# A study memo on implicit function theorem and method of Lagrange multiplier

## Contents

# 1 Inverse function theorem

**Lemma 1.1.** *Let*

*(S1)* $I_b := (-b, b)^{n+1}$ *and* $J_b := (-b, b)^n$.

*(S2)* $a \in I_b$.

*(A1)* $f \in C^1(\bar{I}_b, \mathbb{R})$.

*(A2)* $f(a) = 0$.

*(A3)* $\alpha := inf_{x \in I_b} \frac{\partial f}{\partial x_1}(x)) > 0$.

*then* $r \in C^1(J_b, \mathbb{R})$ *such that*

$$f(r(y), y) = 0 \ (\forall y \in J_b) \tag{1}$$

*Proof.* By (A3), for any $y \in J_b$ there exists only one $r(y) \in (-b, b)$ such that $f(r(y), y) = 0$.

Let us fix arbitary $y \in J_b$ and fix arbitary $i \in \{2, , ..., n\}$.

For $z \in \mathbb{R}$ such that $|z|$ is sufficient small,

$$\begin{aligned}
0 &= 0 - 0 \\
&= f(r(y + z\boldsymbol{e}_i), y + z\boldsymbol{e}_i) - f(r(y + z\boldsymbol{e}_i), y) \\
&\quad + f(r(y + z\boldsymbol{e}_i), y) - f(r(y), y) \\
&= \int_0^1 \frac{d}{dt} f(r(y + z\boldsymbol{e}_i), y + tz\boldsymbol{e}_i) dt \\
&\quad + \int_0^1 \frac{d}{dt} f(r(y) + t(r(y + z\boldsymbol{e}_i) - r(y)), y) dt \\
&= z \int_0^1 \frac{\partial f}{\partial x_i}(r(y + z\boldsymbol{e}_i), y + tz\boldsymbol{e}_i) dt \\
&\quad + (r(y + z\boldsymbol{e}_i) - r(y)) \int_0^1 \frac{\partial f}{\partial x_1}(r(y) + t(r(y + z\boldsymbol{e}_i) - r(y)), y) dt
\end{aligned} \tag{2}$$

By (2), $(r(y + z\boldsymbol{e}_i) - r(y)) \leq |z| \frac{1}{\alpha} sup_{\bar{I}_b} |\frac{\partial f}{\partial x_i}|$. So $r$ is continuous on $J_b$.

By (A1) and (A3), $\int_0^1 \frac{\partial f}{\partial x_1}(r(y) + t(r(y + z\boldsymbol{e}_i) - r(y)), y) dt > 0$. So, by (2),

$$\frac{(r(y + z\boldsymbol{e}_i) - r(y))}{z} = \frac{(\int_0^1 \frac{\partial f}{\partial x_i}(r(y + z\boldsymbol{e}_i), y + tz\boldsymbol{e}_i) dt}{\int_0^1 \frac{\partial f}{\partial x_1}(r(y) + t(r(y + z\boldsymbol{e}_i) - r(y)), y) dt} \tag{3}$$

By (A1) and continuity of $r$ and (3),

$$\lim_{z \to 0} \frac{(r(y + z\boldsymbol{e}_i) - r(y))}{z} = \frac{\partial f}{\partial x_i}(r(y), y) \frac{\partial f}{\partial x_1}(r(y), y) \tag{4}$$

Consequently $r \in C^1(J_b, \mathbb{R})$. □

**Theorem 1.1** (Inverse function theorem)**.** *Let*

*(S1) $U$ is open set in $\mathbb{R}^n$.*

*(S2) $a \in U$.*

*(A1) $f \in C^1(U, \mathbb{R}^n)$.*

*(A2) $det(Jf(a)) > 0$ on $U$.*

*then there is $V \subset U$ such that $V$ and $f(U)$ are open set and $f : V \to f(V)$ is injective and $f^{-1} \in C^1(f(V), V)$.*

*STEP1: case when $n = 1$.* It is easy to show. □

*STEP2-1: f is locally injective(case when $n > 1$).* Let us fix arbitary $n_0 \in \mathbb{N}$. We assume the above theorem is true if $n \leq n_0$. Let us assume $n = n_0 + 1$. By (A2), for any $i \in \{1, 2, ..., n\}$ there is $u_i \in \mathbb{R}^n$ such that $Jf(a)u_i = e_i$. By setting for sufficient $b > 0$ $g : (-b, b)^n \ni \boldsymbol{t} \mapsto f(\sigma_{i=1}^n t_i v_i) \in \mathbb{R}^n$, We can assume $a = 0$ and $f(0) = 0$ and $[-c, c]^n \subset U$ for some $c > 0$. and

$$\frac{\partial f^i}{\partial x_i} > 0 \text{ on } I_c := (-c, c)^n \ (\forall i > 0) \tag{5}$$

By (5), clearly $f$ is injective on $I_c$.

