A study memo on weak law of large numbers and crude Monte Carlo Simulation

1 Law of large numbers

Proposition 1.1 (Weak law of large numbers). *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(A1)* $\{X_i\}_{i=1}^{\infty}$ *is a sequence of independent random variables on* $(\Omega, \mathscr{F}, P)$.
- *(A2) There is a* $\mu \in \mathscr{P}(\mathbb{R})$ *such that* $X_i \sim \mu(\forall i)$.
- *(A3)* $E[\mu] = \nu$ *and* $V[\mu] = \sigma^2$ *exist.*

then the followings hold.

- *(i)* $\{X_i\}_{i=1}^{\infty}$ *stochastic converges to* μ*, i.e., for any* $\epsilon > 0$

$$\lim_{n\to\infty} \mu(|\bar{X} - \mu| \geq \epsilon) = 0 \tag{1}$$

  Hereafter we denote stochastic convergence by $\xrightarrow[N\to\infty]{\mathrm{p}}$ *or* plim.

- *(ii) For any* $\epsilon > 0$,

$$\mu(|\bar{X} - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2} \tag{2}$$

A proof using Chebyshev's inequality. For any $n \in \mathbb{N}$,

$$\begin{aligned} \mu(|\bar{X} - \mu| \geq \epsilon) &= \frac{\epsilon^2 \mu(|\bar{X} - \mu|^2 \geq \epsilon^2)}{\epsilon^2} \\ &\leq \frac{1}{\epsilon^2} \int_{\{|\bar{X}-\mu|^2 \geq \epsilon^2\}} \epsilon^2 dP \\ &\leq \frac{1}{\epsilon^2} V[\bar{X}] = \frac{\sigma^2}{n\epsilon^2} \end{aligned} \tag{3}$$

This implies the above equation. □

A proof using Central limit theorem. By resetting $X_i \to \dfrac{X_i - \mu}{\sigma}$, we can assume $\mu = 0$ and $\sigma = 1$. Let us fix arbitary $\epsilon > 0$ and $\delta > 0$. There is $a > 0$ such that

$$N(0,1)((-\infty, -a) \cup (a, \infty)) < \delta \tag{4}$$

By Central limit theorem, there is $n_0 \in \mathbb{N}$ such that

$$\frac{a}{\sqrt{n_0}} < \delta \tag{5}$$

and for any $n \geq n_0$

$$|\mu(|\sqrt{n}\bar{X}| \geq a) - N(0,1)((-\infty,-a) \cup (a,\infty))| < \delta \tag{6}$$

So for any $n \geq n_0$

$$\begin{aligned} \mu(|\bar{X}| \geq \epsilon) &\leq \mu(|\bar{X}| \geq \frac{a}{\sqrt{n}}) = \mu(\sqrt{n}|\bar{X}| \geq a) \\ &\leq 2\delta \end{aligned} \tag{7}$$

So $\overline{\lim_{n\to\infty}}\mu(|\bar{X}| \geq \epsilon) \leq 2\delta$. Consequently, $\lim_{n\to\infty}\mu(|\bar{X}| \geq \epsilon) = 0$. □

2 Crude Monte Carlo method

Proposition 2.1. *Let*

- *(S1)* $(S := \{1,2,...,M\}, 2^{\Omega}, H)$ *is a probability space.*
- *(S2)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S3)* $\{X_n\}_{n=1}^{\infty}$ *is a sequence of independet random variables on* Ω *such that* $X_n(\Omega) \subset S$ *for any* $n \in \mathbb{N}$.
- *(A1)* $X_n \sim H$ *for any* $n \in \mathbb{N}$. $X_n \sim H)$ *means that* $P(\{X_n = i\}) = H(i)$
- *(S4)* g *is a function on* S.
- *(S5)* $\{Y_n\}_{n=1}^{\infty}$ *is a sequence of independet random variables on* Ω *such that* $Y_n(\Omega) \subset S$ *for any* $n \in \mathbb{N}$.
- *(A2)* $Y_n \sim C$ *for any* $n \in \mathbb{N}$. *Here,* C *is the counting measure of* S.

then

$$\operatorname*{plim}_{N\to\infty} \frac{\Sigma_{i=1}^{N} g(X_i)}{N} = \Sigma_{s\in S}\ g(s)H(\{s\}) = \#S \operatorname*{plim}_{N\to\infty} \frac{\Sigma_{i=1}^{N} g(Y_i)H(\{Y_i\})}{N} \tag{8}$$

STEP1. Showing (the left side)=(the middle side) . Clearly $\{g(X_n)\}_{n=1}^{\infty}$ is a sequence of independet random variables on Ω. By $(A1)$,

$$\int_{\Omega} g(X_n)dP = \Sigma_{s\in S}\ g(s)H(\{s\}) \tag{9}$$

and

$$\int_{\Omega} g(X_n)^2 dP = \Sigma_{s\in S}\ g^2(s)H(\{s\}) \tag{10}$$

So by weak law of large numbers (8) holds. □

STEP2. Showing (the right side)=(the middle side) . We set

$$G : S \ni s \mapsto g(s)H(\{s\})\#S \in \mathbb{R} \tag{11}$$

By applying the method of STEP1 to G and C,

$$\begin{aligned} \operatorname*{plim}_{N\to\infty} \frac{\Sigma_{i=1}^{N} g(Y_i)H(\{Y_i\})\#S}{N} &= \Sigma_{s\in S}\ g(s)H(\{s\})\#SC(\{s\}) \\ &= \Sigma_{s\in S}\ g(s)H(\{s\}) \end{aligned} \tag{12}$$

□

References

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