A study memo on triangularisability of matricies

1 Triangularisability of matricies

Proposition 1.1. *Let*

(S1) $m \in \mathbb{N} \cup [2, \infty)$

(S2) $f_1, ..., f_m \in \mathbb{C}[X] \setminus 0$.

(A1) $f_1, ..., f_m$ *don't have common divisor.*

then there are $h_1, ..., h_m \in \mathbb{C}[X]$ *such that*

$$\Sigma_{i=1}^{m} h_i f_i = 1 \tag{1}$$

Case when $m = 2$. When $\Sigma_{i=1}^{m} deg(f_i) = 0$, $deg(f_1) = deg(f_2) = 1$. In this case, the the claim in this Proposition holds. We assume the claim in this Proposition holds when $\Sigma_{i=1}^{m} deg(f_i) < K$. We can assume $deg(f_1) > 0$ There is $q, r \in \mathbb{C}[X]$ such that $f_1 = qf_2 + r$ and $deg(r) < deg(f_1)$ By the assumption of our mathematical induction, there are $h_1, h_2 \in \mathbb{C}[X]$ such that $h_1 r + h_2 f_2 = 1$. Because $r = qf_2 - f_1$, $-h_1 f_1 + (q + h_2) f_2 = 1$. □

Case when $m > 2$. We assume the claim in this Proposition holds when $m = K$. Let us set q is a maximum diviser of $f_1, ..., f_K$ and $g_i := \frac{f_i}{q_i}$ $(i = 1, 2, ..., K)$. Clearly, $g_1, ..., g_m$ don't have common divisor and f_{K+1} and q don't have common divisor. By the assumption of mathematical induction,
there are $h_1, ..., h_K, h_{K+1}, s \in \mathbb{C}[X]$ such that

$$\Sigma_{i=1}^{K} h_i g_i = 1 \tag{2}$$

and

$$sq + h_{K+1} f_{K+1} = 1 \tag{3}$$

Then $\Sigma_{i=1}^{K} h_i f_i = q$. Consequently,

$$\Sigma_{i=1}^{K} s h_i f_i + h_{K+1} f_{K+1} = 1 \tag{4}$$

□

Proposition 1.2. *Let*

(S1) $A \in M(n, \mathbb{C})$

then the followings hold.

(i) There is $P \in GL(n, \mathbb{C})$ and $\alpha_1, ..., \alpha_K \in \mathbb{C}$ such that

$$P^{-1}AP = \begin{pmatrix} \alpha_1 & * & * & & & & \mathit{0} \\ & \ddots & * & & & & \\ & & \alpha_1 & & & & \\ & & & \ddots & & & \\ & & & & \alpha_m & * & * \\ & & & & & \ddots & * \\ \mathit{0} & & & & & & \alpha_m \end{pmatrix} \quad (5)$$

(ii) If $\alpha_i \neq \alpha_j$ (for any $i \neq j$), A is diagonalizable.

STEP1. Existence of the minimal polynomial of A. Because $E, A, A^2, ..., A^{n^2}$ are linearly dependent, there are $a_0, a_2, ..., a_n$ such that

$$\Sigma_{i=0}^{n^2} a_i A^i = 0 \quad (6)$$

So there is a $\varphi_A \in \mathbb{C}[X]$ such that

$$deg(\varphi_A) = min\{deg(\varphi) | \varphi \in \mathbb{C}[X] \ and \ \varphi(A) = 0\} \quad (7)$$

□

STEP2. Decomposition of $\mathbb{C}^n$ into generalized eigenspaces. By fundamental theorem of algebra, there are distinct $\alpha_1, ..., \alpha_K \in \mathbb{C}$

$$\varphi_A(x) = \Pi_{i=1}^K (x - \alpha_i)^{m_i} \quad (8)$$

We set $f_i \in \mathbb{C}[X]$ by $f_i(x) := \dfrac{\varphi_A(x)}{(x - \alpha_i)^{m_i}}$ $(i = 1, 2, ..., K)$. By Proposition(), then there are $h_1, ..., h_m \in \mathbb{C}[X]$ such that

$$\Sigma_{i=1}^K h_i(A) f_i(A) = E \quad (9)$$

We set $W_{i,j} := \{x \in \mathbb{C}^n | (A - \alpha_i E)^j x = 0\}$ and $W_i := W_{i,m_i}$ $(j = 1, 2, ..., m_i)$ For any $x \in \mathbb{C}^n$, $x = \Sigma_{i=1}^K h_i(A) f_i(A) x$. For each i, $h_i(A) f_i(A) x \in W_i$. So

