A study memo on a proof of the central limit theorem

Contents

1 Introduction

This memo is a study memo on a proof of the central limit theorem. In this memo, I will show the proof using characteristic functions.

Theorem 1.1 (Central limit theorem)**.** *Let*

(S1) $(\Omega, \mathscr{F}, P)$ *is a probability space.*

(S2) $\{X_i\}_{i=1}^{\infty}$ *is a sequence of random variables on* $(\Omega, \mathscr{F}, P)$*.*

(A1) $\exists\mu$ *such that* $X_i \sim \mu$ $(\forall i)$*. Bellow, we fix such* μ*.*

(A2) $\{X_i\}_{i=1}^{N}$ *are independent for any* $N \in \mathbb{N}$*.*

(A3) $E[\mu] = \nu$ *and* $V[\mu] = \sigma^2$ *and* $\sigma \neq 0$*.*

then $P_{\sqrt{n}\bar{X}}$ *weakly converges to* $N(0, 1)$*.*

2 Preliminaries

Notation 2.1 (The set of all probability measures on (R))**.** *Denote the set of all borel sets on* $\mathbb{R}$ *by* $\mathscr{B}(\mathbb{R})$*. Denote the set of all probability measures on* $\mathscr{B}(\mathbb{R})$ *by* $\mathscr{P}(R)$*.*

Notation 2.2 (order relation in $\mathbb{R}^n$)**.** *Let* $x, y \in \mathbb{R}^n$*. Denote* $x \leq y$ *(*$x < y$*) if* $x_i \leq y_i$ $(x_i < y_i)$ $(\forall i)$*.*

Definition 2.1 (A distribution of random variables)**.** *Let* $(\Omega, \mathscr{F}, P)$ *be a probability space and let* $X = (X_1, X_2, ..., X_n)$ *be random variables on* Ω*. We define* $P_X : \mathscr{B}(\mathbb{R}^n) \ni A \mapsto P(X^{-1}(A)) \in [0, 1]$*. We denote the distribution of* X *by* P_X*.*

Definition 2.2 (A distribution function of a probability measure)**.** *Let* $\mu \in \mathscr{R}(\mathbb{R}^n)$*. We define* $F_\mu : \mathbb{R}^n \ni x \mapsto \mu((-\infty, x_1] \times (-\infty, x_2] ... \times (-\infty, x_n]) \in \mathbb{R}$ *and we call* F_μ *the destribution function of* μ*.*

Notation 2.3 (Fourier transform)**.** *Let* $f \in L^1(\mathbb{R}^n)$*. Denote fourier transformation of f by* $\mathscr{F}(f)$ *and denote fourier inverse transformation of f by* $\mathscr{F}^{-1}(f)$*.*

Definition 2.3 (Weakly convergence of probability measures)**.** *Let*

(S1) $(\Omega, \mathscr{F}, P)$ *is a probability space.*

(S2) Let $\{\mu_n\}_{n=1}^{\infty} \in \mathcal{P}(\mathbb{R}^N)$*.*

(S3) Let $\mu \in \mathcal{P}(\mathbb{R}^N)$*.*

$\{\mu_n\}_{n=1}^{\infty}$ *is weakly converges to* μ *if* $\lim_{n\to\infty} F_{\mu_n}(x) = F_\mu(x)$ *for any point* x *at which* F_μ *is continuous. Denote this by* $\mu_n \Longrightarrow \mu$ $(n \to \infty)$

Definition 2.4 (Characteristic function of probability measure)**.** *Let*

(S1) $(\Omega, \mathscr{F}, P)$ *is a probability space.*

(S2) Let $\mu \in \mathcal{P}(\mathbb{R}^n)$.

then call $\varphi_\mu : \mathbb{R}^n \ni t \mapsto \int_{\mathbb{R}^n} exp(itx)d\mu(x) \in \mathbb{C}$ is the characteristic function of μ. Bellow, assume the characteristic function of μ denotes φ_μ unless otherwise noted.

Definition 2.5 (Characteristic function of random variables)**.** *Let*

(S1) $(\Omega, \mathscr{F}, P)$ is a probability space.

(S2) Let $X = (X_1, X_2, ..., X_n)$ be a vector of random variables on $(\Omega, \mathscr{F}, P)$.

then call $\varphi_X : \mathbb{R} \ni t \mapsto \int_\Omega exp(itX)dP \in \mathbb{C}$ is the characteristic function of X. Bellow, assume the characteristic function of X denotes φ_X unless otherwise noted.

