$ is zero.

Proof. If $\#\sigma_+(T) = \infty$, then there is no accumulation points of $\sigma_+(T)$. So, hereafter, we assume $\#\sigma_+(T) = \infty$. By Proposition4.2 and Proposition4.4, there is a sequence of positive eigenvalue $\lambda_1 > \lambda_2 > .... > 0$ and $\{v_i\}_{i=1}^\infty \subset V$ such that v_i is an eigenvector of λ_i $(i = 1, 2, ...)$ and $\lim_{i\to\infty} Kv_i$ exists.

$$\lambda_i^2 \leq \lambda_i^2 + \lambda_{i+1}^2 = ||Kv_i - Kv_{i+1}||^2 \to 0 \ (i \to \infty)$$

□

Lemma 4.4. *Let*

(S1) V is a Hilbert space.

(A1) T is a compact self adjoint operator from V to V.

(S2) V_+ is the minimum closed subspace of V such that V_+ contains all eigenspaces whose eigenvalue is positive. V_- is the minimum closed subspace of V such that V_+ contains all eigenspaces whose eigenvalue is negative.

Then

$$V = V_+ \oplus Ker(T) \oplus V_-$$

Proof. We set $V_* := (V_+ \oplus Ker(T) \oplus V_-)^\perp$. Because $(V_+ \oplus Ker(T) \oplus V_-)$ is T-invariant and T is self-adjoint, V_* is T-invariant. By Proposition4.2, $(Tv, v) = 0$ $(\forall v \in V_*)$. By Proposition4.1, $T|V_* = 0$. So, $V_* = \{0\}$. □

4.4 Matrix coefficient and Character of representation

Definition 4.3 (Character). *Let G be a topological group and (π, V) be a finite dimensional continuous representation of G. Then*

$$\chi_\pi(g) := Trace\pi(g) \ (g \in G)$$

We call χ_π a character of π.

Definition 4.4 (Matrix Coefficient). *Let G be a topological group and (π, V) be a finite dimensional irreducible continuous representation of G and let $v \in V$ and $f \in V^*$.*

$$\Phi_\pi(v, f)(g) := f(\pi(g)^{-1}v)$$

Because π is a continuous representation, $\Phi_\pi(v, f)$ is a continuous function on G.

The following clearly holds.

Proposition 4.6. *We succeed notations in Definition4.4. Then Φ_π is a bilinear form on $\mathbb{C}$.*

Proposition 4.7. *Let*

(S1) G is a topological group.

(S2) (π, V) is a finite dimensional unitary representation of G.

(S3) $\{v_1, v_2, ..., v_m\}$ is an orthonomal basis of V.

(S4) $\pi_{i,j}(g) := (\pi(g)v_j, v_i)$ $(g \in G, i, j \in \{1, 2, ..., m\})$

then

(i) $\chi_\pi = \sum_{i=1}^m \pi_{i,i}$.

(ii) $\pi_{i,j}(gh) = \sum_{k=1}^m \pi_{i,k}(g)\pi_{k,j}(h)$ $(\forall g, h \in G, \forall i, j \in \{1, 2, ..., m\})$.

(iii) $\pi_{i,j}(g^{-1}) = \overline{\pi_{j,i}(g)}$ $(\forall g \in G, \forall i, j \in \{1, 2, ..., m\})$.

Proof of (i). It is clear. □

Proof of (ii).

$$\pi_{i,j}(gh) = (\pi(gh)v_j, v_i) = (\pi(g)\pi(h)v_j, v_i) = (\pi(g)(\sum_{k=1}^m(\pi(h)v_j, v_k)v_k), v_i) = \sum_{k=1}^m(\pi(g)v_k, v_i)(\pi(h)v_j, v_k)$$
$$= \sum_{k=1}^m \pi_{i,k}(g)\pi_{k,j}(h)$$

□

Proof of (iii).

$$\pi_{i,j}(g^{-1}) = (\pi(g^{-1})v_j, v_i) = (v_j, \pi(g)v_i) = \overline{(\pi(g)v_i, v_j)} = \overline{\pi_{j,i}(g)}$$

□

4.5 Schur orthogonality relations

Proposition 4.8. *Let*

(S1) G is a compact Lie group.

(S2) (π_i, V_i) is a continous unitary representation of G on $\mathbb{C}$ $(i = 1, 2)$.

(S3) $f \in Hom_{\mathbb{C}}(V_1, V_2)$.

(S4) We set $\tilde{f}$ by

$$\tilde{f}(v) := \int_G \pi_2(g) \circ f \circ \pi_1(g)^{-1}(v)dg \ (v \in V_1)$$

Then $\tilde{f} \in Hom_G(V_1, V_2)$.

Proof. By Proposition4.2, $\tilde{f}(v)$ exists and

$$||\tilde{f}(v)|| \leq \int_G ||\pi_2(g) f \pi_1(g^{-1}) v|| dg$$

Because π_1 and π_2 are unitary representation,

$$\int_G ||\pi_2(g) f \pi_1(g^{-1}) v|| dg \leq \int_G ||\pi_2(g) f \pi_1(g^{-1})|| ||v|| dg \leq \int_G ||f|| ||v|| dg \leq ||f|| ||v||$$

So $\tilde{f}$ is continuous linear map. Becuase dg is a Haar measure on G, clearly, $\tilde{f}$ is G-invariant. □

Proposition 4.9 (Shur orthogonality relations)**.** *Let*

(S1) G is a compact Lie group.

(S2) (π_i, V_i) is a continous irreducible representation of G on $\mathbb{C}$ $(i = 1, 2)$.

(A1) Either V_1 or V_2 is finite dimensional.

(S3) $(u_i, v_i) \in V_i$ $(i = 1, 2)$.

Then

$$(\Phi(u_1, v_1), \Phi(u_2, v_2))_{L^2(G)} = \begin{cases} 0 & (\pi_1 \not\simeq \pi_2) \\ dim V (Tu_1, u_2)\overline{(Tv_1, v_2)} & (\pi_1 \simeq \pi_2) \end{cases}$$

Here T is a unitary G-isomorphism from V_1 to V_2.

STEP1. Case when $\pi_1 \not\simeq \pi_2$. We set $f \in Hom_{\mathbb{C}}(V_1, V_2)$ by

$$f(v) := (v, v_1)v_2 \ (v \in V_1)$$

Proposition4.8, $\tilde{f} \in Hom_G(V_1, V_2)$ exists. In this case, by Shur Lemma, $\tilde{f} = 0$.

$$0 = (\tilde{f}(u_1), u_2) = \int_G (\pi_2(g) f \pi_1(g)^{-1} u_1, u_2) dg = \int_G (f \pi_1(g)^{-1} u_1, \pi_2(g)^{-1} u_2) dg = \int_G (v_2, \pi_2(g)^{-1} u_2) dg$$

$$= \int_G (\pi_1(g)^{-1} u_1, v_1)(v_2, \pi_2(g)^{-1} u_2) dg = \int_G (\pi_1(g)^{-1} u_1, v_1)\overline{(\pi_2(g)^{-1} u_2, v_2)} dg$$

□

STEP2. Case when $\pi_1 \simeq \pi_2$. In this case, by Shur Lemma, there is $\lambda \in \mathbb{C}$ such that $T^{-1} \circ \tilde{f} = \lambda id_{V_1}$. By the argument in STEP1,

$$(\Phi(u_1, v_1), \Phi(u_2, v_2))_{L^2(G)} = \lambda(Tu_1, u_2)$$

And

$$Trace(T^{-1} \circ \tilde{f}) = \lambda dim V_1$$

By Proposition4.1 and T^{-1} is G-invariant,

$$T^{-1} \circ \tilde{f} = \widetilde{T^{-1} f}$$

So,

$$Trace(T^{-1} \circ \tilde{f}) = Trace(T^{-1} f) = T^{-1} f(\frac{v_1}{||v_1||}) = (T^{-1}(\frac{v_1}{||v_1||}, v_1)v_2, \frac{v_1}{||v_1||}) = ||v_1||(T^{-1} v_2, \frac{v_1}{||v_1||})$$

$$= (T^{-1} v_2, v_1) = (v_2, Tv_1) = \overline{(Tv_1, v_2)}$$

So,

$$(\Phi(u_1, v_1), \Phi(u_2, v_2))_{L^2(G)} = (Tu_1, u_2)\overline{(Tv_1, v_2)}$$

□

By Shur orthogonality Relations, the following three holds.

Proposition 4.10. *Let*

(S1) G is a compact Lie group.

(S2) $R(G) := \left\langle \{\Phi_\pi(u,v) |\ (\pi, V) \in \hat{G}_f, u, v \in V\} \right\rangle$. *Here,* $\hat{G}_f$ *is the set of all finite dimensional irreducible continous unitary representations of* G.

Then

(i) *Let* $\{u_i\}_{i=1}^{dimV}$ *is a orthonormality base of* V. *For any* $(\pi, V) \in \hat{G}_f$,

$$\{\frac{1}{\sqrt{dimV}}\Phi_\pi(u_i, u_j) |\ i, j = 1, 2, ..., dimV\}$$

is a basis of $\Phi(V, V^*)$.

(ii) *The following is well-defined.*

$$\Phi_\pi(u \otimes v) := \Phi_\pi(u, v)$$

(iiii) *The following holds.*

$$R(G) = \oplus_{(\pi,V)\in\hat{G}_f}\Phi_\pi(V \otimes V^*)$$

Proposition 4.11. *Let*

(S1) G *is a compact Lie group.*

(S2) (π, V) *is a finite irreducible continuos representation of* G *and* χ_π *is the character of* π.

Then

$$(\chi_\pi, \chi_\pi) = 1$$

Proposition 4.12. *Let*

(S1) G *is a compact Lie group.*

(S2) (π_i, V_i) *are two finite irreducible continuos representation of* G *and* χ_{π_i} *is the character of* π_i $(i = 1, 2)$.

(A1) $\chi_{\pi_1} = \chi_{\pi_2}$.

Then

$$\pi_1 \simeq \pi_2$$

4.6 Orthogonal projection by character

Proposition 4.13. *We succeed notations in Definition4.3.*

(i) χ_π *is continuous.*

(ii) *If* $\pi_1 \simeq \pi_2$ *then* $\chi_{\pi_1} = \chi_{\pi_2}$.

(iii) $\chi_\pi(gxg^{-1}) = \chi_\pi(x)$ $(\forall g, x \in G)$.

(iv) $\chi_\pi(g^{-1}) = \chi_{\pi^*}(g)$ $(\forall g \in G)$.

Proof of (i). (i) is from Proposition4.13. □

Proof of (ii). Let us take $T : (\pi_1, V_1) \to (\pi_2, V_2)$ be a G-isomorphism. Then $T \circ \pi_1 = \pi_2 \circ T$. This means $T \circ \pi_1 \circ T^{-1} = \pi_2$. So, $\chi_{\pi_1} = \chi_{\pi_2}$. □

Proof of (iii). For any $g, x \in G$,

$$\chi_\pi(gxg^{-1}) = Trace(\pi(gxg^{-1}) = Trace(\pi(g)\pi(x)\pi(g)^{-1}) = Trace(\pi(x)) = \chi_\pi(x)$$

□

Proof of (iv). For any $g \in G$,

$$\chi_\pi(g^{-1}) = Trace(\pi(g^{-1})) = Trace({}^t\pi(g^{-1})) = \chi_{\pi^*}(g)$$

□

Definition 4.5 (τ-component)**.** *Let*

(S1) G *is a topological group.*

(S2) (π, V) *is a continous representation of* G.

(S3) (τ, W) is a continous irreducible representation of G.

We set

$$V_\tau := \sum_{A \in Hom_G(W,V)} ImA$$

We call this τ-component of V.

Proposition 4.14. *We succeed settings in Definition4.5. And if $dimW < \infty$, for any $A \in Hom_G(W,V)$, $ImA = \{0\}$ or $A : (\tau, W) \to (\pi|ImA, ImA)$ is G-isomorphism.*

Proof. Let us assume $ImA \neq \{0\}$. Because W is irreducible, $Ker(A) = \{0\}$. And, because A is G-linear, $Im(A)$ is G-invariant. So, A is bijective and A is G-linear and A^{-1} is G-linear. Because $Im(A)$ is finite dimensional, A^{-1} is continuous. So, $A : (\tau, W) \to (\pi|ImA, ImA)$ is G-isomorphism. □

Definition 4.6 (Projection by character)**.** *Let*

(S1) G is a compact Lie group.

(S2) (τ, V) is a continuous finite dimensional unitary representation of G.

We set

$$P_{\pi,\tau}(v) := P_\tau(v) := dim\tau \int_G \overline{\chi_\tau(g)}\tau(g)vdg$$

We call P_τ the projection by τ.

Lemma 4.5. *Let*

(S1) G is a compact Lie group.

(S2) (τ, W) is a continuous finite dimensional irreducible unitary representation of G.

(S2) (π, V) is a continuous finite dimensional unitary representation of G.

then $ImP_\tau \subset V_\tau$.

Proof. By Proposition1.28, there is $\pi_1, ..., \pi_n \in \hat{G}_f$ such that

$$\pi = \oplus_{i=1}^n \pi_i$$

This implies that

$$P_{\pi,\tau} = \sum_i P_{\pi_i,\tau}$$

Let us fix any $i \in \{1, 2, ..., n\}$. By Shur orthogonality relation, if $\tau \not\cong \pi_i$, $P_{\pi_i,\tau} = 0$. If there is $T : (\tau, W) \to (\pi_i, V_i)$ which is an unitary map and G-isomorphism. Let us take $w_1, ..., w_m$ which is a orthonomality basis of W. By Shur orthogonality relation, for any j,

$$P_{\pi_i,\tau}(Tw_j) = \dim\tau \int_G \overline{\chi_\tau(g)}\pi_i(g)Tw_jdg = \dim\tau \sum_{k,l} \int_G \overline{(\tau(g)w_k, w_k)}(\pi_i(g)Tw_j, Tw_l)Tw_ldg$$

$$= \quad \dim\tau \sum_{k,l} \int_G \overline{(\pi_i(g)Tw_k, Tw_k)}(\pi_i(g)Tw_j, Tw_l)Tw_ldg = Tw_j$$

So, $P_{\pi_i,\tau} = id_{V_i}$. By this, $P_{\pi_i,\tau}(V_i) = ImT \subset V_\tau$. □

Lemma 4.6. *Let*

(S1) G is a compact Lie group.

(S2) (τ, W) is a continuous finite dimensional irreducible unitary representation of G.

(S3) (τ', W) is a continuous finite dimensional irreducible unitary representation of G.

(S4) (π, V) is a continuous finite dimensional unitary representation of G.

(A1) $(\tau, W) \not\cong (\tau', W)$.

then $P_\tau|V'_\tau = 0$.

4.7 Peter-Weyl theorem

4.7.1 Irreducible decomposition

Theorem 4.2. *Let*

(S1) G is a compact Lie group.

(S2) (π, V) is a continous finite dimensional representation of G.

(S3) $(\cdot,\cdot)$ is an inner product of V.

Then

(i) (π, V) is a unitary representation with respect to the following inner product

$$(u, v)_\pi := \int_G (\pi(g)u, \pi(g)v)dg$$

Here, dg is a Haar measure on G. By Proposition2.53, this Haar measure on G.

(ii) (π, V) is irreducible $\iff$ $(\pi, V, (\cdot,\cdot)_\pi)$.

(ii) If π' is a continuous representation of G such that π and π' are equivalent as continuous representations, $(\pi, V, (\cdot,\cdot)_\pi)$ and $(\pi', V', (\cdot,\cdot)_{\pi'})$ are equivalent as unitary representations.

Proof. Because G is unimodular and $C(G) \subset L^\infty(G)$, (i) holds. Because $(\cdot,\cdot)$ and $(\cdot,\cdot)_\pi)$ are equivalent, (ii) holds. □

The following Proposition clealy holds.

Proposition 4.15. *Let*

(S1) G is a topological group.

(S2) (π, V) is a continous finite dimensional representation of G.

(S3) $P \in Hom_G(V, V)$.

Then

(i) Any eigenvalue space of P is G-invariant.

(ii) ImP is G-invariant.

Proposition 4.16. *Let*

(S1) G is a compact Lie group.

(S2) $(\pi, V, (\cdot,\cdot))$ is a unitary representation of G.

(S3) $v_0 \in V$ and $||v_0|| = 1$

(S4) $P : V \ni v \to (v, v_0)v_0 \in V$

(S5) $\Phi : G \ni g \to \pi(g)P\pi(g)^* \in B_0(V)$.

Then

(i) Φ is a continuous. And for any $g \in G$, $\Phi(g)$ is self adjoint.

(ii) Φ is Bochner integrable with respact to a Haar measre on G.

(iii) $\tilde{P} := \int_G \Phi(g)dg$ is G-invariant.

(iv) $\tilde{P}$ is a self-adjoint compact operator.

(v) $\tilde{P}$ is a nonzero map.

(vi) There is $\lambda \neq 0$ such that eigenspace of $\tilde{P}$ with respect to λ is not zero.

Proof of (i). For any $v \in V$ and $g, h \in G$

$$\begin{aligned}
&||\pi(g)P\pi(g)^*v - \pi(h)P\pi(h)^*v|| = ||\pi(g)(\pi(g)^*v, v_0)v_0 - \pi(h)(\pi(h)^*v, v_0)v_0|| \\
= \; & ||(v, \pi(g^{-1})v_0)\pi(g)v_0 - (v, \pi(h^{-1})v_0)\pi(h)v_0|| \\
= \; & ||(v, \pi(g^{-1})v_0)\pi(g)v_0 - (v, \pi(h^{-1})v_0)\pi(g)v_0|| + ||(v, \pi(h^{-1})v_0)\pi(g)v_0 - (v, \pi(h^{-1})v_0)\pi(h)v_0|| \\
\leq \; & ||v||||\pi(g^{-1})v_0 - \pi(h^{-1})v_0||||\pi(g)v_0|| + ||v||||\pi(h^{-1})v_0||||\pi(g)v_0 - \pi(h)v_0|| \\
= \; & ||v||(||\pi(g^{-1})v_0 - \pi(h^{-1})v_0|| + ||\pi(g)v_0 - \pi(h)v_0||)
\end{aligned}$$

So Φ is continuous. By Proposition, for any $g \in G$, $\Phi(g)$ is compact. Because P is self-adjoint and $\pi(g)$ is unitary operator, $\Phi(g)$ is self adjoint. □

Proof of (ii). This is from Proposition4.2 and (i). ☐

Proof of (iii). Let us fix any $h \in G$ and $v, u \in V$. By Proposition4.1,

$$(\pi(h)\int_G \pi(g)P\pi(g)^* dgv, u) = \int_G (\pi(h)\pi(g)P\pi(g)^* v, u)dg = \int_G (\pi(hg)P\pi(hg)^{-1}\pi(h)v, u)dg$$
$$= \int_G (\pi(g)P\pi(g)^{-1}\pi(h)v, u)dg = (\int_G \pi(g)P\pi(g)^{-1}dg\pi(h)v, u)$$

So, $\pi(h)\tilde{P} = \tilde{P}\pi(h)$ ☐

Proof of (iv). By the simila argument to the proof of (iii), $\tilde{P}$ is self-adjoint. By the argument of proof of Proposition4.2, $\tilde{P} \in \overline{B_0(V)}$. By Proposition4.7.1, $\tilde{P} \in B_0(V)$. ☐

Proof of (v).

$$(\int_G \pi(g)P\pi(g)^* dgv_0, v_0) = \int_G (\pi(g)P\pi(g)^* v_0, v_0)dg = \int_G (P\pi(g)^* v_0, \pi(g)^* v_0)dg$$
$$= \int_G (P^*P\pi(g)^* v_0, \pi(g)^* v_0)dg = \int_G (P^*P\pi(g)^* v_0, \pi(g)^* v_0)dg = \int_G (P\pi(g)^* v_0, P\pi(g)^* v_0)dg = \int_G ||P\pi(g)^* v_0||^2 dg$$

Because $||P\pi(e)^* v_0||^2 = 1$, $\int_G ||P\pi(g)^* v_0||^2 dg > 0$. ☐

Proof of (vi). By (v) and Lemma4.4, (vi) holds. ☐

In the following proposition, we give a proof for the general case as well as for the finite group case. The proof of the finite group case shown here follows the same policy as the proof of the general case, but uses only knowledge of linear algebra. Therefore, this proof has the advantage that the essence of the proof of the general case can be easily understood. Note that the finite group case can be easily shown from the fact that $< \pi(G)v >$ is finite dimensional G-invariant subspace for any vector v, apart from the proof given below.

Proposition 4.17. *Let*

(S1) G is a compact Lie group.

(S2) $(\pi, V, (\cdot, \cdot))$ is a unitary representation of G.

Then there is a finie irreducible G-subspace of V.

Proof in general case. By (v) of Proposition4.16, this Proposition holds. ☐

We will show a proof that does not knowledge of bochner integrals and self-adjoint compact operators in the case when G is a finite group.

Proof in the case when G is a finite group. We will succeed notations the proof of Proposition4.16 . Then

$$\tilde{P} = \sum_{g \in G} \pi(g^{-1}) \circ P \circ \pi(g)$$

For any $h \in G$,

$$\tilde{P} \circ \pi(h) = \sum_{g \in G} \pi(g^{-1}) \circ P \circ \pi(gh) = \sum_{g \in G} \pi(h) \circ \pi(gh^{-1}) \circ P \circ \pi(gh) = \pi(h) \circ \tilde{P}$$

So, $\tilde{P}$ is G linear.

For each $g \in G$, $\pi(g^{-1}) \circ P \circ \pi(g)$ is finite rank operator. So, $\tilde{P}$ is G finite rank operator. Then $\{v_1, .., v_m\}$ such that $\sum_{i=1}^m \mathbb{C}\tilde{P}(v_i) = Im(\tilde{P})$. Let us fix $\{w_1, ..., w_n\}$ which is an orthonormal basis of $Im(\tilde{P}) + \sum_{i=1}^n w_i$. Because $\tilde{P}| \sum_{i=1}^m w_i$ is not zero, $\tilde{P}| \sum_{i=1}^n w_i$ has nonzero eiggenvalue $\lambda \neq 0$.

For any $u \in Ker(\tilde{P} - \lambda I)$,

$$u = \frac{1}{\lambda}(\tilde{P}u) = \frac{1}{\lambda}\sum_{i=1}^m (\tilde{P}u, u_i)u_i$$

So,

$$Ker(\tilde{P} - \lambda I) \subset \sum_{i=1}^n \mathbb{C}u_i$$

These imply that $Ker(\tilde{P} - \lambda I)$ is finite dimensional G-invariant subspace. ☐

By Proposition4.17 and the same argument as the proof of Proposition3.7, the following holds.

Theorem 4.3 (Peter-weyl theorem I)**.** *Let (π, V) be a continuous unitary representation of a compact Lie group G. Then there is $\mathcal{W}$ which is a subset of G-invariant finite dimensional irreducible subspaces such that*

$$V = \overline{\bigoplus_{W \in \mathcal{W}} W}$$

In specail, if π is irreducible, $dim(\pi) < \infty$.

4.7.2 Orthonormal basis of $L^2(G)$

Proposition 4.18. *Let*

- *(S1) G is a compact Lie group.*
- *(S2) $(\pi, V, (\cdot, \cdot))$ is a finite dimensional unitary representation of G.*

Then

$$\{\Phi_\pi(u, v) | u, v \in V\}$$

is $G \times G$-invariant subspace of $L^2(G)$.

Proof. For any $x, y, g \in G$,

$$L_x \times R_y \Phi_\pi(u, v)(g) = (\pi(xgy^{-1})^{-1}u, v) = (\pi(g)^{-1}\pi(x)^{-1}u, \pi(y)^{-1}v) = \Phi_\pi(\pi(x)^{-1}u, \pi(y)^{-1}v)(g)$$

So,

$$\{\Phi_\pi(u, v) | u, v \in V\}$$

is $G \times G$-invariant subspace of $L^2(G)$. □

By Proposition4.10, the following two holds.

Proposition 4.19. *Let*

- *(S1) G is a compact Lie group.*
- *(S2) (π, V) is a finite dimensional G-invariant space of $L^2(G)$.*

Then $V \subset \Phi_\pi(V \otimes V^*)$.

Proof. Let us fix $\{f_1, ..., f_m\}$ which is an orthonormal basis of V. Let us fix any i. Then for any $g \in G$

$$L(g^{-1})f_i = \sum_{j=1}^{m} (L(g^{-1})f_i, f_j)f_j$$

So,

$$f_i(g) = L(g^{-1})f_i(e) = \sum_{j=1}^{m} \Phi(f_i, f_j)(g)f_j(e)$$

This means

$$f_i = \sum_{j=1}^{m} f_j(e)\Phi(f_i, f_j)$$

So, $V \subset \Phi_\pi(V \otimes V^*)$. □

Proposition 4.20. *Let*

- *(S1) G is a compact Lie group.*
- *(S2) $R(G) := \oplus_{(\pi,V)\in\hat{G}} \Phi_\pi(V \otimes V^*)$. Here $\hat{G}$ is the set of all equivalent classes of irreducible representation of G.*

Then $R(G)$ is dense in $L^2(G)$.

Proof. Be Proposition4.18, $R(G)^{\perp}$ is G-invariant. Let us assume $R(G)^{\perp} \neq \{0\}$. By Proposition4.17 and Proposition4.19, there are $\{f_1, ..., f_m\} \subset L^2(G)$ such that $\{f_1, ..., f_m\}$ is an orthonormality and $\langle f_1, ..., f_m \rangle$ is a irreducible G-invariant subspace and $\langle f_1, ..., f_m \rangle \subset R(G)$. So,

$$1 = (f_i, f_i) = 0$$

This is contradiction. □

Theorem 4.4 (Peter-Weyl Theorem II)**.** *Let*

(S1) G is a compact Lie group.