$$\mathbb{C}^n = \Sigma_{i=1}^K W_i \quad (10)$$

□

STEP3. $W_{i,k} \cap W_{j,l} = \{0\}$ $(i \neq j)$. We assume $k = l = 1$. Let us fix arbitary $x \in W_{i,1} \cap W_{j,1}$. Because $0 = Ax - Ax = \alpha_i x - \alpha_j x = (\alpha_i - \alpha_j)x$, $x = 0$. So $W_{i,1} \cap W_{j,1} = \{0\}$ $(i \neq j)$. Nextly we assume if $k + l \leq K$ then $W_{i,k} \cap W_{j,l} = \{0\}$ $(i \neq j)$. Let us fix arbitary i, j, k, l such that $i \neq j$. Let us fix arbitary $x_0 \in W_{i,k} \cap W_{j,l}$. We set $s : \mathbb{C}^n \ni x \mapsto [x] \in \mathbb{C}^n/W_{1,1}$. Because $AW_{1,1} \subset W_{1,1}$, $\tilde{A} : \mathbb{C}^n/W_{1,1} \ni [x] \mapsto [Ax] \in \mathbb{C}^n/W_{1,1}$ is well-definied and linear. We set $\tilde{W_{i,k}} := \tilde{A}s(W_{i,k})$ and $\tilde{W_{i,l}} := \tilde{A}s(W_{i,l})$ We can assume $k > 1$. Clearly $\tilde{W_{i,k}} \subset \{[x] \in \tilde{W_{i,k}} | (\tilde{A} - \alpha_i)^{k-1}[x] = 0\}$. So by the assumption of mathematical induction, $\tilde{W_{i,k}} \cap \tilde{W_{j,l}} = \{0\}$. This implies that $W_{i,k} \cap W_{j,l} \subset W_{i,1}$. Similarly, $W_{i,k} \cap W_{j,l} \subset W_{j,1}$. So $W_{i,k} \cap W_{j,l} \subset W_{i,1} \cap W_{j,1} = \{0\}$. □

STEP4. $\Sigma_{i=1}^{K} W_i = \oplus_{i=1}^{K} W_i$. By STEP3, $\Sigma_{i=1}^{2} W_i = \oplus_{i=1}^{2} W_i$. We assume if $K \leq K_0$ then $\Sigma_{i=1}^{K} W_i = \oplus_{i=1}^{K} W_i$. We will show if $K = K_0 + 1$ then $\Sigma_{i=1}^{K} W_i = \oplus_{i=1}^{K} W_i$. By the assumption of mathematicalinduction,

$$\Sigma_{i=1}^{K_0} W_i / W_{K_0+1} = \oplus_{i=1}^{K} W_i / W_{K_0+1} \tag{11}$$

Let us fix arbitary $w_i \in W_i$ $(i = 1, 2, ..., K_0 + 1)$ such that $\Sigma_{i=1}^{K_0+1} w_i = 0$. By (11), $w_i \in W_i \cap W_{K_0+1}$ $(i = 1, ..., K_0)$. By STEP3, $w_i = 0$ $(i = 1, ..., K_0)$. So $w_K = 0$. □

STEP5. Basis of W_i. Let us fix i. We pick up $w_{1,1}, w_{1,2}, ..., w_{1,n_1}$ which is a basis of the eigenspace $W_{i,1}$ corresponding to α_i. We set $s_1 : W_i \ni w \mapsto [w] \in W_i/W_{i,1}$. Because $AW_{i,1} \subset W_{i,1}$, if we set $A_1 : W_i/W_{i,1} \ni [w] \mapsto [Aw] \in W_i/W_{i,1}$ then A_1 is well-defined and linear. We denote V_1 by the eigenspace of A_1 correspondig to α_i. Clearly $s_1^{-1}(V_1) = W_{i,2}$. So there are $w_{1,n_1+1}, ..., w_{1,n_2}$ such that $s_1(w_{1,n_1+1}), ..., s_1(w_{1,n_2})$ is a basis of V_1. Clearly $w_{1,1}, w_{1,2}, ..., w_{1,n_2}$ is a basis of $W_{i,2}$. $Aw_i - \alpha_i w_i \in W_{i,1}$.

Similarly, we can take $w_{i,1}, ..., w_{i,n_1}, ..., w_{i,n_2}, ..., w_{i,n_{m_i}}$ such that $w_{i,1}, ..., w_{i,n_1}, ..., w_{i,n_2}, ..., w_{i,m_i}$ is a basis of W_i and $Aw_i - \alpha_i w_i \in W_{i,k}$ $(i = 1, 2, ..., K-1, n_k < i \leq n_{k+1})$. So the representation matrix of $A|W_i$ is an upper triangular matrix if $w_{i,1}, ..., w_{i,n_1}, ..., w_{i,n_2}, ..., w_{i,m_i}$ is a basis of W_i. □

STEP6. Showing (i). By STEP4, $\{w_{i,j}\}_{\{i=1,...,K, j=1,2,..,dim(W_i)\}}$ is a basis of $\mathbb{C}^n$. Clearly the representation matrix of A is an upper triangular matrix if $\{w_{i,j}\}_{\{i=1,...,K, j=1,2,..,dim(W_i)\}}$ is a basis of $\mathbb{C}^n$. □

STEP6. Showing (ii). (i) implies (ii). □

References

[1] Ichiro Satake, LINEAR ALGEBRA, ISBN-0 8247-1596-9.