Definition 2.6 (Tightness of probability measures)**.** *Let*

(S1) $(\Omega, \mathscr{F}, P)$ is a probability space.

(S2) Let $\{\mu_n\}_{n=1}^\infty \in \mathcal{P}(\mathbb{R}^N)$.

$\{\mu_n\}_{n=1}^\infty$ is tight if for any $\epsilon > 0$ there is a $M > 0$ such that

$$\mu_n(\{x \in \mathbb{R}^N||x| \leq M\}) \geq 1 - \epsilon \tag{1}$$

Definition 2.7 (Weakly compactness of probability measures)**.** *Let*

(S1) $(\Omega, \mathscr{F}, P)$ is a probability space.

(S2) Let $\{\mu_n\}_{n=1}^\infty \subset \mathcal{P}(\mathbb{R}^N)$.

$\{\mu_n\}_{n=1}^\infty$ is weakly compact if for any subsequence $\{\mu_{\alpha(n)}\}_{n=1}^\infty$ of $\{\mu_n\}_{n=1}^\infty$ there is a subsequence of $\{\mu_{\alpha(n)}\}_{n=1}^\infty$ which weakly comverges to a probability measure.

Definition 2.8 (Outer measure)**.** *Let*

(S1) X is a set.

$\Gamma : 2^X \to [0, \infty]$ is an outer measure on X if the followings hold.

(i) $\Gamma(\phi) = 0$

(ii) If $A \subset B$ then $\Gamma(A) \leq \Gamma(B)$

(iii) If $\{A_i\}_{i=1}^\infty \subset 2^X$ then $\Gamma(\cup_{i=1}^\infty A_i) \leq \Sigma_{i=1}^\infty \Gamma(A_i)$

3 Assumptions

In this note, we use the following propositions without proofs.

Proposition 3.1. *Let*

(S1) $(\Omega, \mathscr{F}, P)$ is a probability space.

(S2) X *is a* N*-dimensional vector of random variables on* $(\Omega, \mathscr{F})$.

(S3) Let μ_X *be a probability distribution of* X.

(S4) $f \in L^1(\Omega) \cup L^\infty(\Omega)$

then

$$\int_{\mathbb{R}^N} f d\mu_X = \int_\Omega f \circ X dP \tag{2}$$

Proposition 3.2. *For any* $\eta > 0$,

$$\mathscr{F}(exp(-\eta(\cdot)^2) = \frac{1}{\sqrt{2\eta}} exp(-\frac{(\cdot)^2}{4\eta}). \tag{3}$$

Proposition 3.3. *Let* Σ *be a positive definite symmetric matrix.*

$$\varphi_{N(0,\Sigma)}(\boldsymbol{t}) = exp(-\frac{\boldsymbol{t}^T \Sigma^2 \boldsymbol{t}}{2}) \tag{4}$$

Proposition 3.4. *Let*

(S1) Arbitrarily take $M > 0$ *and fix it.*

(S2) Let $f_n : \overline{D(0,M)} \ni z \mapsto (1 + \frac{z}{n})^n \in \mathbb{C}$, *where* $\overline{D(0,M)} := \{z \in \mathbb{C} | \; |z| \leq M\}$, $(n = 1, 2, ...)$.

then $\{f_n\}_{n=1}^\infty$ *uniformly converges to exp on* $\overline{D(0,M)}$.

Proposition 3.5. *Let*

(A1) Let $F : \mathbb{R} \mapsto \mathbb{R}$ *is monotone increasing.*

then $\{x \mid F$ *is not continuous at* $x\}$ *is at most countable.*

Proposition 3.6. *Let*

(S1) $(\Omega, \mathscr{F}, P)$ *is a probability space.*

(S2) Let $\{\mu_n\}_{n=1}^\infty \subset \mathcal{P}(\mathbb{R})$.

(A1) Let $\mu \in \mathscr{P}(\mathbb{R})$ *such that* $\mu_n \Longrightarrow \mu \; (n \to \infty)$.

then for any bounded continuous function $f : \mathbb{R} \mapsto \mathbb{R}$

$$lim_{n\to\infty} \int_\mathbb{R} f(x) d\mu_n(x) = \int_\mathbb{R} f(x) d\mu(x) \tag{5}$$

Proposition 3.7. *Let*

(S1) $(\Omega, \mathscr{F}, P)$ *is a probability space.*

(S2) μ *is a probability measure on* $\mathbb{R}$.