Then

$$\Phi : (L, \oplus_{\tau \in \hat{G}} V \otimes V^*) \to (L, L^2(G))$$

is an isomorphism as continuous unitary representaions. And $(L, V \otimes V^*)$ is isomorphic to a direct sum of $dim\tau$ of V.

Proof. The first part is directly followed from Proposition4.20. Let us take an orthonormal basis $\{v_1, ..., v_m\}$ of V. Then $V \otimes V^* = \oplus_{i=1}^{m} V \otimes (v_i)^*$ since $V \otimes (v_i)^* \perp V \otimes (v_j)^*$ for any $i \neq j$. Clearly $V \otimes (v_i)^*$ is isomorphic to V as continuous unitarly representations for any i. The latter half part holds. □

Notation 4.1. *Let*

(S1) G is a compact Lie group.

(S2) (τ, W) is an irreducible unitary representation of G.

then we define Φ_τ, Φ'_τ, $\tilde{\Phi}_\tau$

(i) $\Phi_\tau : W \otimes W^* \ni v \otimes w \mapsto (G \ni g \mapsto (\tau(g)v, w) \in \mathbb{C}) \in C(G)$.

(ii) $\Phi'_\tau := dimW\Phi_\tau$.

(iii) $\tilde{\Phi}_\tau := \sqrt{dimW}\Phi_\tau$.

Proposition 4.21. *Let*

(S1) G is a compact Lie group.

(S2) $(\tau, W) \in \hat{G}_f$.

Then

$$(\tau_{i,j}, \tau_{k,l}) = \frac{1}{dim\tau}\delta_{i,j}\delta_{k,l}$$

Proof. Because for any $i, j \in \{1, ..., dim\tau\}$ and $g \in G$

$$\tau_{i,j}(g) = \Phi_\tau(v_i, v_j)(g^{-1})$$

by Proposition2.54 and Shur orthognality relation,

$$(\tau_{i,j}, \tau_{k,l}) = (\Phi_\tau(v_i, v_j), \Phi_\tau(v_k, v_l)) = \frac{1}{dim\tau}\delta_{i,j}\delta_{k,l}$$

□

By Proposition4.20 and Shur orthogonality relations and Proposition4.21, the following holds.

Theorem 4.5 (Peter Weyl Theorem II, matrix coefficient version)**.** *Let*

(S1) G is a compact Lie group.

(S2) $(\tau, W) \in \hat{G}_f$.

Then

(i) The following is a completely orthonomal system of $L^2(G)$.

$$\{\sqrt{\dim\tau}\tau_{i,j} | i, j = 1, 2, .., \dim\tau, (\tau, W) \in \hat{G}_f\}$$

(ii) $\hat{G}$ is at most countable.

(iii) For any $f \in L^2(G)$,

$$f = dim\tau \sum_{\tau \in \hat{G}_f, i,j=1,...,dim\tau} (f, \tau_{i,j})\tau_{i,j} \ (L^2\text{-convergence})$$

Proof of (i). This is followed by Proposition4.20 and Shur orthogonality relations and Proposition4.21. □

Proof of (ii). Because $L^2(G)$ is separable, $L^2(G)$ has a countable complete orhonormal basis. So, this is followed by (i) and Peter-Weyl I and Proposition1.12(iii). □

Proof of (iii). This is followed by (i) and (ii) and Proposition1.12(ii). □

4.7.3 Uniform approximate of continuous function

Theorem 4.6 (Peter-Weyl Theorem III)**.** *Let G be a compact Lie group. Then the $\mathbb{C}$-vector space generated by the following set is dense subset of $C(G)$ in uniformly convergence topology.*

$$\{(\tau(\cdot)v, v)|(\tau, V) \text{ is a continuous finite dimensional irreducible unitary representation of } G,\ v \in V \text{ such that } ||v|| = 1\}$$

Proof. By Peter-Weyl I and Proposition3.23,

$$ex(\mathbb{P}_1) = \{(\tau(\cdot)v, v)|(\tau, V) \text{ is a continuous finite dimensional irreducible unitary representation of } G,\ v \in V \text{ such that } ||v|| = 1\}$$

Because the trivial representation of G is finite dimensional irreducible, $ex(\mathbb{P}_1)$ contains 1 which is (A2) in Theorem4.1. Because $\varphi \in ex(\mathbb{P}_1) \implies \bar{\varphi} \in ex(\mathbb{P}_1)$, $ex(\mathbb{P}_1)$ satisfies (A3) in Theorem4.1. By Proposition3.12, $ex(\mathbb{P}_1)$ satisfies (A4) in Theorem4.1. By Gelfand-Raikov Theorem, $ex(\mathbb{P}_1)$ satisfies (A5) in Theorem4.1. So, by Theorem4.1, the $\mathbb{C}$-vector space generated by $ex(\mathbb{P}_1)$ is dense subset of $C(G)$ in uniformly convergence topology. □

Definition 4.7 (Class function)**.** *Let G be a group and f be a function on G. We say f is a class function if*

$$f(x^{-1}gx) = f(g)\ (\forall x, g \in G)$$

We denote the set of all squared integrable class functions by $L^2(G)^{Ad}$. We denote the set of all continuous class functions by $C(G)^{Ad}$.

Clearly the following holds.

Proposition 4.22. *Any character of compact Lie group is a class function.*

Proposition 4.23. *Let G be a compact group. Then $L^2(G)^{Ad}$ is closed subset of $L^2(G)$ and $C(G)^{Ad}$ is closed subset of $C(G)$.*

Proof. Because $f(x^{-1}gx) = L_x \circ R_x f\ (\forall x, g \in G, \forall f \in C(G))$ and $L_x \circ R_x$ is continuous operator of $L^2(G)$ and $C(G)$. So, this Proposition holds. □

Proposition 4.24. *Let G be a compact Lie group. We set*

$$P(f)(g) := \int_G f(x^{-1}gx)dg(x)\ (g \in G)$$

then

(i) P is the orthogonal projection of $L^2(G)^{Ad}$.

(ii) $P(C(G)) = C(G)^{Ad}$.

(iii) $P : C(G) \to C(G)^{Ad}$ is surjective continuous in uniform convergence topology.

Proof of (i). Clearly $P(L^2(G)) \subset L^2(G)^{Ad}$, and $P \circ P = P$ and P is linear. For any $g, f \in L^2(G)$,

$$\begin{aligned}|(g, P(f))| &= |\int_G g(x)\int_G \overline{f(y^{-1}xy)}dg(y)dg(x)| = |\int_G\int_G g(x)\overline{f(y^{-1}xy)}dg(x)dg(y)| \\ &\leq \int_G ||g||_{L^2}||L_y \circ R_y f||_{L^2}dg(y) = \int_G ||g||_{L^2}||f||_{L^2}dg(y) = ||g||_{L^2}||f||_{L^2}\end{aligned}$$

and

$$\begin{aligned}&(g, P(f)) = \int_G g(x)\int_G \overline{f(y^{-1}xy)}dg(y)dg(x) = \int_G\int_G g(x)\overline{f(y^{-1}xy)}dg(y)dg(x) \\ &= \int_G\int_G g(yxy^{-1})\overline{f(x)}dg(x)dg(y) = \int_G\int_G g(yxy^{-1})dg(y)\overline{f(x)}dg(x) = \int_G\int_G g(y^{-1}xy)dg(y)\overline{f(x)}dg(x) \\ &= (P(g), f)\end{aligned}$$

So, P is continuous and self adjoint. Because of these result, (i) holds. □

Proof of (ii). Clearly $P(C(G)) \subset C(G)^{Ad}$ and and $P|C(G)^{Ad} = id|C(G)^{Ad}$. □

Proof of (iii). For any $f \in C(G)$, f is uniformly continuous. So, $P|C(G)$ is continuous in uniformy convergence topology. By (ii), $P|C(G)$ is surjective. So (iii) holds. □

Proposition 4.25. *We will succeed notations in Proposition4.24. And let* $(\tau, V) \in \hat{G}_f$*. then for any* $i, j \in \{1, 2, ..., dim\tau\}$

$$P(\tau_{i,j}) = \frac{\delta_{i,j}}{dim\tau}\chi_\tau$$

Proof. For any $g \in G$,

$$\begin{aligned}
&P(\tau_{i,j})(g) = \int_G \tau_{i,j}(x^{-1}gx)dg(x) \\
&\text{by Proposition4.7} \\
&= \sum_{a,b} \int_G \tau_{i,a}(x^{-1})\tau_{a,b}(g)\tau_{b,j}(x)dg(x) \\
&\text{by Proposition4.7} \\
&= \sum_{a,b} \int_G \overline{\tau_{a,i}(x)}\tau_{a,b}(g)\tau_{b,j}(x)dg(x) = \sum_{a,b} \tau_{a,b}(g) \int_G \overline{\tau_{a,i}(x)}\tau_{b,j}(x)dg(x) \\
&\text{by Shur orthogonality relations} \\
&= \delta_{i,j}\frac{1}{dim\tau}\sum_{i=1}^{dim\tau}\tau_{i,i}(g) = \delta_{i,j}\frac{1}{dim\tau}\chi_\tau
\end{aligned}$$

□

Theorem 4.7. *Let*

- *(S1)* G *is a compact Lie group.*
- *(S2)* $(\tau, W) \in \hat{G}_f$*.*

Then

- *(i)* $\sum_{\tau \in \hat{G}_f} \mathbb{C}\chi_\tau$ *is dense in* $C(G)^{Ad}$*.*
- *(ii)* $\{\chi_\tau | \tau \in \hat{G}_f\}$ *is an orthonomal basis of* $L^2(G)^{Ad}$*.*

Proof of (i). Let us fix any $f \in C(G)^{Ad}$, $\epsilon > 0$. Because P is continuous, there is $\delta > 0$ such that

$$g \in C(G) \text{ and } ||g - f||_\infty < \delta \implies ||P(g) - P(f)||_\infty < \epsilon.$$

Because $f \in C(G)^{Ad}$, $P(f) = f$. By Theorem4.6, there is $g \in \sum_{\tau \in \hat{G}_f} \sum_{i,j \in \{1,2,...,dim\tau\}} \mathbb{C}\tau_{i,j}$ such that $||g - f||_\infty < \delta$. By Proposition4.25, $P(g) \in \sum_{\tau \in \hat{G}_f} \mathbb{C}\chi_\tau$. □

Proof of (ii). Let us fix any $f \in L^2(G)^{Ad} \setminus \{0\}$. By Theorem4.5, there is $\tau \in \hat{G}_f$ and $i, j \in \{1, 2, ..., dim\tau\}$ such that $(f, \tau_{i,j}) \neq 0$. Because P is the orthogonal projection of $L^2(G)^{Ad}$, there is $g \in (L^2(G)^{Ad})^\perp$ such that $\tau_{i,j} = P(\tau_{i,j}) + g$. So,

$$0 \neq (f, \tau_{i,j}) = (f, P(\tau_{i,j})) = \frac{\delta_{i,j}}{dim\tau}(f, \chi_\tau)$$

This implies $(f, \chi_\tau) \neq 0$. □

4.7.4 Component of irreducible decomposition

Proposition 4.26. *Let*

- *(S1)* G *is a compact Lie group.*
- *(S2)* (π, V) *is an continuous unitary representation of* G*.*
- *(S3)* $(\tau, W) \in \hat{G}_f$*.*
- *(S4)* $A \in Hom_G(W, V)$*.*

Then

$$P_\tau | ImA = id | ImA$$

Proof. By Proposition4.21

$$P_\tau(Aw_i) = dim\tau \int_G \overline{\chi_\tau(g)}\pi(g)Aw_i dg = dim\tau \int_G \overline{\chi_\tau(g)}A\tau(g)w_i dg = dim\tau \sum_{j=1}^{m} \int_G \overline{\chi_\tau(g)}A(\tau(g)w_i, w_j)w_j dg$$

$$= \quad dim\tau \sum_{j=1}^{m} \int_G \overline{\chi_\tau(g)}\tau_{i,j}(g)dgAw_j = dim\tau \sum_{k=1}^{m}\sum_{j=1}^{m} \int_G \overline{\tau_{k,k}(g)}\tau_{i,j}(g)dgAw_j = Aw_i$$

□

Proposition 4.27. *Let*

(S1) G is a compact Lie group.

(S2) $(\tau, W_1), (\pi, W_2) \in \hat{G}_f$.

then

$$\chi_\tau * \chi_\pi = \begin{cases} \frac{1}{dim\tau}\chi_\tau & (\tau \simeq \pi) \\ 0 & (\tau \not\simeq \pi) \end{cases}$$

Proof. For any $h \in G$,

$$\int_G \chi_\tau(g)\chi_\pi(g^{-1}h)dg = \sum_{i,j} \int_G \tau_{i,i}(g)\pi_{j,j}(g^{-1}h)dg$$

For any j,

$$\pi_{j,j}(g^{-1}h) = (\pi(g^{-1}h)v_j, v_j) = (\pi(h)v_j, \tau(g)v_j) = \sum_k \pi_{j,k}(h)(v_k, \pi(g)v_j) = \sum_k \pi_{j,k}(h)\overline{\pi_{j,k}(g)}$$

So, by Shur orthogonality relations,

$$\sum_{i,j} \int_G \tau_{i,i}(g)\pi_{j,j}(g^{-1}h)dg = \sum_{i,j,k} \tau_{j,k}(h) \int_G \tau_{i,i}(g)\overline{\pi_{j,k}(g)}dg = \delta\tau, \pi\frac{1}{dim\tau}\sum_{i=1}^{dim\tau} \tau_{i,i}(h) = \delta\tau, \pi\frac{1}{dim\tau}\chi_\tau(h)$$

□

Proposition 4.28. *Let*

(S1) G is a compact Lie group.

(S2) $(\tau, W), (\pi, V) \in \hat{G}$.

Then

$$P_\tau \circ P_\pi = \begin{cases} P_\tau & (\tau = \pi) \\ 0 & (\tau \not\simeq \pi) \end{cases}$$

Proof. Let us fix an orthonormal basis of V. For any $v_i \in V$, by Shur orthogonality relations,

$$\begin{aligned}
P_\pi(P_\tau(v_i)) &= \textstyle\sum_{j=1}^{dim\pi}(dim\tau)(dim\pi)\int_G \overline{\chi_\tau(g)}\tau(g)\int_G \overline{\chi_\pi(h)}(\pi(h)v_i, v_j)v_j dhdg \\
&= \textstyle\sum_{j,k}(dim\tau)(dim\pi)\int_G \overline{\chi_\tau(g)}\int_G \overline{\chi_\pi(h)}(\pi(h)v_i, v_j)(\tau(g)v_j, v_k)v_k dhdg \\
&= \textstyle\sum_{j,k}(dim\tau)(dim\pi)\int_G \overline{\chi_\tau(g)}\int_G \overline{\chi_\pi(h)}\pi_{j,i}(h)\tau_{k,j}(g)v_k dhdg \\
&= \textstyle\sum_{j,k,a,b}(dim\tau)(dim\pi)\int_G \overline{\tau_{a,a}(g)}\int_G \overline{\pi_{b,b}(h)}\pi_{j,i}(h)\tau_{k,j}(g)dhdgv_k \\
&= (dim\tau)(dim\pi)\textstyle\sum_{j,k,a,b}(\tau_{k,j}, \tau_{a,a})(\pi_{k,j}, \pi_{a,a}) = \delta_{\tau,\pi}v_i
\end{aligned}$$

□

Theorem 4.8. *Let*

(S1) G is a compact Lie group.

(S2) (π, V) *is an continuous unitary representation of G.*

(S3) $(\tau, W) \in \hat{G}$.

then P_τ is the orthogonal projection of V_τ.

Proof. By Proposition4.26,

$$P_\tau|V_\tau = id_{V_\tau}$$

Let us fix any $v \in V$. We will show there is V' which is a finite dimensional G-invariant subspace of V such that $P_\tau(v) \in V'$. Let us fix $\{v_1, ..., v_m\}$ which is a orthogonality basis of (τ, W). For any $x \in G$,

$$\pi(x)P_\tau(v) = \int_G \overline{\chi_\tau(g)}\pi(xg)vdg = \int_G \overline{\chi_\tau(x^{-1}g)}\pi(g)vdg = \sum_i \int_G \overline{\tau_{i,i}(x^{-1}g)}\pi(g)vdg = \sum_i \int_G \overline{(\tau(x^{-1}g)v_i, v_i)}\pi(g)vdg$$

$$= \sum_i \int_G (\tau(x)v_i, \tau(g)v_i)\pi(g)vdg = \sum_{i,j} \tau_{i,j}(x) \int_G (v_j, \tau(g)v_i)\pi(g)vdg \in \sum_{i,j} \mathbb{C} \int_G (v_j, \tau(g)v_i)\pi(g)vdg =: V'$$

By Proposition4.28 and Proposition4.5, $P_\tau(v) = P_\tau(P_\tau(v)) \in P_\tau(V') \subset V'_\tau \subset V_\tau$.

Lastly, we will show $P_\tau^* = P_\tau$. Let us fix any $u, v \in V$. By Proposition2.54 and Proposition4.13,

$$(P_\tau(u), v) = (\int_G \overline{\chi_\tau(g)}\pi(g)udg, v) = \int_G \overline{\chi_\tau(g)}(\pi(g)u, v)dg = \int_G (u, \chi_\tau(g)\pi(g^{-1})v)dg$$

$$= (u, \int_G \overline{\chi_\tau(g^{-1})}\pi(g^{-1})vdg) = (u, P_\tau(v))$$

So, $P_\tau^* = P_\tau$. □

Proposition 4.29. *Let*

(S1) G is a compact Lie group.

(S2) (π, V) is an continuous unitary representation of G.

(S3) (τ, V) is an continuous finite dimensional unitary representation of G.

then $P_{\pi,\tau}$ is G-linear.

Proof. For any $x \in G$ and $v \in V$,

$$\pi(x)P_{\pi,\tau}(v) = \int_G \overline{\chi_\tau(y)}\pi(x)\pi(y)vdg(y) = \int_G \overline{\chi_\tau(xx^{-1}yxx^{-1})}\pi(xyx^{-1})\pi(x)vdg(y)$$
$$= \int_G \overline{\chi_\tau(xyx^{-1})}\pi(y)\pi(x)vdg(y) = \int_G \overline{\chi_\tau(y)}\pi(y)\pi(x)vdg(y) = P_{\pi,\tau}(\pi(x)v)$$

□

Theorem 4.9. *Let*

(S1) G is a compact Lie group.

(S2) (π, V) is a continuous unitary representation of G.

then

$$V = \oplus_{\tau \in \hat{G}_f} V_\tau$$

Proof. By Proposition4.28, $V_\tau \perp V_\pi$ ($\tau \not\simeq \pi$). So, it is enough tho show $\cap_{\tau \in \hat{G}_f} V_\tau^\perp = \{0\}$. Let us fix any $v \in \cap_{\tau \in \hat{G}_f} V_\tau^\perp$. Then for any $x \in G$ and $\tau \in \hat{G}_f$, by Proposition4.29,

$$0 = \int_G (P_\tau(\pi(x^{-1})\pi(x)w), w)dg(x) = \int_G (\pi(x^{-1})P_\tau(\pi(x)w), w)dg(x) == \int_G (P_\tau(\pi(x)w), \pi(x)w)dg(x)$$
$$= \int_G \int_G \overline{\chi_\tau(g)}(\pi(g)\pi(x)w, \pi(x)w)dg(g)dg(x) = (f, \chi_\tau)$$

Here,

$$f(x) := \int_G (\pi(x)\pi(g)v, \pi(g)v)dg \ (x \in G)$$

For any $x, y \in G$,

$$f(y^{-1}xy) = \int_G (\pi(y^{-1}xy)\pi(g)v, \pi(g)v)dg = \int_G (\pi(x)\pi(yg)v, \pi(yg)v)dg = f(x)$$

So, $f \in C(G)^{Ad}$. By Theorem4.7, $f = 0$. So, $||w||^2 = f(e) = 0$. □

4.7.5 Expansion formula of L^2 functions

Proposition 4.30. *Let*

(S1) G is a compact Lie group.

(S2) (τ, W) is an irreducible unitary representation of G.

Then

$$\Phi_\tau(W \otimes W^*) = L^2(G)_\tau$$

Proof. Firstly, we will show that

$$\Phi_\tau(W \otimes W^*) \subset L^2(G)_\tau$$

For each $f \in W^*$, we set $\Phi_{\tau,f} : W \to L^2(G)$ by

$$\Phi_{\tau,f}(w) := \Phi_(w, f) \ (w \in W)$$

Let us fix any $f \in W^*$. Clearly $\Phi_{\tau,f}$ is linear. By shur orthogonality relations, $\Phi_{\tau,f}$ is continous. And for any $h \in G$

$$\Phi_{\tau,f}(\tau(h)w) = f(\tau(\cdot)^{-1}\tau(h)w) = f(\tau(h^{-1}\cdot)^{-1}w) = L_h\Phi_{\tau,f}(\tau(h)w)$$

This means that $\Phi_{\tau,f}$ is G-linear. So, $\Phi_\tau(W \otimes W^*) \subset L^2(G)_\tau$.

Lastly, we will show that

$$L^2(G)_\tau \subset \Phi_\tau(W \otimes W^*)$$

Let us fix $w_1, ..., w_m \in W$ which is a basis of W and $A \in Hom_G(W, V)$. For any i and $x \in G$,

$$(Aw_i)(x) = (L_{x^{-1}}Aw_i)(e) = (A\tau(x^{-1})w_i)(e) = (A(\sum_{j=1}^{m} \tau(x^{-1})w_i, w_j)w_j)(e) = (A(\sum_{j=1}^{m} \Phi_{i,j}(x)w_j)(e)$$

$$= \sum_{j=1}^{m}(Aw_j)(e)\Phi_{i,j}(x)$$

So, $L^2(G)_\tau \subset \Phi_\tau(W \otimes W^*)$. □

Proposition 4.31. *Let*

(S1) G is a compact Lie group.

(S2) $\tau \in \hat{G}$.

for any $f \in L^2(G)$

$$P_{L,\tau}(f)(x) = dim\tau\overline{\chi_\tau} * f(x) \ (a.e. \ x \in G)$$

Proof. For any $f \in L^2(G)$ and $a.e$ $x \in G$,

$$P_{L,\tau}(f)(x) = \int_G \overline{\chi_\tau(g)}f(g^{-1}x)dg = \int_G \overline{\chi_\tau(g^{-1})}f(gx)dg = \int_G \overline{\chi_\tau(xg^{-1})}f(g)dg = \overline{\chi_\tau} * f(x)$$

□

Proposition 4.32 (Operator Valued Fourier Transform)**.** *Let*

(S1) G is a compact Lie group.

(S2) (τ, W) is a continuous unitary representation of G.

(S3) $f \in L^2(G)$.

Then

(i) For each $w \in W$, there is the unique element $I(\tau, f)w$ such that

$$(u, I(\tau, f)w) = \int_G (u, f(g)\tau(g)w)dg_l(g) \ (\forall u \in W)$$

(ii) $I(\tau, f)$ is bounded and $||I(\tau, f)|| \leq ||f||_{L^2(G)}$.

Without fear of misinterpretation, we denote $I(\tau, f)$ by $\tau(f)$. We call $\hat{G} \ni \pi \mapsto I(\pi, f)$ the operator valued fourier transform of f.

Proof of (i).

$$|\int_G (u, f(g)\tau(g)w)dg_l(g)| \leq ||f||_{L^2(G)}||u|| \cdot ||w|| \ (\forall u \in W)$$

So, by Riez representation theorem, (i) holds. □

Proof of (ii). (ii) is followed by the above equation. □

Proposition 4.33. *Let*

(S1) G is a compact Lie group.

(S2) (π, V) is a continuous unitary representation of G.

(S3) $f \in L^2(G)$.

Then

(i) $\pi(f * g) = \pi(f)\pi(g)$ is a compact Lie group.

(ii) $\pi(R_x f) = \pi(f)\pi^*(x)$ $(\forall x \in G)$.

(iii) $\pi(L_x f) = \pi(x)\pi(f)$ $(\forall x \in G)$.

Proof of (i).

$$\pi(f * g) = \int_G f * g(x)\pi(x)dg(x) = \int_G \int_G f(xy^{-1})g(y)dg(y)\pi(x)dg(x) = \int_G \int_G f(y^{-1})g(yx)dg(y)\pi(y^{-1})\pi(yx)dg(x)$$
$$= \int_G f(y^{-1})\pi(y^{-1}) \int_G g(yx)\pi(yx)dg(x)dg(y) = \int_G f(y^{-1})\pi(y^{-1}) \int_G g(x)\pi(x)dg(x)dg(y) = \int_G f(y^{-1})\pi(y^{-1})\pi(g)dg(x)$$
$$= \int_G f(y)\pi(y)\pi(g)dg(x) = \pi(f)\pi(g)$$

□

Proof of (ii).

$$\pi(R_x f) = \int_G f(gx)\pi(g)dg(g) = \int_G f(gx)\pi(gx)\pi(x^{-1})dg(g) = \int_G f(gx)\pi(gx)dg(g)\pi^*(x) = \pi(f)\pi^*(x)$$

□

Proof of (iii).

$$\pi(L_x f) = \int_G f(x^{-1}g)\pi(g)dg(g) = \int_G f(x^{-1}g)\pi(xx^{-1}g)dg(g) = \pi(x) \int_G f(g)\pi(g)dg(g) = \pi(x)\pi(f)$$

□

(i)(ii) in Proposition4.33 characterize the operator valued fourier transformation. See Theorem3.1 in [18].

Proposition 4.34. *Let*

(S1) G is a compact Lie group.

(S2) (τ, W) is a continuous finite dimensional unitary representation of G.

