A study memo on skewness and kurtosis

1 Skewness

Definition 1.1 (Skewness). *Let*

(S1) $\mu \in \mathscr{P}(\mathbb{R})$.

(A1) $\nu := E[\mu]$ *and* $\sigma^2 := V[\mu]$ *exist.*

Let us call $E[\frac{(x-\nu)^3}{\sigma^3}]$ *be the skewness of* μ.

Proposition 1.1. *Let*

(S1) f *is a probability density function on* $\mathbb{R}$.

(A1) $f(x) = f(-x)$ *a.e* $x > 0$.

(A2) $\int_{\mathbb{R}} |x|^i f(x)dx < \infty$ $(i = 1, 2)$.

(A3) $\int_{\mathbb{R}} xf(x)dx = 0$.

Then the skewness of the distribution from f *is zero.*

Proof. We denote S by the skewness of the distribution from f.

$$\begin{aligned} S &= \int_{\mathbb{R}} x^3 f(x)dx \\ &= \int_0^{\infty} x^3 f(x)dx + \int_{-\infty}^0 x^3 f(x)dx \\ &= \int_0^{\infty} x^3 f(x)dx + \int_{\infty}^0 (-y)^3 f(-y)(-1)dy \\ &= \int_0^{\infty} x^3 f(x)dx - \int_0^{\infty} y^3 f(y)dy \\ &= 0 \end{aligned} \tag{1}$$

□

Proposition 1.2. *Let*

(S1) f *is a probability density function on* $\mathbb{R}$.

(A1) $\int_{\mathbb{R}} |x|^i f(x)dx < \infty$ $(i = 1, 2, 3)$.

(S2) $d > 0$.

(A2) *For any* $\epsilon > 0$, *there is* $A, B, a, b \in \mathbb{R}$ *such that* $1 < A < B$ *and* $0 \leq a < b$ *and* $b \leq A$ *and* $(b-a) \leq (B-A)$ *and* $\frac{1}{b-a}\int_a^b xf(-x)dx \leq \frac{1}{B-A}\int_A^B xf(x)dx$ *and* $(A^2-1)\int_A^B xf(-x)dx - (b^2-1)\int_a^b xf(-x)dx \geq d$ *and* $|\int_0^\infty x^i f(x)dx - \int_A^B x^i f(x)dx| < \epsilon$ *and* $|\int_0^\infty x^i f(-x)dx - \int_a^b x^i f(x)dx| < \epsilon$ $(i = 1, 3)$.

(S3) *We denote the skewness of the distribution from* f *by* S.

Then $S \geq d$.

Proof.

$$\begin{aligned}
\int_0^\infty x^3 f(-x)dx &\leq \int_a^b x^3 f(-x)dx + \epsilon \\
&\leq \int_a^b x^3 f(-x)dx - \int_a^b xf(-x)dx + \int_a^b xf(-x)dx + \epsilon \\
&\leq \int_a^b (x^2-1)xf(-x)dx + \int_a^b xf(-x)dx + \epsilon \\
&\leq (b^2-1)\int_a^b xf(-x)dx + \int_0^\infty xf(-x)dx + 2\epsilon \\
&\leq (A^2-1)\int_A^B xf(-x)dx - d + \int_0^\infty xf(-x)dx + 2\epsilon \\
&\leq A^2\int_A^B xf(x)dx - d - \int_A^B xf(-x)dx + \int_0^\infty xf(x)dx + 2\epsilon \\
&\leq A^2\int_A^B xf(x)dx - d - \int_0^\infty xf(-x)dx + \int_0^\infty xf(x)dx + 3\epsilon \\
&\leq \int_A^B x^3 f(x)dx - d + 3\epsilon \\
&\leq \int_0^\infty x^3 f(x)dx - d + 4\epsilon \qquad (2)
\end{aligned}$$

So $S \geq d$. □

2 Kurtosis

Definition 2.1 (Kurtosis)**.** *Let*

(S1) $\mu \in \mathscr{P}(\mathbb{R})$.

(A1) $\nu := E[\mu]$ *and* $\sigma^2 := V[\mu]$ *exist.*

Let us call $E[\frac{(x-\nu)^4}{\sigma^4}] - 3$ *be the kurtosis of* μ *and denote it by* $Kurt(\mu)$.