□

*STEP2-2: f is locally open map(case when $n > 1$).* Next, we will show $f$ is open map in $I_c$ for sufficient small $c > 0$. And by Lemma1.1, there is $c' \in (0, c)$ and $r : C^1(J'_c, \mathbb{R})$ such that $f_1(r(y), y) = 0$ ($\forall y \in J_{c'}$). Here, $J'_c := (-c', c')^{n_0}$. By resetting $c$ to be sufficient smaller, we can assum that $c = c'$.

We set $\tilde{f} = (f_2, ..., f_n)$. Let us set $g : J_c \ni y \mapsto \tilde{f}(r(y), y)\mathbb{R}^{n_0}$. $Jg(0) = \begin{pmatrix} \boldsymbol{e}_2^T \\ \boldsymbol{e}_3^T \\ ... \\ \boldsymbol{e}_n^T \end{pmatrix} \begin{pmatrix} dr \\ \boldsymbol{e}_1^T \\ ... \\ \boldsymbol{e}_{n_0}^T \end{pmatrix} = E_{n_0}$. By the assumption in mathematical induction, there is $c'' \in (0, c)$ such that $g(J_{c''})$ is an open set in $\mathbb{R}^{n_0}$ and $g$ is injective on $J_{c''}$. By resetting $c$ to be sufficient smaller, we can assum that $c = c''$.

Let us fix arbitary connected open interval $(x_1, x_2) \times J \subset I_c$. We set $I := (x_1, x_2)$. We will show $f(I \times J)$ is open set. Let us fix arbitary point $f(x_0, y_0) \in f(I \times J)$. Because (5), For any $y \in J$, $f(x_1, y) < f(x_0, y) < f(x_2, y)$. Because $\bar{J}$ is compact, by (5), there is $d_1$ and $d_2$ such that

$$f_1(x_1, y) \leq d_1 < f_1(x_0, y) < d_2 \leq f_1(x_2, y) \ (\forall y \in J) \tag{6}$$

We set $W := (d_1, d_2) \times g(J)$. Cleary $f(x, y) \in W$. Because $g$ is open map, $W$ is open set. We will show $W \subset f(I \times J)$. Let us fix arbitary $(u, g(y)) \in I \times W$. Because $f_1(\cdot, y)$ is continuous and (6), by intermediate value theorem, there is $x \in I$ such that $f_1(x, y) = u$. So $f(x, y) = (f_1(x, y), g(y)) = (u, g(y))$. This means $W \subset f(I \times J)$. Consequently, $f$ is open map in $I_c$.

We replace $f$ by $f|I_c$.

□

*STEP2-3: For each $i$, $\frac{\partial f^{-1}}{\partial w_i}$ exists(case when $n > 1$).* Let us fix arbitary $(u_0, v_0) \in W$. Let us set $(x, y) := f^{-1}(u_0, v_0)$. By using an approach is same with one in STEP2-1, it is enough to show that for any $i$ $\frac{\partial f^{-1}}{\partial w_i}(x_0, y_0)$ exists we can assume that $Jf(x_0, y_0) = E_n$.

Let us fix arbitary $i \in \{1, 2, ..., n\}$. Let us pick up $j \in (\{1, 2, ..., n\} \setminus \{i\}$. By swaping $x_j$ by $x_1$ and swaping $f_j$ by $f_1$, we can assume $j = 1$. By using an approach is same with one in STEP2-2, there is $R \in C^1(J_c, \mathbb{R})$ such that

$$f_1(R(y), y) = u \ (\forall y \in J_c) \tag{7}$$

and $G : J_c \ni y \mapsto \tilde{f}(R(y), y) \in G(J_c)$ is injective and open map and class $C^1$ and $G^{-1}$ is class $C^1$.

For any $t$ such that $|t|$ is sufficient small,

$$f^{-1}((u, v) + t\boldsymbol{e}_i) = (R(G^{-1}((u, v) + t\boldsymbol{e}_i)), G^{-1}((u, v) + t\boldsymbol{e}_i)) \tag{8}$$

The right side of (8) is differentiable at $t = 0$.

$\frac{\partial f^{-1}}{\partial w_i}(x_0, y_0)$ exists. □

*STEP2-4 $f^{-1}$ is class $C^1$(case when $n > 1$).* Lastly, we will show $f^{-1} \in C^1(W, I_c)$.