Then

$$P_{L,\tau}(f) = dimW\Phi_{L,\tau}(\tau(f)) \ (\forall f \in L^2(G))$$

Proof. For any $y \in G$,

$$\Phi_\tau(\tau(f))(y) = \sum_{i,j} \Phi_\tau((\tau(f)v_j, v_i)v_i \otimes v_j)(y) = \sum_{i,j} \int_G \Phi_\tau((f(x)\tau(x)v_j, v_i)v_i \otimes v_j)dg(x)(y)$$
$$= \int_G \sum_{i,j} f(x)(\tau(x)v_j, v_i)\Phi_\tau(v_i \otimes v_j)dg(x)(y) = \int_G \sum_{i,j} f(x)\tau_{i,j}(x)(\tau(y^{-1})v_j, v_i)dg(x)$$
$$= \sum_{i,j} \int_G f(x)\tau_{i,j}(x)dg(x)(\tau(y^{-1})v_j, v_i) = \sum_{i,j} \int_G f(x)\tau_{i,j}(x)dg(x)\tau_{j,i}(y^{-1})$$
$$= \sum_i \int_G f(x)\tau_{i,i}(xy^{-1}) = \sum_i \int_G f(xy)\tau_{i,i}(x)dg(x) = \int_G f(xy)\overline{\chi_\tau(x^{-1})}dg(x)$$
$$= \int_G f(x^{-1}y)\overline{\chi_\tau(x)}dg(x) = \int_G L_x f\overline{\chi_\tau(x)}dg(x)(y) = \frac{1}{dim\tau}P_{L,\tau}(f)(y)$$

□

Theorem 4.10 (Plancherel formula for compact Lie group)**.** *Let*

(S1) G is a compact Lie group.

(S2) $f \in L^2(G)$.

then

$$f = \sum_{\tau \in \hat{G}_f} \Phi'_\tau(\tau(f)) \ (L^2 \ convergence)$$

We set μ by the counting measure of $\hat{G}_f$. Then

$$f = \int_{\hat{G}_f} \Phi'_\tau(\tau(f))d\mu(\tau)$$

The right side is a bochner integral on the $L^2(G)$ valued function. We call μ the Plancherel measure on $\hat{G}$.

Proof by Peter-Weyl Theorem III.. This is followed by Theorem4.8 and Proposition4.9 and Proposition. □

Proof by Peter-Weyl Theorem II.. By Proposition4.30 and Theorem4.8, $P_\tau(L^2(G)) = \Phi_\tau(V \otimes V^*)$ for any $(\tau, V) \in \hat{G}$. By Proposition4.7.5, $P_\tau(f) = \Phi'_\tau(f)$ $(\forall f \in L^2(G))$. By Peter Weyl Theorem II and Proposition1.17,

$$f = \sum_{\tau \in \hat{G}_f} \Phi'_\tau(\tau(f)) \ (\forall f \in L^2(G))$$

□

Proposition 4.35. *Let*

(S1) G is a compact Lie group.

(S2) (π, V) and (τ, W) are continuous unitary representations of G.

(S3) $T : V \to W$ is an isomorphism as continuous unitary representations of G.

(S4) $f \in L^2(G)$.

Then

$$\pi(f) = T^{-1}\tau(f)T$$

Proof. For any $u, v \in V$,

$$(u, \pi(f)v) = (Tu, T\pi(f)v) = \int_G (Tu, Tf(g)\pi(g)v)dg = \int_G (Tu, f(g)\tau(g)Tv)dg = \int_G (u, T^{-1}f(g)\tau(g)Tv)dg$$
$$= (u, T^{-1}\tau(f)Tv)$$

□

4.7.6 Example:Fourier series expansion

By Lemma2.10, the following holds.

Proposition 4.36. *The following μ is a Haar measure on S^1.*

$$\mu(f) := \frac{1}{2\pi}\int_0^{2\pi} f(exp(i\theta))d\theta \ (f \in C(S^1))$$

Proposition 4.37. *Let*

(S1) (τ, W) is a unitary representation of $\mathbb{T}^1$.

Then (τ, W) is irreducible $\iff$ $dim\tau = 1$ and there is $n \in \mathbb{Z}$ such that

$$\tau(exp(i\theta 2\pi))v = exp(in\theta 2\pi)v \ (\forall\theta \in \mathbb{R}, \forall v \in W)$$

We denote this irreducible representation by τ_n

Proof1 of $\Longrightarrow$. By Shur Lemma, $dim\tau = 1$. Since τ is unitary, $\tau(S^1)$ can been seen as elements of S^1. By Theorem2.2, τ is C^ω-class. We set $f(\theta) := \tau(i\theta 2\pi)$ $(\theta \in \mathbb{R})$. Because $f(\theta + h) = f(\theta)f(h)$ $(\forall\theta, h \in \mathbb{R})$,

$$f'(\theta) = f'(0)f(\theta) \ (\forall\theta \in \mathbb{R})$$

So, taylor series of f converges on $\mathbb{R}$. This implies that there is $\alpha \in C$ such that

$$f(\theta) = exp(i\alpha\theta 2\pi) \ (\forall\theta \in \mathbb{R})$$

Because $Im(f) \subset S^1$, $\alpha \in \mathbb{R}$. Because $f(1) = 1$, $\alpha \in \mathbb{Z}$. □

Proof2 of $\Longrightarrow$ without Theorem2.2. By Shur Lemma, $dim\tau = 1$. Since τ is unitary, $\tau(S^1)$ can been seen as elements of S^1. We set

$$f(\theta) := \tau(i\theta 2\pi) \ (\theta \in \mathbb{R})$$

and

$$\psi(\theta) := exp(i\theta) \ (\theta \in (-\pi, \pi))$$

There is $\delta > 0$ such that $f((-\delta, \delta)) \subset \psi((-\pi, \pi))$ We can assume $f|(-\delta, \delta) \neq 1$. So, there is $t_0 \in (-\delta, \delta) \setminus 0$ such that $f(t_0) \neq 1$. There is $\alpha \in (-\pi, \pi)$ such that $f(t_0) = exp(i\alpha)$. Because ψ is injective,

$$f(\frac{k}{2^m}t_0) = exp(i\frac{k}{2^m}\alpha) \ (\forall m \in \mathbb{Z}_+, \forall k \in \mathbb{Z} \text{ such that } |\frac{k}{2^m}| \leq 1)$$

Because the both sides are continuous,

$$f(\theta) = exp(i\frac{\alpha}{t_0 2\pi}\theta 2\pi) \ (\forall\theta \in (-|t_0|, |t_0|))$$

We set $\beta := \frac{\alpha}{t_0 2\pi}$. Becuase f is homomorphism,

$$f(\theta) = exp(i\beta\theta 2\pi) \ (\forall\theta \in \mathbb{R})$$

Because $f(1) = 1$, $\beta \in \mathbb{Z}$. □

Proof of $\Longleftarrow$. It is clear. □

By Proposition4.37, the following holds.

Proposition 4.38. *Let*

(S1) τ_n is an irreducible unitary representation of $\mathbb{T}^1$ for $n \in \mathbb{Z}$.

(S2) χ_n is the character of τ_n.

(S3) $\tau^n_{1,1}$ is the matrix coefficient of τ_n.

Then

(i)

$$\tau^n_{1,1}(z) = \chi_n(z) = z^n = exp(i \cdot n \cdot arg(z)) \ (\forall z \in S^1)$$

(ii)

$$(f, \tau^n_{1,1}) = \frac{1}{2\pi}\int_0^{2\pi} f(expi\theta)exp(-in\theta d\theta = \hat{f}(n) \ (\forall f \in L^2(S^1), \forall n\mathbb{N})$$

By Peter-Weyl II and Proposition4.38 and Proposition1.12, the following holds.

Theorem 4.11 (Fourier expansion formula)**.** *For any* $f \in L^2([0, 2\pi])$

$$f = \lim_{N\to\infty} \sum_{n=-N}^{N} \hat{f}(n)\chi_n \ (L^2\text{-convergence})$$

By Peter-Weyl III and Proposition4.38 and Proposition1.12, the following holds.

Theorem 4.12 (Wierstrass Theorem)**.** *For any* $f \in C(S^1)$ *and* $\epsilon > 0$*, there is a finite subset* $N \subset \mathbb{N}$ *and* $a_{-N}, a_{-N+1}, ..., a_N$ *such that*

$$||f - \sum_{n=-N}^{N} a_n\chi_n||_\infty < \epsilon$$

4.7.7 Characterization of compact Lie group

Theorem 4.13. *Let us* G *be a compact topological group. Then* G *is a Lie group* $\iff$ G *has a continuous finite dimensional faithful unitary representation. In special, if* G *is a compact Lie group, then there is a* C^ω*-class diffeomorphism from* G *to some closed subgroup of* $U(n)$ *for some* $n \in \mathbb{N}$*.*

Proof of $\Longrightarrow$. By Proposition2.33, there is an open neighborhood U which does not contain subgroups without $\{e\}$. By Peter-Weyl Theorem I, for any $\tau \in \hat{G}$, $Ker(\tau)$ is closed subset of G. By Gelfand-Raikov theorem, $G = \cup_{\tau\in\hat{G}} Ker(\tau)^c \cup U$. Because G is compact, there are finite $\tau_1, ..., \tau_m \in \hat{G}_f$ such that $G = \cup_{i=1}^m Ker(\tau_i)^c \cup U$. Because U does not contain subgroups without $\{e\}$, $\cap_{i=1}^m Ker(\tau_i) = \{e\}$. Then $\oplus_{i=1}^m \tau_i$ is a continuous finite dimensional faithful unitary representation of G. □

Proof of $\Longleftarrow$. Then G is isomorphic to closed subgroup of $U(n) \subset GL(n, \mathbb{C})$ as toplogical groups for some $n \in \mathbb{N}$. So, G is Lie group. □

4.8 Review

The main theorems of this chapter are Peter-Weyl's Theorem I-III, embedding any compact Lie group into $U(n)$, Plancherel formula for compact Lie groups. In this section, we review these theorems, noting their relationship to the Mautner-Teleman theorem. We also explain how this is a generalization of the theory of Fourier series expansions. The key facts in this chapter are various capabilities of 'averaging' by Haar measure in compact Lie groups, Shur Lemma, Gelfand-Raikov Theorem.

The Mautner-Teleman theorem guarantees that any unitary representation of a Lie group can be decomposed into a direct integral of irreducible unitary representations. The following Peter-Weyl Theorem I guarantees that this direct integral is a discrete direct sum of finite-dimensional irreducible unitary representations if the Lie group G is compact. In particular, the irreducible unitary representation of a compact Lie group is always finite-dimensional. This means $\hat{G} = \hat{G}_f$. Here $\hat{G}$ is the set of all equivalent classes of continuous irreducible unitary representation of G, and $\hat{G}_f$ is the set of all equivalent classes of continuous finite dimensional irreducible unitary representation of G.

Theorem 4.14 (Peter-weyl theorem I)**.** *Let* (π, V) *be a continuous unitary representation of a compact Lie group* G*. Then there is* D *which is a subset of* G*-invariant finite dimensional irreducible subspaces such that*

$$V = \overline{\bigoplus_{W\in D} W}$$

The proof of Peter-Weyl's Theorem I, by using Zorn's Lemma, boils down to the proof of the claim that any unitary representation of a compact Lie group has a finite dimensional G-invariant subspace. Such an invariant subspace can be realized as the eigenspace of a G-linear map composed by acting on all group elements in their projection onto a suitable 1-dimensional space and averaging them. If the group is a finite group, this operator is a finite-dimensional matrix, its eigenspace will be one-dimensional. In the general case, this sum is the Bochner integral, and the operator formed by the sum is compact operator, so its eigenspace is finite-dimensional.

The irreducible unitary representation of S^1 is, by Shur's lemma and the real analyticity of finite dimensional representations of Lie groups(Theorem2.2), we find that it is exhausted by homomorphisms of the following form(Proposition4.37).

$$\tau_n : S^1 \ni z \mapsto z^n = exp(i \cdot n \cdot arg(z)) \in S^1 \ (n \in \mathbb{Z})$$

Thus, any unitary representation of S^1 can be decomposed into a direct sum of these representations.

Peter-Weyl's Theorem II gives the irreducible decomposition of $L^2(G)$ using Peter-Weyl's Theorem I.

Theorem 4.15 (Peter-weyl theorem II)**.**

$$\Phi : (L, \oplus_{\tau \in \hat{G}_f} V \otimes V^*) \to (L, L^2(G))$$

Here, for each $(\tau, V) \in \hat{G}_f$ *and* $v \otimes f \in V \otimes V^*$,

$$\Phi(v \otimes f)(g) := f(\tau(g^{-1})v) \ (g \in G)$$

$$L_x(v \otimes f) = \tau(x)v \otimes f \ (x \in G)$$

$$L_x h(g) = h(x^{-1}g) \ (h \in L^2(G), g, x \in G)$$

We set

$$A := \{\sqrt{dim\tau}\tau_{i,j} | (\tau, V) \text{ is an representative of } \hat{G}_f \text{ and } \{v_1, ..., v_{dim\tau}\} \text{ is an orthonormal basis of } V \text{ and } 1 \le i, j \le dim\tau\}$$

Here, $\tau_{i,j}$ is defined as bellow for each i, j.

$$\tau_{i,j}(g) := (\tau(g)v_j, v_i) \ (g \in G)$$

The Peter-Weyl Theorem III guarantees that any continuous function f on G can be uniformly approximated by elements of a vector space B generated from the above set A.

Theorem 4.16 (Peter-Weyl Theorem III)**.** *For any* $\epsilon > 0$, *there is a* $a_1, ..., a_n \in \mathbb{C}$ *and* $\tau_{j_1,j_1}, ..., \tau_{j_n,j_n} \in A$

$$|f(g) - \sum_{i,k=1,...,n} a_i \tau_{j_i,j_k}(g)| < \epsilon \ (\forall g \in G)$$

The proof of this theorem uses Stone Wierestrass's theorem(Theorem4.1) on uniform approximation of continuous functions on compact metric spaces. By Gelfand Raikov's theorem and the theory of positive definite functions, B contains constants and is closed by products and complex conjugates. Stone wierestrass theorem, such a space is , guarantees a uniform approximation of continuous functions on G. By applying Peter-Weyl's Theorem III to the case $G = S^1$, we obtain the following approximate theorem.

Theorem 4.17 (Wierstrass Theorem)**.** *For any* $f \in C(S^1)$ *and* $\epsilon > 0$, *there is a finite subset* $N \subset \mathbb{N}$ *and* $a_{-N}, a_{-N+1}, ..., a_N$ *such that*

$$|f(z) - \sum_{n=-N}^{N} a_n z^n| < \epsilon \ (\forall z \in S^1)$$

By Peter-Weyl Theorem I and Gelfand-Raikov Theorem, the following is shown(Theorem4.13).

Theorem 4.18. *Any compact Lie group is isomophic to a closed subgroup of* $U(n)$ *for some* $n \in \mathbb{N}$

By Peter-Weyl Theorem II and Shur's Lemma, the above set A of matrix coefficients corresponding to all irreducible unitary representations is guaranteed to be an orthonormal basis of $L^2(G)$. Since $L^2(G)$ is separable, by Peter-Weyl's Theorem II, $\hat{G}_f$ is at most countable set. Due to the real analyticity of finite-dimensional representations of Lie groups, each $\tau_{i,j}$ is real analytic. From the above, we can say that this family of functions is an easy-to-handle family of functions. By the theory on orthonormal bases of Hilbert spaces, The square integrable function on G can be expanded by such a tractable function as by such an easy-to-handle function.

$$f = \sum_{\tau \in \hat{G}_f, 1 \le i,j \le dim\tau} dim\tau (f, \tau_{i,j})\tau_{i,j} \ (L^2\text{-convergence})$$

This equation has two other expression. The one is the expression by characters(Proposition4.31 and Theorem4.7).

$$f = \sum_{\tau \in \hat{G}_f} \dim \tau \overline{\chi_\tau} * f \ (L^2\text{-convergence})$$

The another one is the expression by operator valued fourier transform.

Theorem 4.19 (Plancherel formula for compact Lie group)**.** *Let*

(S1) G is a compact Lie group.

(S2) $f \in L^2(G)$.

then

$$f = \sum_{\tau \in \hat{G}_f} \Phi'_\tau(\tau(f)) \ (L^2 \ convergence)$$

Here,

$$\tau(f) := \int_G \overline{\chi_\tau}(g)\tau(g) f dg \ (f \in L^1(G))$$

$$\Phi'_\tau(v \otimes f)(g) := dim\tau f(\tau(g^{-1})v)$$

We set μ by the counting measure of $\hat{G}_f$. Then

$$f = \int_{\hat{G}_f} \Phi'_\tau(\tau(f)) d\mu(\tau)$$

The left side is a bochner integral on the $L^(G)$ valued function. We call μ the Plancherel measure on $\hat{G}$.

The mapping $\hat{G} \ni \tau \mapsto \tau(f)$ is called the operator valued fourier transform of f. Operator valued fourier transform have the following properties.

(i) $\pi(f * g) = \pi(f)\pi(g) \ (\forall f, g \in L^2(G))$.

(ii) $\pi(R_x f) = \pi(f)\pi^*(x) \ (\forall x \in G)$.

It is known operator valued fourier transform is characterized by these properties[18]. In the case when $G = S^1$, $\tau_n(f) = \hat{f}(n) = (f, \tau_n)$ and $P_{\tau_n}(f)(\theta) = \hat{f}(n) exp(in\theta)$.

By applying Peter-Weyl's Theorem II to the case $G = S^1$, we obtain the following Fourier series expansion formula.

Theorem 4.20 (Fourier series expansion formula)**.** *For any $f \in L^2([0, 2\pi])$*

$$f = \lim_{N \to \infty} \sum_{n=-N}^{N} \hat{f}(n)\chi_n \ (L^2\text{-}convergence)$$

5 Homogeneous space

5.1 C^ω-class structure

Theorem 5.1. *Let*

(S1) G_1 *is a Le group which is locally isomorphic to a Lie subgroup of* $GL(n,\mathbb{C})$ G_2*.*

(A1) H *is a closed subgroup of* G_1 *such that* $dimLie(H) > 0$*.*

(S2) $\mathfrak{h} := Lie(H)$*.*

(S3) $\mathfrak{g}_1$ *is a complementary space of* $\mathfrak{h}$ *in* $\mathfrak{g} := Lie(G_1)$*.*

(S4) $k := dim\mathfrak{g}_1$ *and* $l := dim\mathfrak{h}$*.*

Then there is a C^ω*-class manifold structure of* G/H *such that*

(i) $p : G_1 \ni g \mapsto gH \in G_1/H$ *is a continuous map and an open map.*

(ii) $G_1 \times G_1/H \ni (g_1, g_2H) \mapsto g_1g_2H$ *is* C^ω*-class.*

(iii) For any $g \in G$ *and* $h \in H$*, there is* $\epsilon > 0$ *such that*

$$B_k(O,\epsilon) \times B_l(O,\epsilon) \ni (X,Y) \mapsto gExp(X)hExp(Y) \in G$$

and

$$B_k(O,\epsilon) \ni X \mapsto \pi(gExp(X)) \in G/H$$

are C^ω*-class diffeomorphism.*

We call G/H *homogeneous space or homogeneous manifold.*

STEP1. Definition of the topology of G/H*.* We set

$$p : G \ni g \to gH \in G/H$$

and

$$\mathcal{O}(G/H) := \{A \subset G/H | p^{-1}(A) \in \mathcal{O}(G)\}$$

Clearly, p is continuous. Also, for each $O \in \mathcal{O}(G)$,

$$p^{-1}(p(O)) = \cup_{h\in H} Oh$$

So, p is an open map. Because p is surjective, for any $O_1 \in \mathcal{O}(G/H)$, there is $O_2 \in \mathcal{O}(G)$ such that

$$p(O_2) = O_1$$

And clearly, for any $O \in \mathcal{O}(G)$ and $g \in G$,

$$L_g \circ p(O) = p \circ L_g(O)$$

So, L_g is a homeomorphism of G/H.

We will show G/H is a Hausdorff space. Let us fix $g_1, g_2 \in G$ such that $g_1H \neq g_2H$. So, $g_2^{-1}g_1 \notin H$. Because H is a closed set, there is U which is an open neighborhood of e such that

$$U^{-1}g_2^{-1}g_1U \cap H = \phi$$

This implies that

$$g_1UH \cap g_2UH = \phi$$

So, G/H is a Hausdorff space. □

STEP2. Construction of a local coordinate system of G/H*.* There is $\epsilon_0 > 0$ and $\epsilon > 0$ such that $Exp|B(O,\epsilon)$ is a C^ω-class homeomorphism to an open set of G and

$$Exp(B(O,\epsilon))Exp(B(O,\epsilon)) \subset Exp(B(O,\epsilon_0))$$

and

$$\rho : (\mathfrak{g}_1 \cap B(O,\epsilon_0)) \oplus (\mathfrak{h} \cap B(O,\epsilon_0)) \ni X + Y \to Exp(X)Exp(Y)$$

is a C^ω-class homeomorphism. We set for each $g \in G$

$$\rho_g : (\mathfrak{g}_1 \cap B(O,\epsilon)) \ni X \to gExp(X)H \in gExp(B(O,\epsilon_0))H$$

Clearly, $gExp(B(O,\epsilon_0))H \in \mathcal{O}(G/H)$ and ρ_g is surjective. We will show ρ_g is injective. Let us fix any $X_1, X_2 \in \mathfrak{g}_1$ such that $\rho_g(X_1) = \rho_g(X_2)$. Then, becaluse $Exp(B(O,\epsilon))Exp(B(O,\epsilon)) \subset Exp(B(O,\epsilon_0))$,

$$Exp(-X_2)Exp(X_1) \in H \cap Exp(B(O,\epsilon_0))$$

By von-Neumann-Cartan's theorem, we can assume

$$H \cap Exp(B(O,\epsilon_0)) = Exp(B(O,\epsilon_0) \cap \mathfrak{h})$$

So,

$$Exp(X_1) = Exp(X_2)Exp(B(O,\epsilon_0) \cap \mathfrak{h})$$

Because ρ is injective, $X_1 = X_2$.

We can assume for any $X \in B(O,\epsilon)\mathfrak{g}_1$, there is C^ω-class π_1 and π_2 such that for any $Z \in B(O,\epsilon)\mathfrak{g}_1$

$$Exp(X_2 + Z) = Exp(X_2 + \pi_1(Z))Exp(\pi_2(Z)), \pi_1(Z) \in \mathfrak{g}_1, \pi_2(Z) \in \mathfrak{h}$$

Let us fix any $g_1, g_2 \in G$ such that

$$g_1Exp(\mathfrak{g}_1 \cap B(O,\epsilon))H \cap g_2Exp(\mathfrak{g}_1 \cap B(O,\epsilon))H \neq \phi$$

Let us fix any $X_1 \in \rho_{g_1}^{-1}(g_1Exp(\mathfrak{g}_1 \cap B(O,\epsilon))H \cap g_2Exp(\mathfrak{g}_1 \cap B(O,\epsilon))H)$. There is $X_2 \in \mathfrak{g}_1 \cap B(O,\epsilon)$ and $h \in H$ such that

$$g_2^{-1}g_1Exp(X_1)h = Exp(X_2)$$

So, there is $\delta > 0$ such that

$$g_2^{-1}g_1Exp(X_1 + B(O,\delta))h \subset Exp(B(O,\epsilon_0))$$

We set

$$\psi(Y) := log(\tau(g_2^{-1}g_1Exp(X_1 + Y)h)) - X_2 \ (Y \in B(O,\delta) \cap \mathfrak{g}_1)$$

Then ψ is C^ω-class and

$$g_1Exp(X_1 + Y)h = g_2Exp(X_2 + \psi(Y))$$

So,

$$g_2Exp(X_2 + \psi(Y)) = g_2Exp(X_2 + \pi_1(\psi(Y)))Exp(\pi_2(\psi(Y)))$$

This implies that

$$\rho_{g_2}^{-1} \circ \rho_{g_1}(Y) = \pi_1(\psi(Y))$$

Consequently, $\{\rho_g\}_{g \in G}$ defines the C^ω-class structure of G/H. □

STEP3. Showing $G \times G/H \ni (g_1, g_2H) \to g_1g_2H$ is C^ω-class. For any $Y \in Lie(G) \cap B(O,\epsilon)$ and $X_1 \in \mathfrak{g}_1 \cap B(O,\epsilon)$

$$\rho_{g_1g_2}(g_1ExpYg_2Exp(X_1)H) = \rho_{g_1g_2}(g_1g_2Exp(Ad(g^{-1}Y)Exp(X_1)H) = \rho_{g_1g_2}(g_1g_2Exp(\xi(Ad(g^{-1}Y, X_1))) = \xi(Ad(g^{-1}Y, X_1)$$

Here, ξ is C^ω-class mapping such that $Exp(Y')Exp(X_1') = \xi(Y', X_1')$ $(\forall Y' \in Lie(G) \cap B(O,\epsilon), \forall X_1' \in \mathfrak{g}_1 \cap B(O,\epsilon))$. □

STEP4. Proof of (iii). By STEP2., there is $\delta > 0$ such that

$$\sigma : \mathfrak{g}_1 \cap B_k(O,\delta) \times \mathfrak{h} \cap B_l(O,\delta) \ni (X,Y) \mapsto Exp(X)Exp(Y) \in G$$

is C^ω-class diffeomorphism and

$$\mathfrak{g}_1 \cap B_k(O,\delta) \ni X \mapsto \pi(Exp(X)) \in G/H$$

is C^ω-class diffeomorphism. So,

$$B_k(O,\delta) \ni X \mapsto \pi(gExp(X)) \in G/H$$

is C^ω-class diffeomorphism. There is $\epsilon > 0$ such that

$$Ad(h)B_l(O,\epsilon) \subset B_l(O,\delta)$$

Let us fix any $g \in G$ and $h \in H$. We set

$$\rho : B_k(O,\epsilon) \times B_l(O,\epsilon) \ni (X,Y) \mapsto gExp(X)hExp(Y) \in G$$

Then ρ is clearly C^ω-class and $Im\rho$ is an open set. Because $gExp(X)hExp(Y) = gExp(X)Exp(Ad(h)Y)h$,

$$Im\rho \ni x \mapsto (p_1(\sigma^{-1}(g^{-1}xh^{-1})), Ad(h^{-1})p_2(\sigma^{-1}(g^{-1}xh^{-1})) \in \mathfrak{g}_1 \cap B_k(O,\delta) \times \mathfrak{h} \cap B_l(O,\delta)$$

is the inverse of σ and C^ω-class diffeomorphism. □

Definition 5.1 (Involutive automorphism)**.** *Let G be a Lie group. We call $\sigma \in Auto(G)$ a involutive or involution if $\sigma \circ \sigma = id_G$. We set $G^\sigma := \{g \in G | \sigma(g) = g\}$. And we denote the connected component of G^σ which contains the unit element by G_0^σ.*

Clealy the following hold.