Proposition 2.1. *The kurtosis of $N(\mu,\sigma)$ is 0.*

Proof. Let us denote by $C_\sigma := \dfrac{1}{\sigma\sqrt{2\pi}}$.

$$\begin{aligned} E_{N(\mu,\sigma)}[(x-\mu)^4] &= C_\sigma \int_{-\infty}^{\infty} (x-\mu)^4 exp(-\frac{1}{2}(\frac{x-\mu}{\sigma})^2)dx \\ &= C_\sigma \int_{-\infty}^{\infty} (x-\mu)^4 exp(-\frac{1}{2}(\frac{x-\mu}{\sigma})^2)dx \\ &= C_\sigma \int_{-\infty}^{\infty} -\sigma^2(x-\mu)^3\{exp(-\frac{1}{2}(\frac{x-\mu}{\sigma})^2)\}'dx \\ &= 3C_\sigma \int_{-\infty}^{\infty} -\sigma^2(x-\mu)^2 exp(-\frac{1}{2}(\frac{x-\mu}{\sigma})^2)dx \\ &= 3\sigma^4 \end{aligned} \tag{3}$$

□

Proposition 2.2. *For $\tau > 0$ let us denote kurtosis of $h_\tau := \frac{1}{2\tau}\chi_{[-\tau,\tau]}$ by $k(h_\tau)$. Then $\lim_{\tau\to 0} k(h_\tau) = \infty$ and $\lim_{\tau\to\infty} k(h_\tau) = -3$.*

Proof. Because $E[xf] = 0$,

$$k(h_\tau) + 3 = \frac{E[x^4 h_\tau]}{(E[x^2 h_\tau])^2} \tag{4}$$

The followings hold.

$$E[x^4 h_\tau] = \frac{2}{5}\tau^5 \tag{5}$$

and

$$E[x^2 h_\tau] = \frac{2}{3}\tau^3 \tag{6}$$

So there is constant $C > 0$

$$k(h_\tau) + 3 \sim C\frac{\tau^5}{(\tau^3)^2} = C\frac{1}{\tau} \ (\tau \to 0 \ or \ \tau \to \infty) \tag{7}$$

□

Proposition 2.3. *We set for $\epsilon > 0$ and $\delta > 0$*

$$f_{\epsilon,\delta}(x) = \begin{cases} \dfrac{1}{x^{(5+\delta)}} & if\ |x| > 1, \\ \dfrac{1}{\epsilon}(\dfrac{1}{2} - \dfrac{1}{4+\delta}) & if\ |x| \le \epsilon \\ 0 & otherwise \end{cases} \tag{8}$$

Then $f_{\epsilon,\delta}$ is a probability density function. Let us denote the kurtosis of $f_{\epsilon,\delta}$ by $k(f_\delta)$. Then the followings hold.

(i) Then for any $\epsilon > 0$ $\lim_{\delta \to 0} k(f_{\epsilon,\delta}) = \infty$.

(ii) For any $\delta > 0$ $\lim_{\epsilon \to 0} k(f_{\epsilon,\delta}) = \infty$.

Proof. Because

$$\int_1^\infty \frac{1}{x^{(5+\delta)}} dx = \frac{1}{4+\delta} \tag{9}$$

$f_{\epsilon,\delta}$ is a probability density function.

Because $E[x f_{\epsilon,\delta}] = 0$,

$$k(f_{\epsilon,\delta}) + 3 = \frac{E[x^4 f_{\epsilon,\delta}]}{(E[x^2 f_{\epsilon,\delta}])^2} \tag{10}$$

The followings holds.

$$\begin{aligned} E[x^2 f_{\epsilon,\delta}] &= 2(\int_0^\epsilon x^2 f_{\epsilon,\delta}(x)dx + \int_1^\infty x^2 f_{\epsilon,\delta}(x)dx) \\ &= 2(\frac{\epsilon^3}{3}(\frac{1}{2} - \frac{1}{4+\delta}) + \int_1^\infty \frac{1}{x^{(3+\delta)}}dx) \\ &= 2(\frac{\epsilon^3}{3}(\frac{1}{2} - \frac{1}{4+\delta}) + \frac{1}{(2+\delta)})) \end{aligned} \tag{11}$$

$$\begin{aligned} E[x^4 f_{\epsilon,\delta}] &= 2(\int_0^\epsilon x^4 f_{\epsilon,\delta}(x)dx + \int_1^\infty x^4 f_{\epsilon,\delta}(x)dx) \\ &= 2(\frac{\epsilon^5}{3}(\frac{1}{2} - \frac{1}{4+\delta}) + \int_1^\infty \frac{1}{x^{(1+\delta)}}dx) \\ &= 2(\frac{\epsilon^5}{3}(\frac{1}{2} - \frac{1}{4+\delta}) + \frac{1}{\delta})) \end{aligned} \tag{12}$$

So, if we fix δ then there is constant $C > 0$

$$k(f_{\epsilon,\delta}) + 3 \sim C\frac{\epsilon^5}{(\epsilon^3)^2} = C\frac{1}{\epsilon} \ (\epsilon \to 0) \tag{13}$$

and if we fix ϵ then there is constant $C > 0$

$$k(f_{\epsilon,\delta}) + 3 \sim C\frac{1}{\delta} \ (\delta \to 0) \tag{14}$$

Then (i) and (ii) hold. □

References

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