By STEP2-3,

$$ff^{-1}(w) = w \ (\forall w \in W) \tag{9}$$

So,

$$\frac{\partial f^{-1}}{\partial w_i}(w) = Jf(f^{-1}(w))^{-1} e_i \ (\forall i, \forall w \in W) \tag{10}$$

The right side of (10) is continuous with respect to $w$. So $f^{-1}$ is class $C^1$. □

**Remark 1.1.** *There is a map which does not have global inverse map and has nonsingular Jacobi matrix at every point.* $f : (0, \infty) \times \mathbb{R} \ni (r, \theta) \mapsto r \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \end{pmatrix} \in \mathbb{R}^2 \setminus \{0\}$ *is a example of such maps.*

**Remark 1.2.** *[1] gives a sufficient condition for existence of global inverse map.*

The following proposition is easily proved by inverse mapping theorem.

**Corollary 1.1.** *(S1) $U$ is open set in $\mathbb{R}^n$.*

*(S2) $a \in U$.*

*(A1) $f \in C^1(U, \mathbb{R}^n)$.*

*(S3) $V$ is open set in $\mathbb{R}^n$ such that $f(U) \subset V$.*

*(A2) $g \in C^1(V, \mathbb{R}^n)$.*

*(A3) $g \circ f = id_U$.*

*then there is $f(U)$ is open set.*

# 2 Implicit function theorem

**Theorem 2.1** (Implicit function theorem)**.** *Let*

*(S1)* $U$ *is open set in* $\mathbb{R}^{m+n}$.

*(S2)* $a := (a_1, a_2, ..., a_m)$ *and* $b := (b_1, b_2, ..., b_n)$ *and* $c := (a, b)$.

*(S3)* $c \in U$.

*(A1)* $f \in C^1(U, \mathbb{R}^m)$.

*(A2)* $f(c) = 0$.

*(A3)* $det(\{\frac{\partial f_i}{\partial x_j}(a)\}_{1 \leq i,j \leq m}) \neq 0$.

*then there is an open subset in* $\mathbb{R}^n$ $V$ *and* $r \in C^1(V, \mathbb{R}^m)$ *such that* $b \in V$ *and* $r(b) = a$ *and*

$$(r(y), y) \subset U \text{ and } f(r(y), y) = 0 \ (\forall y \in V) \tag{11}$$

*Proof.* By resetting $Bf$ for $B = \{\frac{\partial f_i}{\partial x_j}(a)\}^{-1}_{1 \leq i,j \leq m}$, we can assume.

Let us set $g : U \ni (x, y) \mapsto (f(x), y) \in \mathbb{R}^{m+n}$. Because $det(Jg(c)) \neq 0$, by inverse function theorem, there is an open neighborhood of $c$ $U' := B(a, \epsilon) \times B(b, \epsilon) \subset U$ such that $g(U')$ is open subset and $g : U' \to g(U')$ is class $C^1$ homeomoriphisim.

We set $r : B(b, \epsilon) \ni g^{-1}(0, y) \in U'$. Clearly $r$ satisfies the conditions in the above theorem. □

# 3 Method of Lagrange multiplier

**Theorem 3.1** (Method of Lagrange multiplier)**.** *Let*

*(S1)* $U$ *is open set in* $\mathbb{R}^{m+n}$.

*(S2)* $a := (a_1, a_2, ..., a_m)$ *and* $b := (b_1, b_2, ..., b_n)$ *and* $c := (a, b)$.

*(S3)* $c \in U$.

*(S4)* $g \in C^1(U, \mathbb{R})$.

*(A1)* $f \in C^1(U, \mathbb{R}^m)$.

*(A2)* $f(c) = 0$.

*(A3)* $rank(Jf(c)) = m$.

*(A4)* $a$ *is a maximum point of* $g$ *in* $U$.

*then there is* $\lambda \in \mathbb{R}^m$ *such that*

$$dg(a) = \Sigma_{i=1}^{m} \lambda_i df_i(a) \tag{12}$$

*Proof.* By swapping variables, we can assume (A3) in Theorem2.1. We pick $r$ in Theorem2.1. We define $(s_1, s_2, ..., s_n)$ by $(s_1, s_2, ..., s_n) := \begin{pmatrix} Jr(b) \\ E_n \end{pmatrix}$ Clearly $dim < s_1, s_2, ..., s_n >= n$. So $dim < s_1, s_2, ..., s_n >^{\perp} = m$ Because $f(r(\cdot), \cdot) \equiv 0$ in $U$, $< df_1, df_2, ..., df_m > \subset < s_1, s_2, ..., s_n >^{\perp}$. By (A3), $< df_1, df_2, ..., df_m >=< s_1, s_2, ..., s_n >^{\perp}$. Because $g(r(\cdot), \cdot)$ achives maximum at $b$, $dg \in < s_1, s_2, ..., s_n >^{\perp}$. Consequnently, there is $\lambda \in \mathbb{R}^m$ such that

$$dg(a) = \Sigma_{i=1}^{m} \lambda_i df_i(a) \tag{13}$$

□

## References


[1] D. Gale, H. Nikaido, The Jacobian matrix and global univalence of mappings, Math. Annalen 159, 81-93(1965).