Proposition 5.1. *G^σ and G_0^σ a closed subgroup of G.*

Definition 5.2 (Symmetric space)**.** *Let G be a Lie group and σ be a involution of G. If H is a closed subgroup of G such that $G_0^\sigma \subset H \subset G^\sigma$. Then we call (G, H) be a symmetric pair and G/H be a symmetric space.*

5.2 Invariant measure

5.2.1 Existence of Invariant measure

Definition 5.3 (Invariant measure)**.** *Here are the settings and assumptions.*

(S1) G is a Lie group and $m := Lie(G)$.

(S2) H is a closed subgroup of G.

(S3) μ is a Baire measure on G/H.

We say μ is a invariant measure on G/H if for any $f \in C_c(G/H)$ and any $g_0 \in G$

$$\int_G f(g_0 \cdot x) d\mu(x) = \int_G f(x) d\mu(x)$$

We say μ is a right invariant measure on G

Notation 5.1. *Let G be a Lie group and $g_0 \in G$. For each $x \in G/H$, $\tau_{g_0}(x) := g_0 \cdot x$.*

Lemma 5.1. *Here are the settings and assumptions.*

(S1) G is a Lie group and $\mathfrak{g} := Lie(G)$ and $m := dim\mathfrak{g}$.

(S2) H is a closed subgroup of G and $\mathfrak{h} := Lie(H)$ and $k := dim\mathfrak{h}$.

(S3) $\pi : G \ni g \mapsto gH \in G/H$.

(S4) $\tau_g : G/H \ni xH \mapsto gxH \in G/H$ $(g \in G)$.

(S5) $\mathfrak{q}$ is a complement space of $\mathfrak{h}$ in $\mathfrak{g}$ and $l := dim\mathfrak{q}$.

(S6) $x \in G$.

(S7) $\delta > 0$ such that $\Phi_x : B_l(O, \delta) \cap \mathfrak{q} \ni X \mapsto xexp(X)H \in G/H$ is a local coordinate around $\pi(x)$ in G/H. We set $U := B_l(O, \delta) \cap \mathfrak{q}$.

(S8) $\omega_{\pi(e)}$ is a m-th antisymmetric tensor field on $T_{\pi(e)}(G/H)$.

(S9) For each $X \in U$,

$$\omega^x_{\Phi_x(X)}(v_1, ..., v_l) := \omega_e(((d\tau_{xExp(X)})_{\pi(e)})^{-1} v_1, ..., ((d\tau_{xExp(X)})_{\pi(e)})^{-1} v_l) \ (v_1, ..., v_l \in T_{\Phi_x(X)}(G/H))$$

Then ω^x is C^ω-class l-form on $\Phi_x(U)$.

Proof. It is enough to show a representation matrix $(d\tau_{xExp(X)})_{\pi(e)}$ is C^ω-class. For each $y \in G/H$, we denote the local coordinate around y defined in the proof of 5.1 by ψ_y. So, it is enough to show

$$U \times U \ni (X, Y) \mapsto \psi^{-1}_{\pi(x)}(\tau_{xExp(X)}\psi_{\pi(e)}(Y)) \in \mathfrak{q}$$

is C^ω-class. By the proof of 5.1, there is $\epsilon \in (0, \delta)$ such that

$$\Theta : \mathfrak{q} \cap B_k(O, \epsilon) \times \mathfrak{h} \cap B_l(O, \epsilon) \ni (X, Y) \mapsto exp(X)exp(Y) \in G$$

is a C^ω-class homeomorphism to an open neighborhood of e. We can assume $Exp(U)Exp(U) \in Im\Theta$. For each $(X, Y) \in U \times U$, there is the unique $(\alpha(X), \beta(Y)) \in \mathfrak{q} \cap B_k(O, \epsilon) \times \mathfrak{h} \cap B_l(O, \epsilon)$ such that

$$\tau_{xExp(X)}\psi_{\pi(e)}(Y) = Exp(\alpha(X, Y))Exp(\beta(X, Y))$$

and α and β are C^ω-class. And for any $X, Y \in U$,

$$\psi^{-1}_{\pi(x)}(\tau_{xExp(X)}\psi_{\pi(e)}(Y)) = \alpha(X, Y)$$

So,

$$U \times U \ni (X, Y) \mapsto \psi^{-1}_{\pi(x)}(\tau_{xExp(X)}\psi_{\pi(e)}(Y)) \in \mathfrak{q}$$

is C^ω-class. □

Lemma 5.2. *We will succeed notations in 5.2. And here are the settings and assumptions.*

(A1) For any $x, y \in G$, there is $\sigma \in \{-1, 1\}$ such that

$$\omega^x = \sigma\omega^y \text{ in } \Phi_x(U) \cap \Phi_y(U)$$

(S1) For any $x \in G$, there is $\phi_x \in C^\omega(\Phi_x(U))$ such that for any $q \in \Phi_x(U)$,

$$\omega^x_q = \phi_x(q) d(\Psi^1_x)_q \wedge ... \wedge d(\Psi^k_x)_q$$

Here $\Psi_x := \Phi_x^{-1}$.

(S2) We set

$$\tilde{\omega}_q = |\phi_x(q)| d(\Psi^1_x)_q \wedge ... \wedge d(\Psi^k_x)_q \ (x \in G, q \in \Phi_x(U))$$

and define $\rho : G/H \to \{-1, 1\}$ by

$$\tilde{\omega}_q = \rho(q)\omega_q(x \in G, q \in \Phi_x(U))$$

Then $\tilde{\omega}$ is C^∞-class form on G/H and for any $q \in G/H$ and $g \in G$ there is $\sigma_{g,q} \in \{-1, 1\}$

$$(d\tau_g)\tilde{\omega}_q = \sigma_{g,q}\tilde{\omega}_q$$

and G/H is orientable.

Proof. Let us fix any $g, x \in G$. We set $q := \pi(x)$ and $p := \pi(e)$. Then for any $v_1, ..., v_k \in T_q(G/H)$,

$$((d\tau_g)\tilde{\omega})_q(v_1, ..., v_k) = \tilde{\omega}_{gq}((d\tau_g)_q v_1, ..., (d\tau_g)_q v_k) = \rho(gq)\omega_e((d\tau_{gx})_e^{-1}(d\tau_g)_q v_1, ..., (d\tau_{gx})_e^{-1}(d\tau_g)_q v_k)$$
$$= \omega_e((d\tau_x)_e^{-1} v_1, ..., (d\tau_x)_e^{-1} v_k) = \rho(gq)\rho(q)\tilde{\omega}_q(v_1, ..., v_k)$$

□

Lemma 5.3. *We will succeed notations in 5.2. Then*

$$\omega^x_{xExp(X)H} = det(d\tau_{xExp(X)})^{-1}(d\Psi^1_x)_{xExp(X)H} \wedge ... \wedge (d\Psi^k_x)_{xExp(X)H} \ (\forall X \in U)$$

Proof. Let us fix any $X \in U$. We set $g := xExp(X)$ and $q := \pi(g)$.

$$\omega^x_q = det(\{\omega^x_q((\frac{\partial}{\partial \Psi^j_x})_q e_i)\}^k_{i,j=1})(d\Psi^1_x)_q \wedge ... \wedge (d\Psi^k_x)_q$$

We denote the inverse of jacobi matrix of $(d\tau_g)_p$ with respect to $\{(\frac{\partial}{\partial \Psi^j_x})_q\}_j$ and $\{(\frac{\partial}{\partial \Psi^j_e})_p\}_j$ by $\{a_{j,r}\}^k_{j,r=1}$. Then

$$(d\tau_g)^{-1}_p(\frac{\partial}{\partial \Psi^j_x})_q = \sum^k_{r=1} a_{j,r}(\frac{\partial}{\partial \Psi^r_e})_p$$

So,

$$\omega^x_q((\frac{\partial}{\partial \Psi^j_x})_q e_i) = a_{j,i}$$

Consequently,

$$\omega^x_{xExp(X)H} = det(d\tau_{xExp(X)})^{-1}(d\Psi^1_x)_{xExp(X)H} \wedge ... \wedge (d\Psi^k_x)_{xExp(X)H}$$

□

Lemma 5.4. *We will succeed notations in 5.2. And here are the settings and assumptions.*

(A1) For any $h \in H$,

$$|det((d\tau_h)_p)| = 1$$

Then for any $x, y \in G$, there is $\sigma \in \{-1, 1\}$ such that

$$\omega^x = \sigma\omega^y \text{ in } \Phi_x(U) \cap \Phi_y(U) \tag{5.2.1}$$

Proof. Let us fix any $q \in \Phi_x(U) \cap \Phi_y(U)$. Then there are $X, Y \in U$ such that

$$\pi(xExp(X)) = q = \pi(yExp(Y))$$

We set $x_0 := xExp(X)$ and $y_0 := yExp(Y)$ and $h := y_0^{-1}x_0$. Then by Lemma5.3,

$$\begin{aligned} &(5.2.1) \\ \iff& |det((d\tau_{x_0})_p)| = |det((d\tau_{y_0})_p)| \\ \iff& |det((d\tau_h)_p)| = |det((d\tau_{y_0})_p^{-1})det((d\tau_{x_0})_p)| = 1 \end{aligned}$$

□

Lemma 5.5. *We will succeed notations in 5.2. Then*

$$(d\tau_h)_p = Ad_{\mathfrak{g}/\mathfrak{h}}(h) \ (\forall h \in H)$$

and

$$det((d\tau_h)_p) = \frac{det(Ad_G(h))}{det(Ad_H(h))} \ (\forall h \in H)$$

Proof. Let us fix any $h \in H$. For any $t \in \mathbb{R}$ and $X \in \mathfrak{g}$,

$$\tau_h\pi(exp(tX)) = hExp(tX)H = hExp(tX)h^{-1}H = Exp(tAd(h)X)$$

So,

$$(d\tau_h)_p = Ad_{\mathfrak{g}/\mathfrak{h}}(h)$$

Let A, B, C be the representation matrices corresponding to $Ad_G(h), Ad_{\mathfrak{g}/\mathfrak{h}}$ and $Ad_H(h)$ with respect to $\mathfrak{g}$, respectively. Let us fix any $X \in \mathfrak{g}$. There are $Y \in \mathfrak{q}$ and $Z \in \mathfrak{h}$ such that $X = Y + Z$. $Ad_G(h)X - Ad_{\mathfrak{g}/\mathfrak{h}}(h)X \in \mathfrak{h}$ and $Ad_H(h)Z \in \mathfrak{h}$. So,

$$A = \begin{pmatrix} B & O \\ * & C \end{pmatrix}$$

This implies $det(A) = det(B)det(C)$. □

Lemma 5.6. *We will succeed notations in 5.2. And here are the settings and assumptions.*

(A1) For any $x, y \in G$, there is $\sigma \in \{-1, 1\}$ such that

$$\omega^x = \sigma\omega^y \ \text{in} \ \Phi_x(U) \cap \Phi_y(U)$$

(S1) $g \in G$.

(S2) (U_α, ψ_α) and (U_β, ψ_β) are local coordinates on G/H and $gU_\beta \cap U_\alpha \neq \phi$.

(S5) For any $x \in U_\alpha$ and $y \in U_\beta$

$$\omega_x = \Phi_\alpha(x)d\phi_{\alpha,1} \wedge ... \wedge d\phi_{\alpha,m}, \ \omega_y = \Phi_\beta(y)d\phi_{\beta,1} \wedge ... \wedge d\phi_{\beta,m}$$

Then, for any $x \in U_\beta \cap L_g^{-1}U_\alpha$,

$$\Phi_\beta(x) = |det(J(\psi_\alpha \circ \tau_g \circ \phi_\beta)(\psi_\beta(x)))|\Phi_\alpha(gx)$$

Proof. Let us fix any $x \in U_\beta \cap \tau_g^{-1}U_\alpha$. Then

$$\omega_x = \Phi_\beta(x)(d\phi_{\beta,1} \wedge ... \wedge d\phi_{\beta,m})_x$$

and

$$\omega_{gx} = \Phi_\alpha(gx)(d\phi_{\alpha,1} \wedge ... \wedge d\phi_{\alpha,m})_{gx}$$

So,

$$\omega_x((\frac{\partial}{\partial\psi_{\beta,1}})_x, ..., (\frac{\partial}{\partial\psi_{\beta,m}})_x) = \omega_{gx}(dL_g((\frac{\partial}{\partial\psi_{\beta,1}})_x), ..., dL_g((\frac{\partial}{\partial\psi_{\beta,m}})_x))$$

and

$$\omega_{gx}(dL_g((\frac{\partial}{\partial\psi_{\beta,1}})_x), ..., dL_g((\frac{\partial}{\partial\psi_{\beta,m}})_x)) = |detJ(\psi_\alpha \circ \tau_g \circ \phi_\beta)(\psi_\beta(x))|(d\phi_{\beta,1} \wedge ... \wedge d\phi_{\beta,m})_x$$

These implies that

$$\Phi_\beta(x) = \Phi_\alpha(gx)|detJ(\psi_\alpha \circ \tau_g \circ \phi_\beta)(\psi_\beta(x))|$$

□

Theorem 5.2. *Here are the settings and assumptions.*

(S1) G be a Lie group.

(S2) H be a closed subgroup of G.

(A1) For any $h \in H$,

$$|detAd_G(h)| = |det(Ad_H(h))|$$

Then

(i) There is C^∞-class form $\tilde{\omega}$ on G such that for any $g \in G$ there is $\sigma_g \in C(G/H, \{-1, 1\})$

$$d\tau_g \tilde{\omega} = \sigma_g \tilde{\omega}$$

(ii) G/H is orientable by $\tilde{\omega}$.

(iii) The measure induced from $\tilde{\omega}$ is G invariant. Specially, G/H has a invariant measure.

Proof. (i) is from Lemma5.2. (ii) is from Lemma5.4. We will show (iii). We set $k := dim(G/H)$. Let us fix $f \in C_c^\infty(G/H)$ and $g_0 \in G$. For $x \in G/H$,

$$(\tau_{g_0} f)(x) := f(g_0 x)$$

By (ii) and the second contable axiom, there is $\{U_i, \psi_i, V_i, \Phi_i, \rho_i\}_{i=1}^\infty$ such that $\{U_i, \psi_i\}_{i=1}^\infty$ is a local coordinate system of G/H and $\{U_i, \psi_i\}_{i=1}^\infty$ is local finite and for each i $V_i \in \mathcal{O}(\mathbb{R}^k)$

$$\psi_i : U_i \to V_i$$

is an homeomorphism and $\{U_i, \psi_i\}_{i=1}^\infty$ preserves a orientation of G and for each i and $x \in U_i$

$$\omega_x = \Phi_i(x)(d\psi_{i,1} \wedge ... \wedge d\psi_{i,k})_x$$

and $\Phi_i > 0$ and $\{\rho_i\}_{i=1}^\infty$ is a partition of unity subordinating $\{U_i\}_{i=1}^\infty$. We set for each i, $f_i := f\rho_i$. By Lebesgue's convergence theorem,

$$\int_{G/H} f\omega = \sum_{i=1}^\infty \int_{G/H} f_i\omega, \ \int_{G/H} \tau_{g_0} f\omega = \sum_{i=1}^\infty \int_{G/H} \tau_{g_0} f_i\omega$$

So, it is enough to show for each i

$$\int_{G/H} f_i\omega = \int_{G/H} \tau_{g_0} f_i\omega$$

By Lemma 2.12, we can assume that for each i, there is j such that $supp(\tau_{g_0} f_i) \subset U_j$. Because $supp(f_i)$ is compact, there is an open set U_i' such that

$$supp(f_i) \subset U_i' \subset U_i$$

and

$$supp(\tau_{g_0} f_i) = \tau_{g_0}^{-1} supp(f_i) \subset \tau_{g_0}^{-1} U_i' \subset U_j$$

We set $\phi_i := \psi_i^{-1}$ and $V_i := \psi_i(U_i)$ and $\phi_j := \psi_j^{-1}$ and $V_j := \psi_j(U_j)$. By change-of-variables formula for integral and Lemma5.6,

$$\begin{aligned}
\int_G \tau_{g_0} f_i\omega &= \int_{\psi_j(\tau_{g_0}^{-1} U_i')} f_i(g_0\phi_j(x))\Phi_j(x)dx \\
&= \int_{\psi_j(\tau_{g_0}^{-1} U_i')} f_i(\phi_i(\psi_i(g_0\phi_j(x))))\Phi_j(x)dx \\
&= \int_{\psi_j(\tau_{g_0}^{-1} U_i')} f_i(\phi_i(\psi_i \circ \tau_{g_0} \circ \phi_j(x))) \\
&\quad \times |det(J(\psi_i \circ \tau_{g_0} \circ \phi_j))(\psi_j \circ \tau_{g_0}^{-1} \phi_i \circ \psi_i \circ \tau_{g_0} \circ \phi_j(x))))|^{-1} \\
&\quad \times \Phi_j(\psi_j \circ \tau_{g_0}^{-1} \phi_i \circ \psi_i \circ \tau_{g_0} \circ \phi_j(x)))) \\
&= \int_{V_i'} f_i(\phi_i(y)) det(J(\psi_i \circ \tau_{g_0} \circ \phi_j))(\psi_j \circ \tau_{g_0}^{-1} \circ \phi_i(y))^{-1} \\
&\quad \times \Phi_j(\psi_j \circ \tau_{g_0}^{-1} \phi_i(y))dy \\
&= \int_{V_i'} f_i(\phi_i(y))\Phi_i(y)dy \\
&= \int_G f_i\omega
\end{aligned}$$

□

Proposition 5.2. *Here are the settings and assumptions.*

(S1) G be a Lie group.

(S2) H be a closed subgroup of G such that $dimLie(H) > 0$.

(S3) $\epsilon > 0$.

(S4) $\mathfrak{g} := Lie(G)$, $\mathfrak{h} := Lie(H)$.

(S5) $\mathfrak{q}$ is a complement subspace of $\mathfrak{h}$ in $\mathfrak{g}$.

Then there are $\{g_i\}_{i=1}^{\infty} \subset G$ and $\{U_i\}_{i=1}^{\infty}$ such that U_i is a open neighborhood of 0_k $(\forall i)$ and $U_i \subset B_k(O,\epsilon) \cap \mathfrak{q}$ $(\forall i)$ and $\{\pi(g_i Exp(U_i))\}_{i\in\mathbb{N}}$ is an open covering of G/H and for any $i \in \mathbb{N}$ $\#\{j \in \mathbb{N} | \pi(g_i Exp(U_i)) \cap \pi(g_j Exp(U_j)) \neq \phi\} < \infty$.

Proof. There is V which an open neighborhood of e in G such that $V^4 \subset Exp(B(O,\epsilon))$ and $\bar{V}$ is compact. There are $\{g_{0,i}\}_{i=1}^{N_0}$ and $\{\epsilon_{0,i}\}_{i=1}^{N_0} \subset (0,\infty)$ such that $\pi(\bar{V}^4) \subset \cup_{i=1}^{N_0} \pi(g_{0,i} Exp(B_k(O,\epsilon_{0,i})))$ and $g_{0,i} Exp(B_k(O,\epsilon_{0,i}) \subset Exp(B_k(O,\epsilon) g_{0,i}$ $(\forall i)$.

And for each $s \in \mathbb{N}$ there are $\{g_{s,i}\}_{i=1}^{N_s}$ and $\{\epsilon_{s,i}\}_{i=1}^{N_s} \subset (0,\infty)$ such that $\pi(\bar{V}^{4+s}) \backslash \pi(V^{3+s}) \subset \cup_{i=1}^{N_s} \pi(g_{s,i} Exp(B_k(O,\epsilon_{s,i})))$ and $g_{s,i} Exp(B_k(O,\epsilon_{s,i}) \subset Exp(B_k(O,\epsilon) g_{s,i}$ $(\forall i)$.

We set $\{g_i\}_{i=1}^{\infty} := \{g_{s,i} | s,i \in \mathbb{N}, 1 \leq i \leq N_s\}$ and $\{U_i\}_{i=1}^{\infty} := \{U_{s,i} | s,i \in \mathbb{N}, 1 \leq i \leq N_s\}$. We will show for any $i \in \mathbb{N}$ and $s \in \mathbb{N}$,

$$\pi(g_{s,i}) \notin \pi(V^{s+2})$$

For aiming contradiction, let us assume $s \in \mathbb{N}$ and $i \in \mathbb{N}$ such that $\pi(g_{s,i}) \in \pi(V^{s+2})$. So,

$$\pi(g_{s,i} Exp(B_k(O,\epsilon_{s,i}))) \subset \pi(Exp(B_k(O,\epsilon)) g_{s,i}) \subset \pi(V^{s+3})$$

This contradicts with

$$\pi(g_{s,i} Exp(B_k(O,\epsilon_{s,i}))) \cap \pi(V^{s+3})^c \neq \phi$$

Nextly, we will show for any $i \in \mathbb{N}$ and $s \in \mathbb{N}$,

$$\pi(g_{s,i}) \cap \pi(V^{s+1}) = \phi$$

For aiming contradiction, let us assume $s \in \mathbb{N}$ and $i \in \mathbb{N}$ such that $\pi(g_{s,i} Exp(B_k(O,\epsilon_{0,i})) \cap \pi(V^{s+1}) \neq \phi$. Then there is $X \in B_k(O,\epsilon)$ and $u \in V^{s+2}$ such that $\pi(Exp(X) g_{s,i}) = \pi(u)$. So, $\pi(g_{s,i}) = \pi(Exp(X) u) \in \pi(V^{s+2})$. This is a contradiction. So,

$$(g_{s,i} Exp(B_k(O,\epsilon_{s,i}))) \cap \pi(V^s) = \phi$$

□

By the same argument as the proof of Proposition5.2, the following holds.

Proposition 5.3. *Here are the settings and assumptions.*

(S1) G be a Lie group such that $dimLie(G) > 0$.

(S2) $\epsilon > 0$.

(S3) $\mathfrak{g} := Lie(G)$ and $m := dim\mathfrak{g}$.

Then there are $\{g_i\}_{i=1}^{\infty} \subset G$ and $\{U_i\}_{i=1}^{\infty}$ such that U_i is a open neighborhood of 0_m $(\forall i)$ and $U_i \subset B_m(O,\epsilon) \cap \mathfrak{g}$ $(\forall i)$ and $\{g_i Exp(U_i)\}_{i\in\mathbb{N}}$ is an open covering of G and for any $i \in \mathbb{N}$ $\#\{j \in \mathbb{N} | g_i Exp(U_i) \cap g_j Exp(U_j) \neq \phi\} < \infty$.

Proposition 5.4. *Here are the settings and assumptions.*

(S1) G be a Lie group.

(S2) H be a closed subgroup of G such that $dimLie(H) > 0$.

(S3) $\epsilon > 0$.

(S4) $\mathfrak{g} := Lie(G)$, $\mathfrak{h} := Lie(H)$.

(S5) $\mathfrak{q}$ is a complement subspace of $\mathfrak{h}$ in $\mathfrak{g}$.

Then there are $\{g_i\}_{i=1}^{\infty} \subset G$ and $\{U_i\}_{i=1}^{\infty}$ and $\{h_j\}_{j=1}^{\infty} \subset H$ and $\{V_j\}_{j=1}^{\infty}$ such that U_i is a open neighborhood of 0_k $(\forall i)$ and $U_i \subset B_k(O,\epsilon) \cap \mathfrak{q}$ $(\forall i)$ and V_j is a open neighborhood of 0_l $(\forall j)$ and $V_j \subset B_l(O,\epsilon) \cap \mathfrak{h}$ $(\forall j)$ and V_j is a open neighborhood of 0_l $(\forall j)$ and $g_i Exp(U_i) h_j Exp(V_j) \in \mathcal{O}(G)$ $(\forall i,j)$ and for any $i,j \in \mathbb{N}$

$$U_i \times V_j \ni (X,Y) \mapsto g_i Exp(X) h_j Exp(Y) \in g_i Exp(U_i) h_j Exp(V_j)$$

is a C^{ω}-class diffeomorphism and $\{g_i Exp(U_i) h_j Exp(V_j)\}_{i,j\in\mathbb{N}}$ is a local finite open covering of G and $\{\pi(g_i Exp(U_i))\}_{i\in\mathbb{N}}$ is a local finite open covering of G/H and $\{h_j Exp(V_j))\}_{j\in\mathbb{N}}$ is a local finite open covering of H.