(A1) $E[\mu] = 0$ *and* $V[\mu] = 1$.

then $\varphi_\mu(s) = 1 - \frac{s^2}{2} + o(s^2) \; (s \to 0)$

The following propositions are used for only Section7 and Subsection8.2.

Proposition 3.8. *Let*

(S1) (X, d) *is a metric space.*

then there is a complete metric space $(\tilde{X}, \tilde{d})$ *and an isometry mapping* $i : (X, d) \to (\tilde{X}, \tilde{d})$ *such that* $i(X)$ *is dense in* $\tilde{X}$*. We call* $(\tilde{X}, \tilde{d})$ *is a completion of* (X, d)*.*

Proposition 3.9. *Let*

(S1) X *is a set.*

(S2) Γ *is an outer measure on* X*.*

(S3) $\mathfrak{M}_\Gamma := \{A \subset X |$ *if* $B \subset A$ *and* $C \subset A^c$ *then* $\mu(B) + \mu(C) = \mu(B \cup C)\}$*.*

then the followings holds.

(i) $\mathfrak{M}_\Gamma$ *is a* σ*-algebra.*

(ii) Γ *is a measure on* $\mathfrak{M}_\Gamma$*.*

Proposition 3.10. *Let*

(S1) (X, d) *is a compact metric space.*

then $C(X) \subset C_u(X)$*.*

Proposition 3.11. *Let*

(S1) $(X, d1)$ *is a compact metric space.*

(S2) $(Y, d2)$ *is a compact metric space.*

(A1) $f \in C(X, Y)$*.*

then $f(X)$ *is compact in* Y*.*

Proposition 3.12. $C_c(\mathbb{R}^n)$ *is dense in* $L^1(\mathbb{R}^n)$*.*

4 $L^1(\mathbb{R}^n)$

Proposition 4.1. *Let us fix* $\epsilon > 0$*. Then there is* $j_\epsilon \in C_c(\mathbb{R}^n) \cup C_+(\mathbb{R}^n)$ *such that*

(i) j_ϵ *is a probability density function on* $\mathbb{R}^n$*.*

(ii) *supp*$(j_\epsilon) \subset B(0, \epsilon)$*.*

The following proposition is easy to show.

Proposition 4.2. *Let*

(S1) j_ϵ *is the function in Proposition4.1.*

(S2) $f \in L^1(\mathbb{R}^n)$.

Then

(i) $j_\epsilon * f \in C^\infty(\mathbb{R}^n)$

(ii) $supp(j_\epsilon * f) \subset \{x \in \mathbb{R}^n | d(x, supp(j_\epsilon * f)) \leq \epsilon\}$

(iii) $||j_\epsilon * f||_1 \leq ||f||_1$

(iv) $\lim_{\epsilon \to 0} j_\epsilon * f = f$ *in* $L^1(\mathbb{R}^n)$.

(i) and (ii). It is easy to show. □

(iii) and (iv). It is enable to show by an approach which is similar to the approach in the proof of Proposition5.1. □

By (iv) of Proposition4.2 and Proposition3.12, the following holds.

Proposition 4.3. $C_c^\infty(\mathbb{R}^n)$ *is dense in* $L^1(\mathbb{R}^n)$.

Proposition 4.4. *Let*

(S1) $\{f_n\}_{n=1}^\infty \subset L^1(\mathbb{R}^n)$ *and* $f \in L^1(\mathbb{R}^n)$.

(A1) $\lim_{n\to\infty} f_n = f$ *in* $L^1(\mathbb{R}^n)$.

then $\lim_{n\to\infty} f_n = f$ *(almost everyware pointwise convergence).*

Proof. Let us fix arbitary $m \in \mathbb{N}$. We set

$$E_m := \{x \in \mathbb{R}^n | \underline{\lim}_{n\to\infty} |f_n(x) - f(x)| \geq \frac{1}{m}\} \tag{6}$$

It is enough to show E_m is zero set.

$$\frac{1}{m}\mu(E_m) \quad \leq \quad ||f_n - f||_1 \to 0$$

□

5 Fourier transform

Definition 5.1. *Let* $\epsilon > 0$ *and* $n \in \mathbb{R}$.

$$G_\epsilon(x) := \frac{1}{(2\pi\epsilon^2)^{\frac{n}{2}}} exp(-\frac{|x|^2}{2\epsilon^2}) \ (x \in \mathbb{R}^n) \tag{7}$$

Proposition 5.1. *The followings hold.*

(i) $G_\epsilon > 0$ *on* $\mathbb{R}^n$ $(\forall \epsilon > 0)$.