Proof. Let $\{g_i\}_{i=1}^{\infty}$ and $\{U_i\}_{i=1}^{\infty}$ be the one in Proposition5.2. Let $\{h_j\}_{j=1}^{\infty}$ and $\{V_j\}_{j=1}^{\infty}$ be the one in Proposition5.3. By Theomrem5.1, we can assume for each $i, j \in \mathbb{N}$

$$U_i \times V_j \ni (X, Y) \mapsto g_i Exp(X) h_j Exp(Y) \in G$$

is a C^ω-class diffeomorphism to an open neighborhood of $g_i h_j$. So, it is enough to show $\{g_i U_i h_j V_j\}_{i,j \in \mathbb{N}}$ is local finite. Let us fix any $i, j \in \mathbb{N}$. For each $i', j' \in \mathbb{N}$,

$$g_i U_i h_j V_j \cap g_{i'} U_{i'} h_{j'} V_{j'} \neq \phi \implies \pi(g_i U_i) \cap \pi(g_{i'} U_{i'}) \neq \phi$$

So,

$$\#\{i' \in \mathbb{N} | \exists j' \text{ s.t } g_i U_i h_j V_j \cap g_{i'} U_{i'} h_{j'} V_{j'} \neq \phi\} < \infty$$

We denote this set by I. Let us fix any $i_0 \in I$. Because $(g_{i_0} \bar{U_{i_0}})^{-1} g_i \bar{U_i} h_j \bar{V_j} \cap H$ is compact, there are $j_1, ..., j_M$ such that

$$(g_{i_0} \bar{U}_{i_0})^{-1} g_i \bar{U}_i h_j \bar{V}_j \cap H \subset \cup_{a=1}^{M} h_{j_a} V_{j_a}$$

This implies

$$\{j' | g_i U_i h_j V_j \cap g_{i_0} U_{i_0} h_j V_{j'} \neq \phi\} \subset \cup_{a=1}^{M} \{j' | h_{j_a} V_{j_a} \cap h_{j'} V_{j'} \neq \phi\}$$

So,

$$\#\{j' | g_i U_i h_j V_j \cap g_{i_0} U_{i_0} h_j V_{j'} \neq \phi\} < \infty$$

□

Theorem 5.3. *Here are the settings and assumptions.*

(S1) G be a Lie group.

(S2) H be a closed subgroup of G such that $dimLie(H) > 0$.

(A1) For any $h \in H$,

$$|detAd_G(h)| = |det(Ad_H(h))|$$

(S3) μ_H is a left invariant meausre induced by a left invariant form on H.

(S4) $\mu_{G/H}$ is a invariant measure induced by Theorem5.2.

(S5) μ_G is a left invariant meausre induced by a left invariant form ω_0 on G.

Then there is $c \in \mathbb{R}$ such that for any $f \in C_c(G)$

$$\int_G f(g) d\mu_G(g) = c \int_{G/H} \bar{f}(x) d\mu_{G/H}(x)$$

Here

$$\bar{f}(gH) = \int_H f(gh) d\mu_H(h) \ (gH \in G/H)$$

$\bar{f}$ is well-defined and $\bar{f}$ is continuous.

STEP1. $\bar{f}$ is well-defined and $\bar{f}$ is continuous. If $g_1 H = g_2 H$, because $g_2^{-1} g_1 \in H$,

$$\int_H f(g_1 h) d\mu_H(h) = \int_H f(g_2 g_2^{-1} g_1 h) d\mu_H(h) = \int_H f(g_2 h) d\mu_H(h)$$

So, $\bar{f}$ is well-defined. Because f is uniformly continuous and $gExp(U)H$ is an open neighborhood of gH for any open neighborhood of e U, $\bar{f}$ is continuous. □

STEP2. Construction of a left invariant measure μ from invariant measures on G/H and H. We set

$$I : C_c^+(G) \ni f \mapsto \int_G \bar{f}(x) d\mu_{G/H}(x) \in \mathbb{R}_+$$

By Riez-Markov-Kakutani Theorem, I induces the baire measure μ on G. □

STEP3. Construction of a local coordinates system. We set $\mathfrak{g} := Lie(G)$ and $\mathfrak{h} := Lie(H)$. We fix $\mathfrak{q}$ which is the complement of $\mathfrak{h}$. $k := dim\mathfrak{q}$ and $m := \mathfrak{g}$ and $l := \mathfrak{q}$. There is $\delta_1 > 0$ such that

$$B_k(O,\delta_1) \cap \mathfrak{q} \times B_l(O,\delta_1) \cap \mathfrak{h} \ni (Y,Z) \mapsto exp(Y)exp(Z) \in G$$

is a C^ω-class diffeomorphism to an open neighborhood of e. For each $g \in G$ and $h \in H$,

$$g(Exp(B_k(O,\delta_1) \cap \mathfrak{q})h(B_l(O,\delta_1) \cap \mathfrak{h}) = gh(Exp(Ad_G(h^{-1})B_k(O,\delta_1) \cap \mathfrak{q})(B_l(O,\delta_1) \cap \mathfrak{h})$$

So, there is $\delta_2 > 0$ such that

$$B_k(O,\delta_2) \cap \mathfrak{q} \times B_l(O,\delta_2) \cap \mathfrak{h} \ni (Y,Z) \mapsto gexp(Y)hexp(Z) \in G$$

is a C^ω-class diffeomorphism to an open neighborhood of gh. There are $\{g_i\}_{i=1}^\infty \subset G \setminus H \cup \{e\}$ and $\{h_i\}_{i=1}^\infty \subset H$ and $\{U_i\}_{i=1}^\infty$ and $\{V_i\}_{i=1}^\infty$ such that s U_i is an open neighborhood of 0_k $(\forall i)$ and V_i is an open neighborhood of 0_k $(\forall i)$ and $\{\pi(g_iU_i)\}_{i=1}^\infty$ is a local finite covering of G/H and $\{h_iV_i\}_{i=1}^\infty$ is a local finite covering of H and $\{g_iU_ih_jV_j\}_{i,j=1}^\infty$ is a local finite covering of G. We denote a partition of unity correspoinding to $\{\pi(g_iU_i)\}_{i=1}^\infty$ by $\{\alpha_i\}_{i=1}^\infty$ and denote a partition of unity correspoinding to $\{h_jV_j)\}_{j=1}^\infty$ by $\{\beta_j\}_{j=1}^\infty$. Then clearly $\{\alpha_i\beta_j\}_{i,j=1}^\infty$ is a partition of unity correspoinding to $\{g_iU_ih_jV_j\}_{i,j=1}^\infty$. □

STEP4. Construction of a C^∞-form ω. We set for each $i,j \in \mathbb{N}$,

$$\omega_{g_iExp(X)h_jExp(Y)} := \Phi_{1,i}(g_iExp(X))\Phi_{2,j}(h_jExp(Y))d\phi_{1,i}^1 \wedge d\phi_{1,i}^2 \wedge ... \wedge d\phi_{1,i}^k \wedge d\phi_{2,j}^1 \wedge d\phi_{2,j}^2 \wedge ... \wedge d\phi_{1,j}^l \ (X \in U_i, Y \in V_j, i,j \in \mathbb{N})$$

We will show ω is well-defined. Let us fix any $i_1, j_1, i_2, j_2 \in \mathbb{N}$, $X_1 \in U_{i_1}$, $Y_1 \in V_{j_1}$, $X_2 \in U_{i_2}$, $Y_2 \in V_{j_2}$ such $g_{i_1}Exp(X_{i_1})h_{j_1}Exp(Y_{j_1}) = g_{i_2}Exp(X_{i_2})h_{j_2}Exp(Y_{j_2})$. We set

$$g_1 := g_{i_1}Exp(X_{i_1}), g_2 := g_{i_2}Exp(X_{i_2}), h_1 := h_{j_1}Exp(Y_{j_1}), h_2 := h_{j_2}Exp(Y_{j_2})$$

Because $h_0 := g_2^{-1}g_1 \in H$, $\pi(g_1) = \pi(g_2)$. So, by Lemma5.2,

$$\Phi_{1,i_1}(g_1)d\phi_{1,i_1}^1 \wedge d\phi_{1,i_1}^2 \wedge ... \wedge d\phi_{1,i_1}^k = \Phi_{1,i}(g_2)d\phi_{1,i_2}^1 \wedge d\phi_{1,i_2}^2 \wedge ... \wedge d\phi_{1,i_2}^k$$

So, $h_0h_1 = h_2$. Because μ_H is left invariant, by Lemma2.9,

$$\begin{aligned}&\Phi_{2,j_2}(h_2)d\phi_{2,j_2}^1 \wedge d\phi_{2,j_2}^2 \wedge ... \wedge d\phi_{2,j_2}^l = \Phi_{2,j_2}(h_0h_1)d\phi_{2,j_2}^1 \wedge d\phi_{2,j_2}^2 \wedge ... \wedge d\phi_{2,j_2}^l \\ &= det(J(\phi_1 \circ \L_{h_0^{-1}} \circ \psi_2)(\phi_2(h_1)))\Phi_{1,j_1}(h_1)d\phi_{2,j_2}^1 \wedge d\phi_{2,j_2}^2 \wedge ... \wedge d\phi_{2,j_2}^l \\ &= \Phi_{1,j_1}(h_1)d\phi_{1,j_1}^1 \wedge d\phi_{1,j_1}^2 \wedge ... \wedge d\phi_{1,j_1}^l\end{aligned}$$

So, ω is well-defined. □

STEP5. The measure induced by ω is equal to μ. Let us fix any $f \in C_c(G)$.

$$\begin{aligned}&\int_G f\omega = \sum_{i,j=1}^\infty \int_{g_iU_ih_iV_i} f\alpha_1\alpha_2\omega \\ &= \sum_{i,j=1}^\infty \int_{\psi_{1,i}(U_i)\times\psi_{2,j}(V_j)} f(g_iExp(X)h_jExp(Y))\alpha_1(g_iExp(X))\alpha_2(h_jExp(Y))\Phi_{1,i}(g_iExp(X))\Phi_{2,i}(h_jExp(Y))dXdY \\ &= \sum_{i=1}^\infty \int_{\psi_{1,i}(U_i)} \Phi_{1,i}(g_iExp(X))\alpha_1(g_iExp(X)) \sum_{j=1}^\infty \int_{\psi_{2,j}(V_j)} f(g_iExp(X)h_jExp(Y))\alpha_2(h_jExp(Y))\Phi_{2,i}(h_jExp(Y))dYdX \\ &= \sum_{i=1}^\infty \alpha_1(g_iExp(X)) \int_{\psi_{1,i}(U_i)} \Phi_{1,i}(g_iExp(X)) \int_H f(g_iExp(X)h)d\mu_H(h)dX \\ &= \sum_{i=1}^\infty \int_{\psi_{1,i}(U_i)} \alpha_1(g_iExp(X))\Phi_{1,i}(g_iExp(X))\bar{f}(g_iExp(X))dX = \int_{G/H} \bar{f}(x)d\mu_{G/H}(x) = I(f)\end{aligned}$$

So, ω introduces μ. By Proposition2.52, ω is left invariant form. Consequently, there is $c \in \mathbb{R}$ such that $\omega = c\omega_0$. This implies $\mu = c\mu_G$. □

In speciality, the following holds.

Proposition 5.5. *Here are the settings and assumptions.*

(S1) G be a compact Lie group.

(S2) H be a closed subgroup of G.

Then G/H has a invariant measure induced by a C^∞ form.

5.2.2 $L^p(G/H)$

By the same argument as the proof of Proposition2.56, the following holds.

Proposition 5.6. *Here are the settings and assumptions.*

- *(S1) G be a Lie group.*
- *(S2) H be a closed subgroup of G.*
- *(A1) For any $h \in H$,*
$$|detAd_G(h)| = |det(Ad_H(h))|$$

Then $L^p(G/H)$ is separable for any $p \in \mathbb{N} \cap [1, \infty)$.

By the proof of Proposition5.6, the following holds.

Proposition 5.7. *Here are the settings and assumptions.*

- *(S1) G be a Lie group.*
- *(S2) H be a closed subgroup of G.*
- *(A1) For any $h \in H$,*
$$|detAd_G(h)| = |det(Ad_H(h))|$$

Then there is at most countable subset of $C_c(G/H)$ which is dense in $L^p(G/H)$ for any $p \in \mathbb{N} \cap [1, \infty)$.

5.3 Homogeneous Vector Bundle

Definition 5.4 (local cross-section)**.** *Let G be a Lie group and H be a closed subgroup of G and $\pi : G \to G/H$ be the projection and U be an open neighborhood of $\pi(e)$. We say $s : U \to G$ is a local cross-section if s is C^∞-class and $p \circ s = id|U$.*

Theorem 5.4. *Let G be a Lie group and H be a closed subgroup of G and $\pi : G \to G/H$ be the projection. Then the followings hold.*

- *(i) There is an open neighborhood of $\pi(e)$ U such that $\mathcal{B} := \{G, G/H, H, \{gU\}_{g\in G}, H\}$ is a principal bundle.*
- *(ii) $\mathcal{B}$ has a local cross-section.*

Proof of (i). We set $\mathfrak{h} := Lie(H)$ and denote a complement of $\mathfrak{h}$ by $\mathfrak{q}$.

By the proof of Theorem5.1, there exists $r > \epsilon > 0$ such that

$$\psi : B(O, r) \cap \mathfrak{q} \times B(O, r) \cap \mathfrak{h} \ni (X, Y) \mapsto exp(X)exp(Y) \in G$$

is a C^ω-class diffeomorphism to an open neighborhood of $p := \pi(e)$ and $exp(B(O, \epsilon))exp(B(O, \epsilon)) \subset exp(B(O, r)$.

We set $U := \pi(exp(B(O, \epsilon) \cap \mathfrak{q}))$.

We set
$$\phi_p : U \times H \ni (\pi(exp(X)), h) \mapsto exp(X)h \in G$$

Since ψ is a diffeomorphism, X is identified uniquely. So, ϕ_p is well-defined and C^ω-class. And clearly $\pi \circ \phi_p = id|U$ and $Im\phi_p \subset \pi^{-1}(U)$. Let us fix any $g \in \pi^{-1}(U)$. Then $\exists X \in B(O, \epsilon) \cap \mathfrak{q}$ and $h \in H$ such that $g = exp(X)h = \phi_p(\pi(exp(X)), h)$. So, ϕ_p is surjective. Let uf fix any $X_1, X_2 \in \mathfrak{q} \cap B(O, \epsilon)$ and $h_1, h_2 \in H$ such that $exp(X_1)h_1 = exp(X_2)h_2$. Then $exp(X_1) = exp(X_2)h_2h_1^{-1}$ and $h_2h_1^{-1} = exp(-X_2)exp(X_1) \in exp(B(O, r)$. Since ψ is injective, $h_2h_1^{-1} = e$. That implies $X_1 = X_2$. For each $h \in H$, by von-Neuman Cartan Theorem, $\phi_p|U \times exp(\mathfrak{h} \cap B(O, \epsilon))h$ is a C^ω-class diffeomorphism to an open neighborhood of h. So, ϕ_p itself is C^ω-class diffeomorphism to $\pi^{-1}(U)$.

For each $g \in G$, we set
$$\phi_{\pi(g)} : gU \times H \ni (\pi(gexp(X)), h) \mapsto gexp(X)h \in G$$

As same as the above argument, $\phi_{\pi(g)}$ is a C^ω-class diffeomorphism from $gU \times H$ to an open subset $\pi^{-1}(gU)$.

Nextly, let us fix any $x \in g_1U \cap g_2U$. Then $w_1 \in g_1exp(B(O, \epsilon) \cap \mathfrak{q})$ and $w_2 \in g_2exp(B(O, \epsilon) \cap \mathfrak{q})$ such that $\pi(w_1) = \pi(w_2)$. Then $h_0 := w_2^{-1}w_1 \in H$. So,
$$\phi_{\pi(g_1)}(w_1, h) = w_1h = w_2h_0h = \phi_{\pi(g_2)}(w_2, h_0h)$$

This means that
$$\phi^{-1}_{\pi(g_2),x} \circ \phi_{\pi(g_1),x}(h) = (w_2^{-1}w_1)h$$

and $\phi_{\pi(g_2),x}^{-1} \circ \phi_{\pi(g_1),x} = L_{(w_2^{-1}w_1)}$. Since

$$\pi(g_1 Exp(B(O,\epsilon) \cap \mathfrak{q})) \ni \pi(w_1) \mapsto w_1 \in g_1 Exp(B(O,\epsilon) \cap \mathfrak{q})$$

and

$$\pi(g_2 Exp(B(O,\epsilon) \cap \mathfrak{q})) \ni \pi(w_2) \mapsto w_2 \in g_2 Exp(B(O,\epsilon) \cap \mathfrak{q})$$

are C^ω class,

$$g_1 U \cap g_2 U \ni x \mapsto \phi_{\pi(g_2),x}^{-1} \circ \phi_{\pi(g_1),x} = L_{w_2^{-1}w_1} \in H$$

is C^ω class. Consequently, $\pi : G \to G/H$ is a C^ω class principal bundle whose structure group is H. □

Proof of (ii). We succeed notations in the proof of (i). We set

$$s : \pi(exp(B(O,\epsilon) \cap \mathfrak{q})) \ni \pi(exp(X)) \mapsto exp(X) \in G$$

Then s is clearly a local cross-section. □

Theorem 5.5 (Homogeneous vector bundle)**.** *The followings are settings and assumptions.*

(i) G *is a Lie group.*

(ii) H *is a closed subgroup of* G*.*

(iii) (π, V) *is a continuous representation of* H*.*

(iv) $(g_1, v_1) \sim (g_2, v_2) :\iff \exists h \in H$ *s.t* $g_1 = g_2 h$ *and* $v_1 = \pi(h)^{-1} v_2$*.*

(v) $p : G \times V \ni (g, v) \mapsto [g, v] \in G/\sim$*. Let us define* $\mathcal{O}(G/\sim)$ *by* p*. We set* $G \times_H V := G/\sim$*.*

(vi) $q : G \times_H V \ni [g, v] \mapsto gH \in G/H$*.*

Then

(i) $\sim$ *is an equivalent relation on* $G \times V$*.*

(ii) q *is a vector bundle whose fibre is* V *and whose structure group is* H*.*

(iii) G *acts on* $G \times_H V$ *by* $g \cdot [x, v] := [gx, v]$ $g, x \in G, v \in V$*.*

(iv) For each $g \in G, v \in V$*,* $\{p(gU \times (v + B))\}_{U:nei.\ of\ e,\ B:nei.\ of\ 0}$ *is a basis of neiborhoods of* $[g, v]$*.*

Proof of (i). It is clear from the def. of $\sim$. □

Proof of (ii):q is well-defined and continuous. We set $\mathfrak{h} := Lie(H)$. Let $\mathfrak{q}$ denote a complement of $\mathfrak{h}$. Firstly, from the def. of $\sim$, q is well-defined. By the proof of Theorem5.1, there is $\epsilon > 0$ such that for each $g \in G$ $\phi_g : \mathfrak{q} \cap B(O,\epsilon) \ni X \mapsto gexp(X)H \in G/H$ is a homeomorphism from $\mathfrak{q}_\epsilon := \mathfrak{q} \cap B(O,\epsilon)$ to an open neighborhood of gH.

For each $g \in G$, $q^{-1}(\phi_g(\mathfrak{q}_\epsilon)) = p(B(O,\epsilon) \times V)$. Because $p^{-1}(p(B(O,\epsilon) \times V)) = B(O,\epsilon)H \times V$ and $B(O,\epsilon)H \times V$ is an open set, $q^{-1}(\phi_g(\mathfrak{q}_\epsilon))$ is an open set. So, q is a continuous. □

Proof of (ii):Local trivializations. For each $g \in G$, we set $\psi_g : gexp(\mathfrak{q}_\epsilon)H \times V \ni (gexp(X)H, v) \mapsto [gexp(X), v] \in G \times_H V$. Clearly, ψ_g is well-defined and continuous and $Im\psi_g \subset q^{-1}(\phi_g(\mathfrak{q}_\epsilon))$ and $q \circ \psi_g(gexp(X)H, v) = gexp(X)H$ $(\forall X \in \mathfrak{q}_\epsilon)$. Let us fix any $[x, v] \in q^{-1}(\phi_g(\mathfrak{q}_\epsilon))$. Then $\exists h \in H$ and $X \in \mathfrak{q}_\epsilon$ such that $xh = gexp(X)$. So, $[x, v] = [gexp(X), \pi(h^{-1})v] = \psi_g(gexp(X), \pi(h^{-1})v)$. Consequently, ψ_g is a local trivialization. □

Proof of (ii):A system of coordinate transformation. Let us fix any

$$\psi_{g_1}(g_1 exp(X_1)H, v_1) = \psi_{g_2}(g_2 exp(\iota(X_1))H, v_2) \in q^{-1}(\phi_{g_1}(\mathfrak{q}_\epsilon)) \cap q^{-1}(\phi_{g_2}(\mathfrak{q}_\epsilon))$$

Then $v_2 = \pi((g_2 exp(X_2))^{-1} g_1 exp(X_1)) v_1$. So, $\{\psi_g\}_{g \in G}$ defines a system of coordinate transformation with the Lie group H. □

Proof of (iii). It is clear from the def. of action. □

Proof of (iv). It is clear from the def. of topology of $G \times_H V$. □

Theorem 5.6. *The followings are settings and assumptions.*

(i) G *is a Lie group.*

(ii) H *is a closed subgroup of* G*.*

(iii) (π, V) *is a continuous representation of* H*.*

(iv) $\Gamma(G/H, G \times_H V)$ *is the set of all cross sections of* q.

(v) $\iota : H \ni h \mapsto (1, h, h) \in G \times H \times H$.

(vi) $\iota : H \ni h \mapsto (1, h, h) \in G \times H \times H$.

(vii) $(g, h_1, h_2) \cdot f(x) := \pi(h_2) f(g^{-1} x h_1)$ $(g, h_1, h_2) \in G \times H \times H, x \in G, f \in C(G, V)$.

(vii) $C(G, V)^{\iota(H)} := \{f \in C(G, V) | \iota(h) f = f \ (\forall h \in H)\}$. *In this note, we sometimes may denote* $C(G, V)^{\iota(H)}$ *by* $C(G, V)^H$.

Then

(i) $G \times H \times H$ *acts on* $C(G, V)$ *based on the def. of (vi).*

(ii) $C(G, V)^{\iota H} \simeq \Gamma(G/H, V)$ *as purely algebraic representation of* G. *Remark that here we don't care about any topology of them and* G *acts on* $\Gamma(G/H, V)$ *by* $g \cdot s(xH) := gs(g^{-1} xH)$ *for* $g, x \in G, s \in \Gamma(G/H, V)$.

Proof of (i). It is clear from the def. of action. □

Proof of (ii). Let us fix any $\phi \in C(G, V)^{\iota H}$. And let us $\Phi(\phi)(\bar{g}) := [g, \phi(g)]$. We will show $\Phi(\phi)$ is well-defined. Let us fix any $g_1, g_2 \in G$ such that $g_1 \sim g_2$. Then there is $h \in H$ such that $g_1 = g_2 h$. So,

$$\Phi(\phi)(g_1 H) = [g_1, \phi(g_1)] = [g_2 h, \phi(g_2 h)] = [g_2 h, \pi(h)^{-1} \phi(g_2)] = [g_2, \phi(g_2)] = \Phi(\phi)(g_2 H)$$

We set $\mathfrak{h} := Lie(H)$. Let $\mathfrak{q}$ denote a complement of $\mathfrak{h}$. Because $\Phi(\phi)(gexp(X)H) = [gexp(X), \phi(gexp(X))]$ $g \in G, X \in \mathfrak{q}$ such that $||X|| \ll 1$, $\Phi(\phi) \in C(G/H, G \times_H V)$. Clearly $q \circ \Phi(\phi) = id_{G/H}$, therefore $\Phi(\phi) \in \Gamma(G/H, G \times_H V)$.

Let us fix any $s \in \Gamma(G/H, G \times_H V)$. Let us fix any $g \in G$. Then there $\exists! v \in V$ such that $s(gH) = [g, v]$. We set $\Psi(s)(g) := v$. $\Psi(s)(g) := v$. Let us fix any $\epsilon > 0$. By (iv) of Theorem5.5, there is $\delta > 0$ such that for any $X \in \mathfrak{q}_\delta := \mathfrak{q} \cap B(O, \delta)$, $s(gexp(\mathfrak{q}_\delta)) \subset p(gexp(\mathfrak{q}_\epsilon \times (v + B(O, \epsilon)))$. So, there is $Y \in \mathfrak{q}_\epsilon$ and $u \in v + B(O, \epsilon)$ such that

$$s(gexp(X)) = [gexp(Y), u]$$

Because $s(gexp(X)) = [gexp(X), \Psi(s)(gexp(X))]$, there is $h \in H$ such that $gexp(X)h = gexp(Y)$ and $\pi(h)^{-1} u = \Psi(s)(gexp(X))$. Because of the proof of Theorem5.1, if we take δ to be sufficient small, then $h = e$. So, $\Psi(s)(gexp(X)) \in (v + B(O, \epsilon))$. Therefore, $\Psi(s)$ is continuous. And clearly $\Psi(s) \in C(G, V)^{\iota(H)}$.

Clearly, $\Phi \circ \Psi = id_{\Gamma(G/H, V)}$ and $\Psi \circ \Phi = id_{C(G, V)^{\iota(H)}}$. And

$$\begin{aligned} \Phi(g \cdot \phi)(x) &= [x, g \cdot \phi(x)] = [x, \phi(g^{-1} x)] = [gg^{-1} x, \phi(g^{-1} x)] = g \cdot [g^{-1} x, \phi(g^{-1} x)] \\ &= g\Phi(\phi)(g^{-1} x) = (g \cdot \Phi(\phi))(x) \ (\forall g, x \in G, \forall \phi \in C(G, V)^{\iota(H)}) \end{aligned}$$

□

5.4 Induced representation

Theorem 5.7 (Induced Representation)**.** *The followings are settings and assumptions.*

(i) G *is a compact Lie group.*

(ii) H *is a closed subgroup of* G.

(iii) (π, V) *is a continuous unitary representation of* H.

(iv) *For each* $f_1, f_2 \in C(G, W)$, $(f_1, f_2) := \int_G (f_1(g), f_2(g))_W d\mu(g)$. *Here,* μ *is the normalized Haar measre on* G.

Then

(i) $C(G/H, W)^{\iota(H)}$ *is a pre-Hilbert space and is an unitary representation space of* G *with the inner product. We call the completion of it the induced representation from* π *and denote the completion by* $L^2(G, \mathcal{W})$ *and denote the representation by* L^2*-Ind*$(H \uparrow G)(\pi)$ *or* L^2*-Ind*$_H^G$.

(ii) *For any* $f_1, f_2 \in C(G/H, W)^{\iota(H)}$,

$$(f_1, f_2) = \int_{G/H} (f_1(g), f_2(g)) d\mu(gH)$$

Proof of (i). It is clear. □

Proof of (ii). It is clear from Theorem5.3. □

Induced Representation can be defined with homogeneous bundle as below.

Theorem 5.8. *The followings are settings and assumptions.*

(i) G *is a compact Lie group.*

(ii) H *is a closed subgroup of* G*.*

(iii) (π, V) *is a continuous unitary representation of* H*.*

(iv) For each $g \in G$ *and* $[g', v_1], [g', v_2] \in q^{-1}(gH)$*, we set* $([g', v_1], [g', v_2]) := (v_1, v_2)$*.*

(v) For $s_1, s_2 \in \Gamma(G/H, G \times_H V)$*,* $(s_1, s_2) := \int_{G/H} (s_1(gH), s_2(gH)) d\mu(gH)$*. Here* μ *is the normalized invariant measure on* G/H*.*

Then

(i) The inner product defined in (iv) is well-defined.

(ii) $\Gamma(G/H, G \times_H V)$ *is a pre-Hilbert space and is an unitary representation space of* G *with the inner product defined in (v).*

(iii) The completion is isomorphic to $L^2(G, \mathcal{W})$ *as continuous unitary representations.*

Proof of (i). For each $[g', v_1] = [g'', v_3], [g', v_2] = [g'', v_4] \in q^{-1}(gH)$,

$$([g', v_1], [g', v_2]) = (v_1, v_2) = (\pi(g'^{-1}g'')^{-1}v_3, \pi(g'^{-1}g'')^{-1}v_4) = (v_3, v_4) = ([g'', v_3], [g'', v_4])$$

Therefore, the inner product is well-defined. □

Proof of (ii). Clearly $\Gamma(G/H, G \times_H V)$ is a $\mathbb{C}$-linear space and G acts on $\Gamma(G/H, G \times_H V)$. Since G is compact, the inner product converges in any case. Since μ is G-invariant, G acts $\Gamma(G/H, G \times_H V)$ as unitary operator. Let denote the isomophism from $C(G, V)^{\iota(H)}$ to $\Gamma(G/H, G \times_H V)$ by Φ. Clearly, for each $s \in \Gamma(G/H, G \times_H V)$,

$$||s|| = 0 \iff \Phi^{-1}(s) = 0$$

Consequently, (ii) holds. □

Proof of (iii). It is clear from (i). □

Clearly the following holds.

Example 5.1. *The followings are settings and assumptions.*

(i) G *is a compact Lie group.*

(ii) H *is a closed subgroup of* G*.*

Then L^2*-Ind*$(H \uparrow G)(1) \simeq L^2(G/H)$*. Here,* 1 *is the trivial representation of* H*.*

5.4.1 Frobenius Reciprocity

Proposition 5.8. *The followings are settings and assumptions.*

(i) G *is a compact Lie group.*

(ii) H *is a closed subgroup of* G*.*

(iii) (π, V) *is a finite dimensional continuous representation of* H*.*

Then

(i) $C(G, V) \simeq C(G) \otimes V$ *as representation of* $G \times G \times H$*.*

(ii) $C(G, V)^{\iota(H)} \simeq (C(G) \otimes V)^{\iota(H)}$ *as representation of* G*.*

(iii) If π *is an unitary representation,* $L^2(G/H, \mathcal{W})^{\iota(H)} \simeq (L^2(G) \otimes V)^{\iota(H)}$ *as representation of* G*.*

Proof of (i). Let $\{v_i\}_{i=1}^m$ denote a basis of V. Let us fix $f \in C(G,V)$. Then for each $g \in G$, there are $\exists! f_1(g), ..., f_m(g)$ such that $f(g) = \sum_{i=1}^m f_i(g)v_i$. We set $\Phi(f) := \sum_{i=1}^m f_i \otimes v_i$.

Let us fix $\phi \in C(G) \otimes V$. By Proposition1.2, there exists $\{f_i\}_{i=1}^m \subset C(G)$ such that $\phi = \sum_{i=1}^m \phi_i \otimes v_i$. We set $\Psi(\phi) := (f_1, ..., f_m)$.

Clearly Φ, Ψ are $\mathbb{C}$-linear and $\Phi \circ \Psi = id_{C(G)\otimes V}$ and $\Psi \circ \Phi = id_{C(G,V)}$.

$$\pi(h)L_{g_1}R_{g_2}f(g) = \sum_{i=1}^m L_{g_1}R_{g_2}f_i(g)\pi(h)v_i = \sum_{i=1}^m \sum_{j=1}^m L_{g_1}R_{g_2}f_i(g)(\pi(h)v_i, v_j)v_j = \sum_{j=1}^m \sum_{i=1}^m L_{g_1}R_{g_2}f_i(g)(\pi(h)v_i, v_j)v_j$$

So,

$$\Phi((g_1,g_2,h)\cdot f) = \Phi(\pi(h)L_{g_1}R_{g_2}f) = \sum_{j=1}^m \sum_{i=1}^m L_{g_1}R_{g_2}f_i(\pi(h)v_i, v_j) \otimes v_j = \sum_{i=1}^m L_{g_1}R_{g_2}f_i \otimes \sum_{j=1}^m (\pi(h)v_i, v_j)v_j$$
$$= \sum_{i=1}^m L_{g_1}R_{g_2}f_i \otimes \pi(h)v_i = (g_1,g_2,h)\cdot \sum_{i=1}^m f_i \otimes v_i = (g_1,g_2,h)\Phi(f)$$

Consequently, Φ is G-invariant. □

Proof of (ii). (ii) is clearly from (i). □

Proof of (iii). (iii) is clearly from (i). □

Proposition 5.9. *The followings are settings and assumptions.*

(i) G is a compact Lie group.

(ii) H is a closed subgroup of G.

(iii) (π, W) is a finite dimensional continuous representation of H.

(iv) (τ, V_τ) is an irreducible continuous representation of G.

Then, for each $\pi \in \hat{G}$,

$$Hom_G(V_\pi, V_\tau \otimes Hom_H(V_\pi|H, W)) \simeq \begin{cases} 0 & \tau \not\cong \pi \\ Hom_H(V_\pi|H, W) & \tau = \pi \end{cases}$$

as vector spaces.

STEP1: When $\tau \not\cong \pi$: By Peter-Weyl theorem, $Hom_H(V_\pi|H,W)$ is finite dimensional. Let us fix a basis of $Hom_H(V_\pi|H,W)$ $\{\psi_i\}_{i=1}^m$. Let us fix any $\phi \in Hom_G(V_\pi, V_\tau \otimes Hom_H(V_\pi|H,W))$. We define $\phi_1, ..., \phi_m$ by

$$\phi(v) = \sum_{i=1}^m \phi_i(g) \otimes \psi_i \ (v \in V_\pi).$$

Clearly, $\phi_1, ..., \phi_m \in Hom_G(V_\pi, V_\tau)$. By Shur Lemma, $\phi_1 = ... = \phi_m = 0$. □

STEP2: When $\tau = \pi$: I continue to use the notations from STEP1. In the case, by Shur Lemma, there exist $c_1, ..., c_m \in \mathbb{C}$ such that $\phi_i = c_i id_{V_\tau}$ $(\forall i)$. Therefore,

$$\phi = id_{V_\tau} \otimes \sum_{i=1}^m c_i\psi_i.$$

This means

$$Hom_G(V_\pi, V_\tau \otimes Hom_H(V_\pi|H,W)) \simeq Hom_H(V_\pi|H,W)$$

□

Proposition 5.10. *The followings are settings and assumptions.*

(i) H is a topological group.

(iii) (π, W) is a finite dimensional continuous representation of H.

(iv) (τ, V) is a continuous representation of H.

(v) $\eta(H) := \{(h,h)|h \in H\}$.

Then,

$$(V^* \times W)^{\eta(H)} \simeq Hom_H(V, W)$$

Proof. That can be proved from the same thought as the proof of Proposition5.8. □

Theorem 5.9 (Frobenius Reciprocity Theorem.)**.** *The followings are settings and assumptions.*

(i) *G is a compact Lie group.*

(ii) *H is a closed subgroup of G.*

(iii) *(π, W) is an irreducible continuous representation of H.*

(iv) *(τ, V_τ) is an irreducible continuous representation of G.*

Then,

(i)

$$Hom_H(\pi|H, \tau) \simeq Hom_G(\pi, Ind_H^G \tau)$$

(ii)

$$[V_\pi|H : W] = [Ind_H^G \tau : W]$$

(iii)

$$Ind_H^G \tau = \oplus_{\pi \in \hat{G}} [\pi|H : \tau]\pi$$

Proof of (i). By Peter-Weyl Theorem,

$$L^2(G) \simeq \oplus_{\sigma \in \hat{G}} V_\sigma \otimes V_\sigma^*$$

Then

$$L^2(G/H, \mathcal{W})$$

$$\text{by Proposition5.8}$$

$$= L^2(G) \otimes W \simeq (\oplus_{\sigma \in \hat{G}} V_\sigma \otimes V_\sigma^* \otimes W)^{\iota(H)} \simeq \oplus_{\sigma \in \hat{G}} V_\sigma \otimes (V_\sigma^* \otimes W)^{\eta(H)} \simeq \oplus_{\sigma \in \hat{G}} V_\sigma \otimes Hom_H(V_\sigma, W)$$

So, by Proposition5.9,

$$Hom_G(\pi, L^2(G/H, \mathcal{W})) \simeq Hom_H(\tau|H, \pi)$$

□

Memo 5.1. *Frobenius Reciprocity Theorem can be purely algebraicly proved. The proof needs only Peter-Weyl Theorem and Shur Lemma and Expressing induced representation as tensor space.*

6 Classification of irreducible representations of compact classical groups

6.1 Facts without proof

Proposition 6.1. *Here are settings and assumptions.*

(S1) $A' = \{a_{i,j}\}_{i,j} \in M(n, \mathbb{C})$.

(S2) $A = \begin{pmatrix} Re(a_{1,1}) & -Im(a_{1,1}) & ... & Re(a_{1,n}) & -Im(a_{1,n}) \\ Im(a_{1,1}) & Re(a_{1,1}) & ... & Im(a_{1,n}) & Re(a_{1,n}) \\ ... & ... & ... & ... & ... \\ Re(a_{n,1}) & -Im(a_{n,1}) & ... & Re(a_{n,n}) & -Im(a_{n,n}) \\ Im(a_{n,1}) & Re(a_{n,1}) & ... & Im(a_{n,n}) & Re(a_{n,n}) \end{pmatrix}$.

Then

$$detA = |detA'|^2$$

6.2 Complex Analysis

Proposition 6.2. *Here are settings and assumptions.*

(S1) $\{a_\alpha\}_{\alpha \in \mathbb{Z}^n} \subset \mathbb{C}$ *such that* $\#\{\alpha | a_\alpha \neq 0\} < \infty$.

(S2) $P(t) := \sum_\alpha a_\alpha t^\alpha$ $(t \in \mathbb{C}^n)$.

(A1) $P = 0$ *in* T^n.

Then $P = 0$ *in* $\mathbb{C}^n$.

Proof. For aiming contradiction, le us assume $a_\alpha \neq 0$ for some α. Let β the biggest index of $\{\alpha | a_\alpha \neq 0\}$. with respect to lexicographic order. We can assume $\beta_1 \neq 0$. For any $r > 0$,

$$|P(r, 1, ..., 1)| = |r^{\beta_1} P(1, ..., 1)| = 0$$

By increasing $r \to \infty$, we get $\infty = 0$. This is contradiction. □

6.3 Complexification

From the definition and property of $\mathbb{C}$, the following holds.

Proposition 6.3 (Complexification)**.** *Here are settings and assumptions.*

(S1) $\mathfrak{g} \subset M(n, \mathbb{C})$ *is a Lie algebra.*

Then

$$\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} := \{X + iY | X, Y \in \mathfrak{g}\}$$

is a $\mathbb{C}$ *vector space with respect to*

$$(a + ib)(X + iY) := (aX - bY)$$

We call $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ *the complexification of* $\mathfrak{g}$.

From the definition and property of $\mathbb{C}$ and the definition of complexification, the following holds.

Proposition 6.4. *Here are settings and assumptions.*

(S1) $\mathfrak{g} \subset M(n, \mathbb{C})$ *is a Lie algebra.*

(S2) $f : \mathfrak{g} \to \mathfrak{g}$ *is a* $\mathbb{R}$ *linear map.*

If we define $F : \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ *by*

$$\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} := \{X + iY | X, Y \in \mathfrak{g}\}$$

then F *is a* $\mathbb{C}$ *linear map.*

Clearly the following holds by Proposition6.1.

Proposition 6.5. *Here are settings and assumptions.*

(S1) $\mathfrak{g} \subset M(n, \mathbb{C})$ *is a Lie algebra.*

(S2) $f : \mathfrak{g} \to \mathfrak{g}$ is a $\mathbb{R}$ linear map.

(A1) There is a basis of $\mathfrak{g}$ which is $\{X_i\}_{i=1}^n \cup \{iX_i\}_{i=1}^n$ for some $\{X_i\}_{i=1}^n \subset \mathfrak{g}$.

(A2) $\{X_i\}_{i=1}^n$ is a basis of the complexification of $\mathfrak{g}$.

(S3) F is the complexification of f.

(A3) All eigenvalues of F are distinct.

Then

$$det(f) = |det(F)|^2$$

6.4 A_{n-1} type case

6.4.1 Main theorem

The propositions shown in this section will not be presented with proofs in this subsection, but will be presented with proofs in the subsections that follow.

Definition 6.1 (Torus, Maximal Torus)**.** *Here are settings and assumptions.*

(S1) G is a compact Lie group.

Then

(i) We say $T \subset G$ is a torus of G if T is a connected commutative closed subgroup of G.

(i) We say $T \subset G$ is a maximal torus of G if T is a torus and there is no torus which contains T as a proper subset.

Notation 6.1 (Diagonal Matrix)**.** *We set*

$$diag(t_1, t_2, ..., t_n) := \begin{pmatrix} t_1 & 0 & ... & 0 \\ 0 & t_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & ... & 0 & t_n \end{pmatrix}$$

Notation 6.2 (Lexicographical order on $\mathbb{Z}^n$)**.** *We denote the lexicographical order on $\mathbb{Z}^n$ by $\prec$.*

Proposition 6.6 (Maximal torus of $U(n)$)**.**

$$T := \{diag(t_1, t_2, ..., t_n) | \; |t_1| = ... = |t_n| = 1\}$$

is a maximal trus of $U(n)$.

The following is clear.

Proposition 6.7 (Irreducible representation of maximal torus of $U(n)$)**.** *Let us $\alpha \in \mathbb{Z}n$.*

$$\chi_\alpha : T \ni diag(t_1, t_2, ..., t_n) \mapsto t_1^{\alpha_1} ... t_n^{\alpha_n} \in S^1$$

is a continuous homomoriphism.

Proposition 6.8 (Weight, Weight vector)**.** *We will succeed notations in Proposition6.7. Let*

(S1) G is a compact Lie group.

(S2) (π, V) is a finite dimensional continuous representation of G.

(S3) For each $\lambda \in \mathbb{Z}$, we denote χ_λ component of $\pi|T$ by V_λ.

Then

(i) We say $\lambda \in \mathbb{Z}$ is a weight of V with respect to T if $V_\lambda \neq \{0\}$. We call an element of V_λ a weight vector for each weight λ.

(ii) We say $\lambda \in \mathbb{Z}$ is the highest weight of V with respect to T if λ is the maximum weight with $\prec$. We define the highest weight vector in the same way.

(iii) We call the multiplicity of χ_λ in V_λ the multiplicity of the weight λ.

Notation 6.3 ($(\mathbb{Z}^n)_+$). *We set*

$$(\mathbb{Z}^n)_+ := \{\lambda \in \mathbb{Z}^n | \lambda \text{ is monotone decreasing.}\}$$

The following is the main theorem in this section. In the last part of this section, we give a proof of this theorem.

Theorem 6.1 (Cartan-Weyl theorem of the highest weight)**.** *The followings hold.*

(i) Let us assume (π, V) be a continuous irreducible unitary representation of $U(n)$ and λ be the highest weight of π. Then $\lambda \in (\mathbb{Z}^n)_+$ and the multiplicity of λ is 1.

(ii) Let us fix any $\lambda \in (\mathbb{Z}^n)_+$. Then there is the unique continuous irreducible unitary representation (π, V) whose highest weight is λ, ignoring isomorphism as continous unitary representation.

6.4.2 General topics on compact Lie group

By Zorn's Lemma, the following holds.

Proposition 6.9 (Maximal torus of a compact Lie group)**.** *For any compact Lie group G, there is a maximal torus of G.*

Proof. We set

$$\mathfrak{T} := \{T \subset G | T \text{ is an abelian subgroup of } G\}$$

For any $\mathfrak{A}$ is any totally ordered subset of $\mathfrak{T}$, $\cup\mathfrak{A} \in \mathfrak{T}$. So, $\mathfrak{T}$ has a maximal element T. Because $\bar{T}$ is an abelian subgroup of G, $\bar{T} = T$. So T is a maximal torus of G. □

Proposition 6.10 (Weyl group)**.** *Let*

(S1) G is a compact Lie group.

(S2) T is a maximal torus of G.

(S3) We set

$$N_G(T) := \{g \in G | gtg^{-1} \in T \ (\forall t \in T)\}$$

(S4) We set

$$Z_G(T) := \{g \in G | gt = tg \ (\forall t \in T)\}$$

Then

(i) $N_G(T)$ is a compact subgroup of G.

(ii) $Z_G(T) = T$.

(iii) $Z_G(T)$ is a compact normal subgroup of $N_G(T)$.

We call the quotient compact group $N_G(T)/Z_G(T)$ the weyl group of G. We define the action of the weyl group on T by

$$w \cdot t := wtw^{-1} \ (w \in N_G(T)/Z_G(T), t \in T)$$

Proof of (i). Let us fix any $g_1, g_2 \in N_G(T)$ and $t \in T$. Because $g_1^{-1}tg_1 = (g_1t^{-1}g_1^{-1})^{-1}$ and $t, g_1t^{-1}g_1^{-1} \in T$, $g_1^{-1}tg_1 \in T$. So, $g_1^{-1} \in N_G(T)$. Because $(g_1g_2)^{-1}t(g_1g_2) = g_1^{-1}(g_2^{-1}tg_2)g_1^{-1}$ and $g_2^{-1}tg_2 \in T$, $(g_1g_2)^{-1}t(g_1g_2) \in T$. So, $g_1g_2 \in N_G(T)$. Consequently, $N_G(T)$ is a subgroup of G.

For each $t \in T$, we set $\sigma_t(g) = gtg^{-1}$ $(g \in G)$. σ_t is continuous for any $t \in T$. Because $N_G(T) = \cap_{t\in T}\sigma_t^{-1}(T)$, $N_G(T)$ is closed subset of G. □

Proof of (ii). Clearly $Z_G(T)$ is abelian compact subgroup of T and $T \subset Z_G(T)$. So, $T = Z_G(T)$. □

Proof of (iii). For any $g \in N_G(T)$, $gZ_G(T)g^{-1} = Z_G(T)$. So, $Z_G(T)$ is a normal subgroup of $N_G(T)$. □

Definition 6.2 (Flag variety)**.** *Let G be a compact Lie group and T be a maximal torus of G. We call G/T the flag variety.*

6.4.3 The maximal torus and Weyl group of $U(n)$

Proposition 6.11 (Maximal torus of $U(n)$)**.**

$$Z_{U(n)}(T) := \{g \in U(n) | gt = tg \ (\forall t \in T)\}$$

is equal to T. In special, T is the maximal torus of $U(n)$.

Proof. Let us fix any $g \in U(n)$. We take $t \in T$ such that $t_i \neq t_j \ (\forall i \neq \forall j)$.Then

$$g_{i,j} t_j = g_{i,j} t_i \ (\forall i, j)$$

So, $g_{i,j} = \delta_{i,j} g_{i,i} \ (\forall i, j)$. Then $g = diag(g_{1,1}, ..., g_{n,n})$. Because $g \in U(n)$, $g \in T$. So, $Z_{U(n)}(T) = T$. □

By the proof of Proposition6.11, the following holds.

Proposition 6.12. *We set*

$$T_{reg} := \{t \in T | t_i \neq t_j \ (\forall i \neq \forall j)\}$$

Then for every $t \in T_{reg}$, $Z_G(t) = T$.

Proposition 6.13 (Weyl group of $U(n)$)**.** *Let*

(S1) For compact group G and the maximal torus T, we set

$$N_G(T) := \{g \in G | gtg^{-1} \in T \ (\forall t \in T)\}$$

(S2) We set

$$\pi_0(w)(t) := (t_{w^{-1}(1)}, ..., t_{w^{-1}(n)}) \ (w \in \mathfrak{G}_n, t \in \mathbb{C}^n)$$

Here, $\mathfrak{G}_n$ is the symmetric group of degree n. We set $W := \pi_0(\mathfrak{G}_n)$.

(S3)

$$\Phi : W \ltimes T \ni (w, t) \mapsto wt \in GL(n\mathbb{C})$$

Then the followings hold.

(i) For any $\omega \in \mathfrak{G}_n$ and $t \in T$,

$$\pi_0(\omega) t \pi_0(\omega)^{-1} = diag(t_{\omega^{-1}(1)}, ..., t_{\omega^{-1}(n)})$$

So, $\pi_0(\omega) \in N_G(T)$.

(ii) $\Phi : W \times T \ni (\sigma, t) \mapsto \sigma t \in N_G(T)$ is a bijection.

(iii) W and $N_G(T)/T$ are isomorphic as groups.

Proof of (i). It is clear. □

Proof of (ii). Let us fix any $\sigma \in W$ and $t \in T$. For any $s \in T$, $\sigma t s (\sigma t)^{-1} = \sigma s \sigma^{-1} \in T$ by (i). So, $\Phi(W \times T) \subset N_G(T)$.
Let us fix any $g \in N_G(T)$. Let us fix $t \in T_{reg}$. We set $s := gtg^{-1}$.
Because s and t have the same set of eigenvalues. So, there is $\omega \in \mathfrak{G}_n$ such that

$$s = (t_{\omega^{-1}(1)}, ..., t_{\omega^{-1}(n)})$$

By (i), this means that $s = \pi_0(\omega) t \pi_0(\omega^{-1})$. So, $t = \pi_0(\omega^{-1}) g t g^{-1} \pi_0(\omega)$. We set $t_1 := \pi_0(\omega^{-1}) g$. By Proposition6.12, $t_1 \in Z_G(T)$. $t = \Phi(\pi_0(\omega), t_1)$. So, Φ is surjective.

Let us fix any $\sigma_1, \sigma_2 \in W$ and any $t_1, t_2 \in T$ such that $\sigma_1 t_1 = \sigma_2 t_2$. Then $\sigma_2^{-1} \sigma_1 = t_2 t_1^{-1} \in W \cap T = \{e\}$. This implies $\sigma_1 = \sigma_2$ and $t_1 = t_2$. □

Proof of (iii). We set $\Psi := \Phi^{-1}$ and $P : W \times T \ni (w, t) \mapsto w \in W$ and $\varphi := P \circ \Psi$. Clearly φ is surjective and $\varphi^{-1}(e) = T$. So it is enough to show φ is homomorphism. For any $\sigma_1, \sigma_2 \in W$ and any $t_1, t_2 \in T$,

$$\sigma_1 t_1 \sigma_2 t_2 = \sigma_1 \sigma_2 \sigma_2^{-1} t_1 \sigma_2 t_2 = \Phi(\sigma_1 \sigma_2, \sigma_2^{-1} t_1 \sigma_2 t_2)$$

So, φ is homomorphism. □

By Shur Lemma, the following clearly holds.

Proposition 6.14. *Let*

(S1) G *is an abelian Lie group.*

(S2) $C := \{\varphi \in C(G, S^1) | \varphi$ *is a continuous homomorphism between groups.*$\}$

(S3) $\pi_\varphi(g)v := \varphi(g)z$ $(g \in G, z \in \mathbb{C}, \varphi \in C)$.

Then

(i) *For any* $\tau \in \hat{G}$, $\chi_\tau \in C$.

(ii) $\Phi : C \ni \varphi \mapsto \pi_\varphi \in \hat{G}$ *is bijective whose inverse is* $\Psi : \hat{G} \ni \pi \mapsto \chi_\pi \in C$.

Hereafter, we equate $\varphi \in \hat{G}$ *and* $\Phi(\varphi)$.

Proposition 6.15. *Let* T *be the maximal torus of* $U(n)$. *Then*

$$\hat{T} = \{\chi_\lambda | \lambda \in \mathbb{Z}^n\}$$

Hereafter, we equate $\lambda \in \mathbb{Z}^n$ *and* $\chi_\lambda \in \hat{G}$.

Proof. This proof is similar to the proof of Proposition4.37. We set

$$f(\theta_1, ..., \theta_n) := \tau(exp(i\theta_1 2\pi), ..., exp(i\theta_n 2\pi)) \ (\theta_1, ..., \theta_n \in \mathbb{R})$$

Then

$$f(\theta + he_i) = f(\theta)f(he_i) \ (\forall\theta \in \mathbb{R}^n, \forall h \in \mathbb{R}, \forall i)$$

So,

$$\frac{\partial f}{\partial \theta_i}(\theta) = \frac{\partial f}{\partial \theta_i}(\mathbf{0})f(\theta) \ (\forall\theta \in \mathbb{R}^n, \forall h \in \mathbb{R}, \forall i)$$

Because $f(\mathbf{0}) = 1$ and $Im(f) \subset S^1$, there are $\alpha_1, ..., \alpha_n \in \mathbb{R}$ such that

$$f(\theta) = exp(i\theta_1\alpha_1 2\pi)...exp(i\theta_n\alpha_n 2\pi) \ (\forall\theta \in \mathbb{R}^n)$$

Because $f(e_i) = 1$ $(\forall i)$, $\alpha_1, ..., \alpha_n \in \mathbb{Z}$. Consequently,

$$\hat{T} = \{\chi_\lambda | \lambda \in \mathbb{Z}^n\}$$

We denote the inverse of

$$\mathbb{Z}^n \ni \lambda \mapsto \chi_\lambda \in C$$

by Ψ. □

The following clearly holds.