(ii) $\int_{\mathbb{R}^n} G_\epsilon dx = 1$.

(iii) *For any* $\delta > 0$, $\lim_{\epsilon\to 0}\int_{|x|>\delta} G_\epsilon dx = 0$.

(iv) *For any* $f \in L^1(\mathbb{R}^n)$, $||G_\epsilon * f||_1 \leq ||f||_1$.

(v) *For any* $f \in L^1(\mathbb{R}^n)$, $\lim_{\epsilon\to 0} G_\epsilon * f = f$ *in* $L^1(\mathbb{R}^n)$.

(vi) $\mathscr{F}^{-1}(\mathscr{F}(G_\epsilon)) = G_\epsilon$ $(\forall \epsilon > 0)$

(i) and (ii). Because G_ϵ is the probability denity function of $N(0, \epsilon E_n)$, (i) and (ii) hold. □

(iii). Because $\int_{|x|\leq\delta} G_\epsilon(x)dx = \int_{|x|\leq\frac{\delta}{\epsilon}} G_1(x)dx$, (iii) holds. □

(iv). By (i) and (ii),

$$\begin{aligned}\int_{\mathbb{R}^n} |G_\epsilon * g(x)|dx &= \int_{\mathbb{R}^n} |\int_{\mathbb{R}^n} G_\epsilon(x-y)g(y)dy|dx \\ &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} G_\epsilon(x-y)|g(y)|dydx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} G_\epsilon(x-y)dx|g(y)|dy \\ &= \int_{\mathbb{R}^n} |g(y)|dy\end{aligned}$$

□

(v). By (iv) and Proposition3.12, we can assume $f \in C_c(\mathbb{R}^n)$.

By Lebesugue's convergence theorem and (iv), it is enough to show $G_\epsilon * f$ pointwize converges to f.

Let us fix arbitary $\epsilon > 0$. Because f is uniform continuous on $\mathbb{R}^n$, $|f(x) - f(y)| < \frac{\epsilon}{2}$ (for any x, y such that $|x - y| < \delta$).

By (iii), there is $\tau_0 > 0$ such that $\int_{|x|>\delta} G_\tau dx < \frac{\epsilon}{2(2||f||_\infty+1)}$ (for any $\tau < \tau_0$).

By (ii), for any $x \in \mathbb{R}^n$

$$\begin{aligned}|G_\epsilon * f(x) - f(x)| &= |\int_{\mathbb{R}^n} G_\epsilon(y)(f(x-y) - f(x))dy| \\ &\leq \int_{|x|<\delta} G_\epsilon(y)|f(x-y) - f(x)|dy \\ &+ \int_{|x|\geq\delta} G_\epsilon(y)|f(x-y) - f(x)|dy \\ &\leq \frac{\epsilon}{2} + 2||f||_\infty \int_{|x|\geq\delta} G_\epsilon(y)dy \\ &\leq \epsilon \end{aligned} \tag{8}$$

□

(vi). By Proposition3.2, (vi) holds. □

Proposition 5.2 (Inverse formula)**.** *For any $f \in L^1(\mathbb{R}^n)$ such that $\mathscr{F}(f) \in L^1(\mathbb{R}^n)$,*

$$f = \mathscr{F}^{-1}(\mathscr{F}(f)) \tag{9}$$

Proof. By (v) in Proposition5.1 and Proposition4.4, it is enough to show $G_\epsilon * f$ pointwize converges to $\mathscr{F}^{-1}(\mathscr{F}(f))$ on $\mathbb{R}^n$.