Proposition 6.16. *We succeed in notations of Proposition6.14 and Proposition6.15.*

(S1) $W \subset U(n)$ *is the weyl group of* $U(n)$.

(S2) $(w \cdot \varphi)(t) := \varphi(w^{-1} \cdot t)$ $(w \in W, \varphi \in C, t \in T)$.

Then W *continuously acts on* C *and*

$$w \cdot \varphi = w^{-1}\Psi(\varphi) \ (\forall w \in W, \forall\varphi \in C)$$

Proposition 6.17. *Here are the settings and assumptions.*

(S1) T *is the maximal torus of* $U(n)$.

(S2) (π, V) *is a continuous unitary representation of* $U(n)$.

(S3) $\lambda \in \hat{U(n)}$.

Then

$$V_\lambda = \{w \in V | \pi(g)w = \chi_\lambda(g)w \ (\forall g \in T)\}$$

Proof. We denote the right side of the above equation by W. Let us fix any $w \in \sum_{A \in Hom_G(\chi_\lambda, \pi)} ImA$. Then there are $A_1, ..., A_m \in Hom_G(\chi_\lambda, \pi)$ and $v_1, ..., v_m \in V$ such that $w = \sum_{i=1}^m A_i v_i$. So, for any $g \in G$,

$$\pi(g)w = \sum_{i=1}^m \pi(g)A_i v_i = \sum_{i=1}^m A_i\chi_\lambda(g)v_i = \chi_\lambda(g)\sum_{i=1}^m A_i v_i = \chi_\lambda(g)w$$

So, $\sum_{A \in Hom_G(\chi_\lambda, \pi)} ImA \subset W$. Because W is closed, $V_\lambda \subset W$.

Let us fix any $w \in W$. We set $P_\lambda := P_{\chi_\lambda}$. By Proposition6.14 ,

$$P_\lambda w = \int_G \overline{\chi_\lambda(g)}\pi(g)wdg = \int_G \overline{\chi_\lambda(g)}\chi_\lambda(g)wdg = \int_G wdg = w$$

By Theorem4.8, $w \in V_\lambda$. □

6.4.4 Weyl Integral Formula

Notation 6.4 (G_{reg}, T_{reg})**.** *Here are the settings and assumptions.*

(S1) T *is the maximal torus of* $G := U(n)$.

Then $G_{reg} := \{g \in G | g \text{ has no duplicate eigenvalues.}\}$ *and* $T_{reg} := T \cap G_{reg}$.

Proposition 6.18. *Here are the settings and assumptions.*

(S1) $G := U(n)$.

(S2) T *be the maximal torus of* G.

(S3) $\epsilon > 0$.

(S4) $\mathfrak{g} := Lie(G)$, $\mathfrak{h} := Lie(T)$.

(S5) $\mathfrak{q}$ *is a complement subspace of* $\mathfrak{h}$ *in* $\mathfrak{g}$.

Then there are $\{g_i\}_{i=1}^{\infty} \subset G$ *and* $\{U_i\}_{i=1}^{\infty}$ *such that* U_i *is a open neighborhood of* 0_k $(\forall i)$ *and* $U_i \subset B_k(O, \epsilon) \cap \mathfrak{q}$ $(\forall i)$ *and* $\{\pi(g_i Exp(U_i) w)\}_{i \in \mathbb{N}, w \in W}$ *is an open covering of* G/H *and for any* $i \in \mathbb{N}$, $w_0 \in W$ $\#\{(j, w) \in \mathbb{N} \times W | \pi(g_i Exp(U_i) w_0) \cap \pi(g_j Exp(U_j) w) \neq \phi\} < \infty$.

Proof. There is V which an open neighborhood of e in G such that $V^4 \subset Exp(B(O, \epsilon))$ and $\bar{V}$ is compact. There are $\{g_{0,i}\}_{i=1}^{N_0}$ and $\{\epsilon_{0,i}\}_{i=1}^{N_0} \subset (0, \infty)$ such that $\pi(\bar{V}^4 \cdot W) \subset \cup_{i=1}^{N_0} \pi(g_{0,i} Exp(B_k(O, \epsilon_{0,i})))$ and $g_{0,i} Exp(B_k(O, \epsilon_{0,i}) \subset Exp(B_k(O, \epsilon) g_{0,i}$ $(\forall i)$.

And for each $s \in \mathbb{N}$ there are $\{g_{s,i}\}_{i=1}^{N_s}$ and $\{\epsilon_{s,i}\}_{i=1}^{N_s} \subset (0, \infty)$ such that $\pi(\bar{V}^{4+s} W) \backslash \pi(V^{3+s} W) \subset \cup_{i=1}^{N_s} \pi(g_{s,i} Exp(B_k(O, \epsilon_{s,i})))$ and $g_{s,i} Exp(B_k(O, \epsilon_{s,i}) \subset Exp(B_k(O, \epsilon) g_{s,i}$ $(\forall i)$.

We set $\{g_i\}_{i=1}^{\infty} := \{g_{s,i} | s, i \in \mathbb{N}, 1 \leq i \leq N_s\}$ and $\{U_i\}_{i=1}^{\infty} := \{U_{s,i} | s, i \in \mathbb{N}, 1 \leq i \leq N_s\}$. We will show for any $i \in \mathbb{N}$ and $s \in \mathbb{N}$,

$$\pi(g_{s,i}) \notin \pi(V^{s+2} W)$$

For aiming contradiction, let us assume $s \in \mathbb{N}$ and $i \in \mathbb{N}$ such that $\pi(g_{s,i}) \in \pi(V^{s+2} W)$. So,

$$\pi(g_{s,i} Exp(B_k(O, \epsilon_{s,i}))) \subset \pi(Exp(B_k(O, \epsilon)) g_{s,i}) \subset \pi(V^{s+3} W)$$

This contradicts with

$$\pi(g_{s,i} Exp(B_k(O, \epsilon_{s,i}))) \cap \pi(V^{s+3} W)^c \neq \phi$$

Nextly, we will show for any $i \in \mathbb{N}$ and $s \in \mathbb{N}$,

$$\pi(g_{s,i} Exp(B_k(O, \epsilon_{0,i})) W) \cap \pi(V^{s+1} W) = \phi$$

For aiming contradiction, let us assume $s \in \mathbb{N}$ and $i \in \mathbb{N}$ such that $\pi(g_{s,i} Exp(B_k(O, \epsilon_{0,i})) W) \cap \pi(V^{s+1} W) \neq \phi$. Then there is $X \in B_k(O, \epsilon)$ and $u \in V^{s+2}$ and $w_1, w_2 \in W$ such that $\pi(Exp(X) g_{s,i} w_1) = \pi(u w_2)$. So,

$$\pi(g_{s,i}) = g_{s,i} T = g_{s,i} w_1 T w_1^{-1} = Exp(-X) u w_2 w_1^{-1} w_1 T w_1^{-1} = Exp(-X) u w_2 w_1^{-1} T \in \pi(V^{s+2} W)$$

This is a contradiction. □

Notation 6.5 ($\Delta(t)$ $(t \in T_{reg})$, T_σ $(\sigma \in \mathfrak{S}_n)$)**.** *Here are the settings and assumptions.*

(S1) $G := U(n)$.

(S2) T *is the maximal torus of* G.

Then

(i) $\Delta(t) := \min\{|arg(t_i) - arg(t_j)| \; | i \neq j\}$. *Here, let us assume* $arg(z) \in [0, 2\pi)$.

(ii) For $\sigma \in \mathfrak{S}_n$, *we set*

$$T_\sigma := \{t \in T_{reg} | arg(t_{\sigma(i)}) < arg(t_{\sigma(i+1)}) \; (\forall i)\}$$

Theorem 6.2. *Here are the settings and assumptions.*

(S1) T *is the maximal torus of* $G := U(n)$.

(S2) $A : G/T \times T \ni (gT, t) \mapsto g t g^{-1} \in G$.

(S3) W *is the weyl group of* G.

Then

(i) *A is well-defined and surjective C^ω-class map.*

(ii) *$A|G/T \times T_{reg}$ is a surjective map onto T_{reg}.*

(iii) *For each $g, g' \in G$ and $t, t' \in T$,*

$$A(gT, t) = A(g'T, t') \iff \exists w \in W \ s.t\ g'T = gw^{-1}T \text{ and } t' = w \cdot t$$

Here, $w \cdot t_2 := wt_2w^{-1}$.

Proof of (i). Because T is commutative, if $g_1, g_2 \in G$ and $t_1, t_2 \in T$ and $(g_1T, t_1) = (g_2T, t_2)$ then

$$g_1t_1g_1^{-1} = g_2g_2^{-1}t_1(g_2g_2^{-1}g_1)^{-1} = g_2g_2^{-1}g_1t_1g_1^{-1}g_2g_2^{-1} = g_2t_2g_2^{-1}g_1g_1^{-1}g_2g_2^{-1} = g_2t_2g_2^{-1}$$

So A is well-defined. And clearly A is surjective.

We take $\{\pi(g_iExp(U_i))\}_i$ and $\{h_jExp(V_j)\}_j$ as the coverings in Proposition5.4. For each i, j and $X \in U_i$ and $Y \in V_j$,

$$A(g_iExp(X), h_jExp(Y)) := g_iExp(X)h_jExp(Y)Exp(-X)g_i^{-1}$$

So, A is C^ω-class. ☐

Proof of (ii). Because for any $g \in G$ and $t \in T$ gtg^{-1} has no duplicate eigenvalues $\iff$ t has no duplicate eigenvalues, (ii) holds. ☐

Proof of (iii). The $\Longleftarrow$ part is clear. We will show the $\Longrightarrow$ part. Let us fix any $g_1, g_2 \in G$ and $t_1, t_2 \in T_{reg}$ such that $g_1t_1g_1^{-1} = g_2t_2g_2^{-1}$. We set $g_3 := g_2^{-1}g_1$. Then

$$t_1 = g_3t_2g_3^{-1}$$

Because $t_1, t_2 \in T_{reg}$, there is $w \in W$ such that

$$t_2 = w^{-1}g_3t_2(w^{-1}g_3)^{-1}$$

So, $w^{-1}g_3 \in Z_G(t_2)$. By Proposition6.10, $t_3 := w^{-1}g_3 \in T$. So, $g_3 = wt_3$. Then

$$g_2T = g_1g_3^{-1}T = g_1w^{-1}wt_3^{-1}w^{-1}T = g_1w^{-1}T, t_1 = wt_3t_2t_3^{-1}w^{-1} = wt_2w^{-1} =: w \cdot t_2$$

☐

By Theorem6.2 and Proposition4.12 and Proposition4.13, the following holds.

Proposition 6.19. *Here are the settings and assumptions.*

(S1) *$G := U(n)$.*

(S2) *T is the maximal torus of G.*

(S3) *(π_i, V_i)(i=1,2) are two continuous finite dimensional representation of G.*

(A1) *$\chi_{\pi_1}|T = \chi_{\pi_2}|T$.*

Then $\pi_1 \simeq \pi_2$.

Proposition 6.20. *Here are the settings and assumptions.*

(S1) *$G := U(n)$.*

(S2) *T is the maximal torus of G.*

(S3) *$\mathfrak{t} := Lie(T)$, $\mathfrak{g} := Lie(G)$.*

(S4) *$\mathfrak{g}_1 := \{X \in \mathfrak{g} | X_{i,i} = 0\ (\forall i)\}$*

Then

$$\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{t}$$

Proof. Clearly, $\mathfrak{g}_1 \cap \mathfrak{t} = \phi$ and $\mathfrak{g} \supset \mathfrak{g}_1 + \mathfrak{t}$ Let us fix any $X \in \mathfrak{g}$. Then

$$X = Y + diag(X_{1,1}, ..., X_{n,n})$$

Here,

$$Y_{i,j} = (1 - \delta_{i,j})X_{i,j} \ (i, j = 1, 2, ..., n)$$

Then $Y \in \mathfrak{g}_1$. Because X is skew-Hermitian, $X_{j,j} \in i\mathbb{R}$ $(\forall j)$. So, $diag(X_{1,1}, ..., X_{n,n}) \in \mathfrak{t}$. So,

$$\mathfrak{g} \subset \mathfrak{g}_1 + \mathfrak{t}$$

□

Proposition 6.21. *Here are the settings and assumptions.*

(S1) $G := U(n)$.

(S2) T *is the maximal torus of* G.

Then there is $\{V_j\}_{i=1}^{\infty}$ *such that* $\{w \cdot V_j\}_{j\in\mathbb{N}, w\in W}$ *is a local finite open covering of* T_{reg} *and for any* i, j $\sup\{|arg(t_i) - arg(t_i)||t \in V_j\} \leq \frac{1}{2}\inf\{\Delta(t)|t \in V_j\}$ *and for any* $s \in \mathbb{N}$ $\#\{j|\Delta(t) \geq \frac{1}{2^s}\ (\exists t \in U_j)\} < \infty$ *and* $V_i \subset T_e$ $(\forall i)$.

Proof. We set

$$T_s := \{t \in T_{reg}|\Delta(t) \leq \frac{1}{2^s}\}, T_{s,\sigma} := T_s \cap T_\sigma \ (s \in \mathbb{N}, \sigma \in \mathfrak{S}_n)$$

Because $T_{1,e}$ is compact, there are $\{U_{1,i}\}_{i=1}^{N_1}$ which is a open covering of $T_{1,e}$ and N_1 is the minimum numver of open covering of $T_{1,e}$. Let us fix $s \in \mathbb{N} \cap [2, \infty)$. Because $T_{s,e} \setminus T^{\circ}_{s-1,e}$ is compact, there are $\{U_{s,i}\}_{i=1}^{N_s}$ which is a open covering of $T_{s,e} \setminus T^{\circ}_{s-1,e}$ and N_s is the minimum numver of cardinalities of all open coverings of $T_{s,e} \setminus T^{\circ}_{s-1,e}$. Clearly, $\cup_{w\in W} \cup_{s=1}^{\infty} \{w \cdot U_{s,i}\}_{i=1}^{N_s}$ is a local finite open covering of T_{reg} and satisfies the condition in the claim of this Proposition. □

Proposition 6.22. *Here are the settings and assumptions.*

(S1) $G := U(n)$.

(S2) T *is the maximal torus of* G.

(S3) $\mathfrak{g} := Lie(G)$, $\mathfrak{h} := Lie(T)$.

(S4) $\mathfrak{q}$ *is a complement subspace of* $\mathfrak{h}$ *in* $\mathfrak{g}$.

(S5) We set

$$A : G/T \times T \ni (gT, t) \mapsto gtg^{-1} \in T$$

Then there are $\{g_i\}_{i=1}^{\infty} \subset G$ *and* $\{U_i\}_{i=1}^{\infty}$ *such that* U_i *is a open neighborhood of* 0_k $(\forall i)$ *and* $U_i \subset B_k(O, \epsilon) \cap \mathfrak{q}$ $(\forall i)$ *and* $\{\pi(g_i Exp(U_i)w^{-1}) \times w \cdot V_j\}_{i\in\mathbb{N}, w\in W, j\in\mathbb{N}}$ *is a local finite open covering of* $G/H \times T_{reg}$ *and* $\{A\pi(g_i Exp(U_i)w^{-1}) \times w \cdot V_j\}_{i\in\mathbb{N}, w\in W, j\in\mathbb{N}}$ *is a local finite open covering of* G_{reg}.

Proof of the first part. We will succeed in notations of Propositions6.22 and Proposition6.18. Let us fix any $(gT, t) \in G/H \times T_{reg}$. There is $w \in W$ such that $(gwT, w^{-1} \cdot t) \in G/H \times T_e$. Then there are i, j such that $(gwT, w^{-1} \cdot t) \in \pi(g_i Exp(U_i)) \times V_j$. Then $t \in w \cdot V_j$. And there is $u \in Exp(U_i)$ such that $gwT = g_i uT$. Because $gwT = gTw$ and $g_i uw^{-1}Tw$,

$$gT = g_i uw^{-1}T$$

So, $(gT, t) \in \pi(g_i Exp(U_i)w^{-1}) \times w \cdot V_j$. Consequently, $\{\pi(g_i Exp(U_i)w^{-1}) \times w \cdot V_j\}_{i\in\mathbb{N}, w\in W, j\in\mathbb{N}}$ is an open covering of $G/H \times T_{reg}$.

Let us fix any $i_0, j_0 \in \mathbb{N}$ and $w_0 = \pi_0(\sigma_0) \in W$. Let us fix any $i, j \in \mathbb{N}$ and $w = \pi_0(\sigma) \in W$ such that

$$\pi(g_{i_0} Exp(U_{i_0})w_0^{-1}) \times w_0 \cdot V_{j_0} \cap \pi(g_i Exp(U_i)w^{-1}) \times w \cdot V_j \neq \phi$$

Because $V_{j_0}, V_j \subset T_e$, $w_0 = w$. So, $V_{j_0} \cap V_j \neq \phi$. Because $g_{i_0} uw_0^{-1}T = g_{i_0} uTw_0^{-1}$ and $g_i vw^{-1}T = g_i vTw^{-1}$ for any $u \in Exp(U_{i_0})$ and $v \in Exp(U_i)$, $\pi(g_{i_0} Exp(U_{i_0})) \cap \pi(g_i Exp(U_i)) \neq \phi$. So,

$$(i, j, w) \in B := \{(i, j, w)|\pi(g_i U_i) \cap \pi(g_{i_0} U_{i_0}) \neq \phi, w = w_0, V_j \cap V_{j_0} \neq \phi\}$$

Because B is finite, $\{\pi(g_i Exp(U_i)w^{-1}) \times w \cdot V_j\}_{i\in\mathbb{N}, w\in W, j\in\mathbb{N}}$ is a local finite open covering of T_{reg}. □

Proof of the last part. By the first part, clearly $\{A\pi(g_i Exp(U_i)w^{-1}) \times w \cdot V_j\}_{i\in\mathbb{N},w\in W,j\in\mathbb{N}}$ is an open covering of T_{reg}. We set $X_i := g_i Exp(U_i)$ $(i \in \mathbb{N})$.

Let us fix any $i_0, j_0 \in \mathbb{N}$ and $w_0 \in W$. We set

$$W_0 := \{w \in W | \exists i, \exists j \text{ s.t } A\pi(X_{i_0} w_0^{-1}) \times w_0 \cdot V_{j_0} \cap A\pi(X_i w^{-1}) \times w \cdot V_j \neq \phi\}$$

Clearly, W_0 is a finite set.

We set

$$J_0 := \{j \in \mathbb{N} | \exists i, \exists w \text{ s.t } A\pi(X_{i_0} w_0^{-1}) \times w_0 \cdot V_{j_0} \cap A\pi(X_i w^{-1}) \times w \cdot V_j \neq \phi\}$$

and

$$\epsilon := \inf\{\Delta(t) | t \in V_{j_0}\}$$

Then

$$\Delta(t) \geq \frac{\epsilon}{2^2} \ (\forall t \in V_j, \forall j \in J_0)$$

So, from the definition of $\{V_j\}_{j\in\mathbb{N}}$, J_0 is a finite set.

We set

$$I_0 := \{i \in \mathbb{N} | \exists j, \exists w \text{ s.t } A\pi(X_{i_0} w_0^{-1}) \times w_0 \cdot V_{j_0} \cap A\pi(X_i w^{-1}) \times w \cdot V_j \neq \phi\}$$

From the definition of $\{X_i\}_{i\in\mathbb{N}}$, I_0 is a finite set. Consequently, $\{A\pi(g_i Exp(U_i)w^{-1}) \times w \cdot V_j\}_{i\in\mathbb{N},w\in W,j\in\mathbb{N}}$ is local finite. □

Proposition 6.23. *Here are the settings and assumptions.*

(S1) T is the maximal torus of $G := U(n)$.

Then

(i) $T \setminus T_{reg}$ is a zero set with respect to a Haar measure on T.

(ii) $G \setminus G_{reg}$ is a zero set with respect to a Haar measure on G.

Proof of (i). Clearly, $T \setminus T_{reg} \subset \cup_{i,j} T_{i,j}$. Here, $T_{i,j} := \{t \in T | t_i = t_j\}$. So, it is enough to show $T_{i,j}$ is a zero set for any i, j. We can assume $i = n-1, j = n$. We set

$$\varphi : T \ni t \mapsto (t_1, ..., t_{n-1}, t_{n-1}) \in T, C := \{t \in T | rank(J\varphi(t)) < n\}$$

Clearly $C = T$ and $T_{n-1,n} \subset \varphi(C)$. By Sard's Theorem(See [10]), $\varphi(C)$ is a zero set. So, $T_{n-1,n}$ is a zero set. □

Proof of (ii). By (i), $G/T \times T \setminus T_{reg}$ is a zero set. And $A : G/T \times T \to G$ is a C^ω-class surjective and $G_{reg} = A(G/T \times T \setminus T_{reg})$. So, by a Lemma for Sard's Theorem(See [10]), G_{reg} is a zero set. □

Proposition 6.24. *Here are the settings and assumptions.*

(S1) T is the maximal torus of $G := U(n)$.

Then for any $f \in C(G)$

$$\int_G f(g)dg = \frac{1}{n!} \int_{G/T} \int_T f(gtg^{-1}) |det(dA_{(gT,t)})| dt d(gT)$$

Here,

$$det(dA_{(gT,t)}) := det(dL_{gt^{-1}g^{-1}} \circ dA_{(gT,t)} \circ j \circ d\tau_g \times dL_t \circ i)$$

$i : T_e(g) = \mathfrak{g}_1 \oplus \mathfrak{t} \to \mathfrak{g}_1 \times \mathfrak{t}$ is the natural isomorphism and $j : T_{gT}(G/T) \times T_t(T) \to T_{(gT,t)}(G/T \times T)$ is the natural isomorphism.

STEP1. Construction of a partition of unity. By Proposition6.23, it is enough to show

$$\int_{G_{reg}} f(g)dg = \frac{1}{n!} \int_{G/T} \int_{T_{reg}} f(gtg^{-1}) det(dA_{(gT,t)}) dt d(gT)$$

Let $\{\pi(g_i U_i w^{-1}) \times w \cdot V_j\}_{i,j\in\mathbb{N},w\in W}$ be the open covering of $G/T \times T_{reg}$ and $\{f_{i,j,w}\}_{i,j\in\mathbb{N},w\in W}$ be a partition of unity with respect to $\{\pi(g_i U_i w^{-1}) \times w \cdot V_j\}_{i,j\in\mathbb{N},w\in W}$.

We set

$$g_{i,j,w}(A(gT,t)) := \frac{1}{n!} f_{i,j,w}(gw^{-1}T, w \cdot t) \ ((gT,t) \in \pi(g_i Exp(U_i)) \times V_j, i, j \in \mathbb{N}, w \in W)$$

We will show $g_{i,j,w}$ is well-defined. Let us fix any $g_1, g_2 \in \pi(g_i Exp(U_i))$ and $t_1, t_2 \in V_j$ and $w \in W$ and $i, j \in \mathbb{N}$. such that $A(g_1T, t_1) = A(g_2T, t_2)$. This means that $g_1 t_1 g_1^{-1} = g_2 t_2 g_2^{-1}$. Because $t_1, t_2 \in T_e$, by Theorem6.2, $t_1 = t_2$ and $g_1T = g_2T$. So, $w \cdot t_1 = w \cdot t_2$. And

$$g_1 w^{-1} T = g_1 T w^{-1} = g_2 T w^{-1} = g_2 w^{-1} T$$

So, $g_{i,j,w}$ is well-defined.

We will show $\{g_{i,j,w}\}_{i,j \in \mathbb{N}, w \in W}$ is a partition of unity on G_{reg} with respect to $\{A\pi(g_i U_i w^{-1}) \times w \cdot V_j\}_{i,j \in \mathbb{N}, w \in W}$. Let us fix any $x \in G_{reg}$. We set

$$I := \{(i, j) \in \mathbb{N}^2 | x \in A\pi(g_i U_i) \times V_j\}$$

Then, by Theorem6.2,

$$I \times W = \{(i, j, w) \in\in \mathbb{N}^2 | x \in A\pi(g_i U_i w^{-1}) \times w \cdot V_j\}$$

So,

$$\sum_{i,j \in \mathbb{N}, w \in W} g_{i,j,w}(x) = \sum_{(i,j) \in I, w \in W} g_{i,j,w}(x) = \sum_{w \in W} \sum_{(i,j) \in I} g_{i,j,w}(x)$$

Let us fix any $w = \pi_0(\sigma) \in W$. And let us fix any i_1, i_2, j_1, j_2 and $h_{i_1} \in g_{i_1} Exp(U_{i_1})$ and $h_{i_2} \in g_{i_2} Exp(U_{i_2})$ and $t_{j_1} \in V_{j_1}$ and $t_{j_2} \in V_{j_2}$ such that $x = (\pi(h_{i_1} w^{-1}), w \cdot t_{j_1}) = (\pi(h_{i_2} w^{-1}), w \cdot t_{j_2})$. Then, because $t_{i_1}, t_{i_2} \in T_e$, $t_{i_1} = t_{i_2}$. And

$$h_{i_1} w^{-1} T = h_{i_1} T w^{-1} = h_{i_2} T w^{-1} = h_{i_2} w^{-1} T$$

So, there is the unique $x_w \in G/T \times T_\sigma$ such that $Ax_w = x$ and

$$\sum_{w \in W} \sum_{(i,j) \in I} g_{i,j,w}(x) = \sum_{w \in W} \frac{1}{n!} \sum_{(i,j) \in I} f_{i,j}(x_w) = \sum_{w \in W} \frac{1}{n!} = 1$$

□

STEP2. Proof of our integeral formula. We set $W_i := g_i Exp(U_i)$ $(i \in \mathbb{N})$.

$$\int_{G_{reg}} f(g)dg = \sum_{i,j,w} \int_{A\pi(W_i w^{-1}) \times w \cdot V_j} f(g) g_{i,j,w}(g) dg = \sum_{i,j,w} \int_{A\pi(W_i w^{-1}) \times w \cdot V_j} f(g) g_{i,j,w}(g) dg$$
$$= \frac{1}{n!} \sum_{i,j,w} \int_{\pi(W_i w^{-1}) \times w \cdot V_j} f(\pi(hw^{-1}), w \cdot t) f_{i,j,w}(\pi(hw^{-1}), w \cdot t) |det(dA_{(\pi(hw^{-1}), w \cdot t)})| dg$$
$$= \frac{1}{n!} \int_{G/T \times T} f(gT, t) |det(dA_{(\pi(hw^{-1}), w \cdot t)})| d\mu_{G/T}(gT) \mu_T(t)$$

□

The following clearly holds.

Proposition 6.25. *We succeed notations in Proposition6.20. Here are the settings and assumptions.*

(S1) $X_{i,j} = E_{i,j} - E_{j,i}$ $(i < j)$.

Then $B_0 := \{X_{i,j}\}_{i<j}$ *is a basis of the complexification of* $\mathfrak{g}_1$ *and* $B_0 \cup iB_0$ *is a basis of* $\mathfrak{g}_1$.

Lemma 6.1. *We succeed notations in Proposition6.24. Then*

(i) $det(dA_{(gT,t)}) = det(Ad(t)^{-1}|_{\mathfrak{g}_1} - id|_{\mathfrak{g}_1})$.

(ii) $det(Ad(t)^{-1}|_{\mathfrak{g}_1} - id|_{\mathfrak{g}_1}) = |D(t)|^2$.

Proof of (i). Let us fix any $X \in \mathfrak{g}_1$ and $Y \in \mathfrak{t}$. Then

$$dA_{(gT,t)} \circ j \circ d\tau_g \times dL_t \circ i(X + Y) = dA_{(gT,t)} \circ j \circ d\tau_g \times dL_t(X, 0) + dA_{(gT,t)} \circ j \circ d\tau_g \times dL_t(0, Y)$$

Here,

$$dA_{(gT,t)} \circ j \circ d\tau_g \times dL_t(X, 0) = \frac{d}{ds}_{|s=0} A(gexp(sX)T, t) = \frac{d}{ds}_{|s=0} gexp(sX)texp(-sX)g^{-1}$$
$$= \frac{d}{ds}_{|s=0} gtg^{-1}gt^{-1}exp(sX)texp(-sX)g^{-1} = \frac{d}{ds}_{|s=0} gtg^{-1}gexp(sAd(t^{-1})X)exp(-sX)g^{-1}$$
$$= dL_{gtg^{-1}} Ad(g)(Ad(t^{-1})X - X)$$

and

$$dA_{(gT,t)} \circ j \circ d\tau_g \times dL_t(0,Y) = \frac{d}{ds}_{|_{s=0}} A(gT, texp(sY)) = \frac{d}{ds}_{|_{s=0}} gtexp(sY)g^{-1}$$
$$= \frac{d}{ds}_{|_{s=0}} gtg^{-1}gt^{-1}exp(sY)g^{-1} = dL_{gtg^{-1}}Ad(g)(Y)$$

So,

$$det(dA_{(gT,t)}) = det(Ad(g))det(F)$$

Here,

$$F : \mathfrak{g}_1 \times \mathfrak{t} \ni (X,Y) \mapsto (Ad(t^{-1})X - X, Y) \in \mathfrak{g} \times \mathfrak{t}$$

Because clearly $T \cdot \mathfrak{g}_1 \subset \mathfrak{g}_1$ and $\mathfrak{g}_1 \cdot T \subset \mathfrak{g}_1$, $Ad(t^{-1})X \in \mathfrak{g}_1$ $(\forall t \in T, \forall X \in \mathfrak{g}_1)$. So, $ImF \in \mathfrak{g}_1 \times \mathfrak{t}$. This implies that $det(F) = det(Ad(t^{-1})|\mathfrak{g}_1 - id_{\mathfrak{g}_1})$. And, by Proposition2.48, $det(Ad(g)) = 1$ $(\forall g \in G)$. □

Proof of (ii). It is enough to show that (ii) holds for any $t \in T_{reg}$. Let us fix any $t \in T_{reg}$. We succeed notations in Proposition6.25.

$$(Ad(t)^{-1} - id)X_{i,j} = (\frac{t_j}{t_i} - 1)X_{i,j} \ (\forall i < \forall j)$$

So, by Proposition6.5,

$$det(Ad(t)^{-1} - id) = (\Pi_{i<j}|(\frac{t_j}{t_i} - 1)|)^2$$
$$\text{by } |t_i| = 1 \text{ and } \overline{\frac{t_j}{t_i}} = \frac{t_i}{t_j} \ (\forall i < \forall j)$$
$$= (\Pi_{i<j}|(t_i - t_j)|)^2 = |D(t)|^2$$

Lemma6.1 and Proposition6.24 implies the following.

Theorem 6.3 (Weyl Integral Formula). *For any $f \in C(U(n))$,*

$$\int_{U(n)} f(g)d\mu_{U(n)}(g) = \frac{1}{n!}\int_{G/T}\int_T f(gtg^{-1})|D(t)|^2 d\mu_T t d\mu_{G/T}(gT)$$

□

6.4.5 The highest weight of $U(n)$

Definition 6.3 (Multiplicity of weight). *We will succeed notations in Proposition6.7. Let*

- *(S1) G is a compact Lie group.*
- *(S2) (π, V) is a finite dimensional continuous representation of G.*
- *(S3) $\lambda \in \mathbb{Z}^n$.*

We call $m_\lambda := dimV_\lambda$ the multiplicity of λ.

Definition 6.4 (Symmetric function). *Let T be the maximal torus of $U(n)$. We say $f \in C(T, \mathbb{C})$ is a symmetric function if*

$$f(x) = f(wx) \ (\forall x \in T, \forall w \in W)$$

We denote the set of all symmetric functions by $C(T)_1$.

Definition 6.5 (Alternating function). *Let T be the maximal torus of $U(n)$. We say $f \in C(T, \mathbb{C})$ is a altenating function if*

$$f(x) = sign(w)f(wx) \ (\forall x \in T, \forall w \in W)$$

We denote the set of all symmetric functions by $C(T)_{sgn}$.

Definition 6.6 (Laurant polynomial). *Let T be the maximal torus of $U(n)$. We say $f \in C(T, \mathbb{C})$ is a Laurant function if*

$$f(x) = \sum_{K \in \mathbb{Z}^n} a_K t^K \ (x \in T), \#\{K \in \mathbb{Z}^n | a_K \neq 0\} < \infty$$

We denote the set of all Laurant polynomials by $R(T)$. We set

$$R_{\mathbb{Z}}(T) := \{f \in R(T) | \textit{Every coefficient of } f \textit{ are in } \mathbb{Z}\}$$

and

$$R_{\mathbb{Z}}(T)_1 := R_{\mathbb{Z}}(T) \cap C(T)_1$$

Proposition 6.26. *Here are the settings and assumptions.*

(S1) T *is th maximal torus of* $U(n)$.

(S2) $W := \pi_0(\mathfrak{S}_n)$.

(S3) (π, V) *is a finite dimensional continuous representation of* G.

(S4) $\Delta(V,T) := \{\lambda \in \hat{T} | V_\lambda \neq \{0\}\}$.

(S5) $\lambda \in \mathbb{Z}^n$ *is the highest weight of* (π, V).

Then

(i) For any $w \in W$ *and* $\lambda \in \mathbb{Z}^n$, $\pi(w)|V_\lambda$ *is a bijection fo* $V_{w\lambda}$.

(ii) $W \cdot \Delta(V,T) \subset \Delta(V,T)$.

(iii) For any $\sigma \in \mathbb{Z}^n$, $m_\sigma = m_{w\sigma}$.

(iv) $\Delta(V,T)$ *is finite set.*

(v) $V_\lambda \simeq m_\lambda \chi_\lambda$ *as continuous unitary representation of* T. *The right side is a discrete direct sum.*

(vi) $\chi_{\pi|T} = \sum_{\lambda \in \Delta(V,T)} m_\lambda \chi_\lambda$

(vii) $\chi_{\pi|T} \in R_{\mathbb{Z}}(T)_1$.

(viii) $\lambda \in (\mathbb{Z}^n)_+$.

Proof of (i). Firstly we will show $\pi(w)|V_\lambda \subset V_{w\cdot\lambda}$ $(\forall w \in W, \forall \lambda \in \hat{T})$. Let us fix any $w \in W$ and any $\lambda \in \mathbb{Z}^n$ and any $v \in V_\lambda$ and any $t \in T$.

$$\pi(t)\pi(w)v = \pi(w)\pi(w^{-1} \cdot t)v = \pi(w)\chi_\lambda(w^{-1} \cdot t)v = \chi_\lambda(w^{-1} \cdot t)\pi(w)v = \chi_{w\cdot\lambda}(t)\pi(w)v$$

So, by Proposition6.16, $\pi(w)v \in V_{w\cdot\lambda}$. Because $\pi(w^{-1})$ is the inverse of $\pi(w)$, $\pi(w)|V_\lambda$ is bijective. □

Proof of (ii). For any $w \in W$ and any $\lambda \in \Delta(V,T)$, by (i), $V_{w\cdot\lambda} = \pi(w) \cdot V_\lambda$. Because $\pi(w) \cdot V_\lambda \neq \{0\}$, $V_{w\cdot\lambda} \neq \{0\}$. So, $w \cdot \lambda \in \Delta(V,T)$. □

Proof of (iii). This is followed by (i). □

Proof of (iv). Because $\chi_{\lambda_1} \not\simeq \chi_{\lambda_2}$ $(\forall\lambda_1 \neq \forall\lambda_2)$, by Theorem4.9, $V = \oplus_{\lambda\in\mathbb{Z}^n} V_\lambda$. Because $dimV < \infty$, $\Delta(V,T)$ is a finite set. □

Proof of (v). Clearly V_λ is finite dimensional T-invariant space. Let us fix $w_1, ..., w_m$ which is the orthonormal basis of V_λ. We set

$$P_i z := z w_i \ (z \in \mathbb{C}, i \in \{1, 2, ..., m\})$$

By Proposition6.16,

$$P_i \chi_\lambda(t) z = z\chi_\lambda(t) w_i = z\pi(t) w_i = \pi(t) z w_i = \pi(t) P_i(z)$$

and $\mathbb{C}w_i$ is T-invariant. So, $P_i : (\chi_\lambda, \mathbb{C}) \to (\pi|\mathbb{C}w_i, \mathbb{C}w_i)$ is an isomorphism as continuous unitary representations of T. Consequently, (v) holds. □

Proof of (vi). (vi) is followed by (v) and Theorem4.9. □

Proof of (vii). By (vi), $\chi_{\pi|T} \in R_{\mathbb{Z}}(T)$. By (i), $\chi_{\pi|T} \in C(T)_1$. So, $\chi_{\pi|T} \in R_{\mathbb{Z}}(T)$. □

Proof of (viii). (viii) is followed by (i). □

Notation 6.6 (S_α, A_α)**.** *For* $\alpha \in \mathbb{Z}^n$,

$$S_\alpha(t) := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} t^{\sigma\alpha}$$

$$A_\alpha(t) := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} sign(\sigma) t^{\sigma\alpha}$$

Proposition 6.27.

(i) $\{S_\alpha\}_{\alpha\in\mathbb{Z}^n}$ *is a basis of* $R_{\mathbb{Z}}(T)_1$.

(ii) $\{A_\alpha\}_{\alpha\in\mathbb{Z}^n}$ *is a basis of* $R_{\mathbb{Z}}(T)_{sgn}$*.*

Proof of (i). Let us fix any $\frac{1}{n!}\sum_\alpha a_\alpha t^\alpha \in R_{\mathbb{Z}}(T)_1$. Let us fix any $\alpha \in \mathbb{Z}^n$ such that $\alpha_1 \geq ... \geq \alpha_n$. Then

$$a_{\sigma\alpha} = a_\alpha \ (\forall \sigma \in \mathfrak{S}_n)$$

So,

$$\frac{1}{n!}\sum_\alpha a_\alpha t^\alpha = \sum_{\alpha_1\geq...\geq\alpha_n} a_\alpha S_\alpha(t)$$

□

Proof of (ii). Let us fix any $\frac{1}{n!}\sum_\alpha a_\alpha t^\alpha \in R_{\mathbb{Z}}(T)_1$. If there are i, j such that $\alpha_i = \alpha_j$, then $a_\alpha = 0$ by the definition of the altenating function. Let us fix any $\alpha \in \mathbb{Z}^n$ such that $\alpha_1 > ... > \alpha_n$. Then

$$a_{\sigma\alpha} = sign(\alpha) a_\alpha \ (\forall \sigma \in \mathfrak{S}_n)$$

So,

$$\frac{1}{n!}\sum_\alpha a_\alpha t^\alpha = \sum_{\alpha_1\geq...\geq\alpha_n} a_\alpha A_\alpha(t)$$

□

Proposition 6.28.

(i) $\{S_\alpha\}_{\alpha\in\mathbb{Z}^n}$ *is a basis of* $R_{\mathbb{Z}}(T)_1$*.*

(ii) $\{A_\alpha\}_{\alpha\in\mathbb{Z}^n}$ *is a basis of* $R_{\mathbb{Z}}(T)_{sgn}$*.*

By the orthogonality of trigonometric functions, the following holds.

Proposition 6.29. *For* $\alpha_1 > ... > \alpha_n$ *and* $\beta_1 > ... > \beta_n$*,*

$$(A_\alpha, A_\beta)_{L^2(T)} = \begin{cases} n! & \alpha = \beta. \\ 0 & \alpha \neq \beta. \end{cases}$$

6.4.6 Weyl Character Formula

Theorem 6.4 (Weyl character formula)**.** *Here are the settings and assumptions.*

(S1) T *is the maximal torus of* $U(n)$*.*

(S2) (π, V) *is a finite dimensional irreducible continuous representation of* G*.*

(S3) λ *is the highest weight of* π*.*

Then

(i)

$$\chi_\pi(t) = \frac{\sum_{\sigma\in\mathfrak{S}_n} sgn(\sigma) t^{\sigma\cdot(\lambda+\rho)}}{\Pi_{1\leq i<j\leq n}(t_i - t_j)}$$

Here, $\rho := (n-1, n-2, ..., 1, 0)$*.*

(ii) $dim(V_\lambda) = 1$*.*

Proof. We set

$$D(t) := \Pi_{1\leq i<j\leq n}(t_i - t_j) \ (t \in T)$$

Then $\chi_\pi(t)D(t)$ is an alternating laurant polynomial, there is $\{a_\alpha\}_{\alpha\in\mathbb{Z}^n}$ such that $\#\{\alpha | a_\alpha \neq 0\} < \infty$ and

$$\chi_\pi(t)D(t) = \sum_\alpha a_\alpha A_\alpha(t) \ (\forall t \in T)$$

By Proposition6.29,

$$1 = \sum_\alpha |a_\alpha|^2$$

By Proposition6.26(vii), for any α $a_\alpha \in \mathbb{Z}$. So $\exists ! \alpha$ such that $|a_\alpha| = 1$. By Proposition6.2,

$$\chi_\pi D = A_\alpha \text{ (in } \mathbb{C}^n)$$

or

$$\chi_\pi D = -A_\alpha \text{ (in } \mathbb{C}^n)$$

Let m_λ denote the multiplicity of λ. And we can assume $\alpha_1 > ... > \alpha_n$. The maximal index of $D(t)$ with respect to lexicographic order is $(n-1, ..., 1)$. And the maximal index of χ_π with respect to lexicographic order is $m_\lambda \lambda$. So,

$$m_\lambda t^{(\lambda_1+n-1,...,\lambda_n+1)} = +t^\alpha \text{ (in } \mathbb{C}^n)$$

and

$$m_\lambda t^{(\lambda_1+n-1,...,\lambda_n+1)} = -t^\alpha \text{ (in } \mathbb{C}^n)$$

This implies that $m_\lambda = 1$ and

$$(\lambda_1 + n - 1, ..., \lambda_n + 1) = \alpha$$

□

6.4.7 Cartan-Weyl Highest Weight Theory

Theorem 6.5. *The followings hold.*

(i) For any $\phi \in R_\mathbb{Z}(T)_1$, $\Phi(\phi) := D\phi \in R_\mathbb{Z}(T)_{sgn}$.

(ii) $\Phi : R_\mathbb{Z}(T)_1 \to R_\mathbb{Z}(T)_{sgn}$ is surjective.

Proof of (i). It is clear. □

Proof of (ii). Let us fix any $\phi \in R_\mathbb{Z}(T)_{sgn}$. There is $N \in \mathbb{N}$ such that $p(t) = \sum_\alpha t^\alpha = t^{(N,...,N)}\phi \in P_\mathbb{Z}(T)_{sgn}$. For any $\alpha \in \mathbb{Z}^n$ such that $\alpha_1 = \alpha_2$, $a_\alpha = 0$.

For any $t \in T$ such that $t_1 = t_2$, $p(t) = 0$. By Proposition6.2, For any $z \in \mathbb{C}^n$ such that $z_1 = z_2$, $p(z) = 0$.

For each $\alpha \in \mathbb{Z}^n$ such that $\alpha_1 > \alpha_2$, $a_\alpha = -a_{S_{1,2}\alpha}$. Here, $S_{1,2}$ is the permutate of 1 and 2. So, there is $q \in P_\mathbb{Z}(T)$ such that

$$p(t) = (t_1 - t_2)q(t)$$

For any $t \in T$ such that $t_1 = t_3$,

$$q(t) = 0$$

So, by the same argument as the above, there is $r \in P_\mathbb{Z}(T)$ such that

$$q(t) = (t_1 - t_3)r(t)$$

By repeating this argument, we find that there is $\psi \in P_\mathbb{Z}(T)$ such that

$$\phi = D\psi$$

□

Theorem 6.6. *The followings hold.*

(i) For any $\phi \in C(U(n))^{Ad}$, $\Phi(\phi) := \phi|T \in C(T)_1$.

(ii) $\Phi : C(U(n))^{Ad} \to C(T)_1$ is surjective.

Proof of (i). It is clear. □

Proof of (ii). We set $G := U(n)$. Let us fix any $\phi \in C(T)_1$. For each $g \in G$, let denote the set of all eigenvalues of g by $\{\lambda_1(g), ..., \lambda_n(g)\}$. And

$$\psi(g) := \phi(\lambda_1(g), ..., \lambda_n(g))$$

Because ϕ is symmetric, ψ is well-defined. We will show ψ is continuous. Let us fix any $g_0 \in G$. Let denote $\lambda_1, ..., \lambda_m$ the distinct set of eigenvalues of g_0. Denote the degree of λ_i as zero point of characteristic polynomial of g by k_i.

By Rouche's Theorem(see [6]), for any $\epsilon > 0$, there is $\delta > 0$ such that g has just k_i eigenvalues(allow multiplicity) of g in $B(\lambda_i, \epsilon)$ for any $g \in B(g_0, \delta)$. So, ψ is continuous. Clearly, $\Phi(\psi) = \phi$. So, Φ is surjective. □

Theorem 6.7 (Cartan-Weyl Highest Weight Theory)**.** *The followings hold.*

(i) Let us assume (π, V) be a continuous irreducible unitary representation of $U(n)$ and λ be the highest weight of π. Then $\lambda \in (\mathbb{Z}^n)_+$ and the multiplicity of λ is 1.

(ii) Let us fix any $\lambda \in (\mathbb{Z}^n)_+$. Then there is the unique continuous irreducible unitary representation (π, V) whose highest weight is λ, ignoring isomorphism as continous unitary representation.

Proof of (i). (i) is from Weyl Character Formula(Theorem6.4) and Proposition6.26. □

Proof of (ii). The uniqueness is from Proposition6.19. We will show the existence. For aiming contradiction, let us assume that there exists $\lambda \in \mathbb{Z}^n_+$ such that λ is different from the highest weight of any irreducible continuous unitary representation of $U(n)$. We set

$$\rho := (n-1, ..., 1)$$

Because $A_{\lambda+\rho} \in R_{\mathbb{Z}}(T)_{sgn}$, by Theorem6.5 and Theorem6.6, there is $\psi \in C(U(n))^{Ad}$ such that $D(t)\psi = A_{\lambda+\rho}$. For any $\pi \in U\hat{(n)}$, by Weyl Integral Formula

$$\int_{U(n)} \chi_\tau(g)\bar{\psi}(g)dg = \int_T \chi_\tau(t)\bar{\psi}(t)|D(t)|^2 dt = \int_T A_{\alpha(\pi)+\rho}(t)\overline{A_{\lambda+\rho}(t)}dt = 0$$

Here, $\alpha(\pi)$ is the highest weight of π. By Theorem4.7, ψ is zero function. This is contradiction. □

6.4.8 Review

In this subsection, we show the result of classification of irreducible continuous unitary representations of $U(n)$. By Peter Weyl Theorem, it is enough to classify finite dimensional irreducible continuous unitary representation of $U(n)$.

We focus the set of all the set of all eigenvalues of $g \in U(n)$, $T := \mathbb{T}^n$. We can simplify discussions about $U(n)$ to discussions about T in some cases. In specialty, Weyl Integral Formula is really usefull.

Theorem 6.8 (Weyl Integral Formula)**.** *For any $f \in C(U(n))$,*

$$\int_{U(n)} f(g)d\mu_{U(n)}(g) = \frac{1}{n!}\int_{G/T}\int_T f(gtg^{-1})|D(t)|^2 d\mu_T t d\mu_{G/T}(gT)$$

By this theorem, we can simply integral of class function on $U(n)$ to simply integral of symmetric function on T. Let recall the proof of Weyl Integral Formula.

$$A : G/T \times T \ni (gT, t) \mapsto gtg^{-1} \in G$$

is $n!$-th covering map of G and $\mathfrak{S}_n$ acts on $A^{-1}(g)$ for each $g \in G$. That implies

$$\int_{U(n)} f(g)d\mu_{U(n)}(g) = \frac{1}{n!}\int_{G/T}\int_T f(gtg^{-1})|det(dA_{(gT,t)})|d\mu_T t d\mu_{G/T}(gT)$$

In the proof of this equation, we need take a good partition of unity of $U(n)$. By focusing the decomposition

$$\mathfrak{u}(n) = \mathfrak{u}(n)_1 \oplus \mathfrak{t}$$

and action on $\mathfrak{u}(n)_1$ and $\mathfrak{t}$, we get

$$det(dA_{(gT,t)}) = det(Ad(t^{-1})|\mathfrak{u}(n)_1 - id|\mathfrak{u}(n)_1)$$

Here,

$$\mathfrak{u}(n)_1 = \{X \in \mathfrak{u}(n) | X_{i,i} = 0 \ (\forall i)\}$$

By complexifying $\mathfrak{u}(n)_1$ and showing $E_{i,j}$ are eigenvector of the complexification of $Ad(t^{-1})|\mathfrak{u}(n)_1 - id|\mathfrak{u}(n)_1$ with respect to $(\frac{t_j}{t_i} - 1)$ $(\forall i \neq \forall j)$, we get

$$det(Ad(t^{-1})|\mathfrak{u}(n)_1 - id|\mathfrak{u}(n)_1) = |D(t)|^2$$

Consequently, we get Weyl Integral Formula. By Weyl Integral Formula and Shur Orthogonality Relation, we can simplify the classification of continuous finite dimensional irreducible unitary representations of $U(n)$ to the classification of $\{\chi_\pi|T | \pi$ is a continuous finite dimensional irreducible unitary representations of $U(n)\}$.

We focus the fact $D\chi_\pi|T$ is an alternating Laurant polynomial on T with $\mathbb{Z}$-coefficients. We can show $\{A_\alpha\}_{\alpha_1>...>\alpha_n}$ is an orthonomal system of $L^2(T)$ and a basis of $R_{\mathbb{Z}}(T)_{sgn}$. Here,

$$A_\alpha = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} sign(\sigma)t^{\sigma\cdot\alpha}, R_{\mathbb{Z}}(T)_{sgn} := \{p|p \text{ is an alternating Laurant polynomial on } T \text{ with } \mathbb{Z}\text{-coefficients.}\}$$

It is important that the decompositions of $D\chi_\pi$ with $\{A_\alpha\}_{\alpha_1>...>\alpha_n}$ correspoinds to the decompositions of $\pi|T$ as continuous unitary representation of T. The last decomposition is called a branching rule. Thanks to these insight, we can classify $\hat{U(n)}$ by the highest weight of each $\pi \in \hat{U(n)}$. In specialty, we get the following Weyl character formula.

Theorem 6.9 (Weyl character formula)**.** *Here are the settings and assumptions.*

(S1) T *is the maximal torus of* $U(n)$*.*

(S2) (π, V) *is a finite dimensional irreducible continuous representation of* G*.*

(S3) λ *is the highest weight of* π*.*

Then

(i)

$$\chi_\pi(t) = \frac{\sum_{\sigma\in\mathfrak{S}_n} sgn(\sigma)t^{\sigma\cdot(\lambda+\rho)}}{\Pi_{1\leq i<j\leq n}(t_i - t_j)}$$

Here, $\rho := (n-1, n-2, ..., 1, 0)$*.*

(ii) $dim(V_\lambda) = 1$*.*

Inversely, for each $\lambda \in (\mathbb{Z})^n_+ := \{\alpha \in (\mathbb{Z})|\alpha_1 \geq ... \geq \alpha\}$, there is $\psi \in C(U(n))^{Ad}$ such that $(\psi|T)D = A_{\lambda+\rho}$. Here $\rho := (n-1, ..., 0)$. That facts from the correspondance

$$U(n) \ni g \mapsto \frac{A_{\lambda+\rho}(\lambda_1(g), ..., \lambda_n(g))}{D(\lambda_1(g), ..., \lambda_n(g))} \in \mathbb{C}$$

By completeness of character about $\hat{U(n)}$, we can show there is $\pi \in \hat{U(n)}$ such that the highest weight of π is λ.

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