By (vi) in Proposition5.1 and Proposition3.2, for any $x \in \mathbb{R}^n$

$$\begin{aligned}
G_\epsilon * f(x) &= \mathscr{F}^{-1}(\mathscr{F}(G_\epsilon)) * f(x) \\
&= \int_{\mathbb{R}^n} \mathscr{F}^{-1}(\mathscr{F}(G_\epsilon)(x-y))f(y)dy \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathscr{F}(G_\epsilon)(\xi)exp(i(x-y)\xi)d\xi f(y)dy \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathscr{F}(G_\epsilon)(\xi)exp(ix\xi)exp(-iy\xi)d\xi f(y)dy \\
&= \int_{\mathbb{R}^n} \mathscr{F}(G_\epsilon)(\xi)exp(ix\xi) \int_{\mathbb{R}^n} \frac{1}{(2\pi)^{\frac{n}{2}}} exp(-iy\xi)f(y)dyd\xi \\
&= \int_{\mathbb{R}^n} \mathscr{F}(G_\epsilon)(\xi)exp(ix\xi)\mathscr{F}(f)(\xi)d\xi \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} (2\pi)^{\frac{n}{2}} \mathscr{F}(G_\epsilon)(\xi)\mathscr{F}(f)(\xi)exp(ix\xi)d\xi \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} exp(-\frac{\epsilon^2}{2}|\xi|^2)\mathscr{F}(f)(\xi)exp(ix\xi)d\xi
\end{aligned} \tag{10}$$

By Lebesuge's convergence theorem,

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} exp(-\frac{\epsilon^2}{2}|\xi|^2)\mathscr{F}(f)(\xi)exp(ix\xi)d\xi \to \mathscr{F}^{-1}(\mathscr{F}(f))(x) \tag{11}$$

□

Proposition 5.3 (Differential formula)**.** *Let*

- *(S1)* $f \in C_c^\infty(\mathbb{R}^n)$.
- *(S2)* $\boldsymbol{\alpha} \in \mathbb{Z}^n \cup [0,\infty)^n$.
- *(S3)* $m := \Sigma_{i=1}^n \alpha_i$.

Then

- *(i)* $D^{\boldsymbol{\alpha}} f \in C_c^\infty(\mathbb{R}^n)$ *and*

$$\mathscr{F}(f)(D^{\boldsymbol{\alpha}} f) = (i\xi)^{\boldsymbol{\alpha}} \mathscr{F}(f) \tag{12}$$

- *(ii)* $\mathscr{F}(f) \in L^1(\mathbb{R}^n)$.

(i). It is enable to show by using integration by parts. □

(ii). It is enable to show by (i). □

6 Metric space

This section introduces definitions and propositions which are used for only Section7 and Subsection8.2.

6.1 the case of general metric space

Definition 6.1 (Totally bouded metric space)**.** *Let*

> *(S1)* (X, d) *is a metric space.*

(X, d) *is totally bounded if for any* $\epsilon > 0$ *there are finite points* $\{x_i\}_{i=1}^N$ *such that* $X = \cup_{i=1}^N B(x_i, \epsilon)$.

Proposition 6.1. *Let*

> *(S1)* (X, d) *is a metric space.*

then the followings are equivalent.

> *(i)* (X, d) *is a totally bounded metric space.*
>
> *(ii) For any sequence* $\{x_i\}_{i=1}^\infty \subset X$ *there is a subsequence* $\{x_{\varphi(i)}\}_{i=1}^\infty$ *which is a cauchy sequence.*

(i) $\implies$ *(ii).* It is easy to show. □

(ii) $\implies$ *(i).* Let us assume (X, d) is not totally bounded. Then there is $\epsilon > 0$ such that for any finite subset $\{x_i\}_{i=1}^N$ $X \supsetneq \cup_{i=1}^{\varphi(n)} B(x_i, \epsilon)$.

Let us fix $x_1 \in X$. Because $X \supsetneq B(x_1, \epsilon)$. Let us fix $x_2 \in X \setminus \cup_{i=1}^1 B(x_i, \epsilon)$. By repeating the procedure in the same way below, there is $\{x_i\}_{i=1}^\infty$ such that $x_{n+1} \notin \cup_{i=1}^n B(x_i, \epsilon)$ $(\forall n)$. Clearly $\{x_i\}_{i=1}^\infty$ does not contain subsequence which is a cauchy sequence. □

Proposition 6.2. *Let*

> *(S1)* (X, d) *is a totally bounded metric space.*

(X, d) *is separable.*

Proof. For each $n \in \mathbb{N}$, $\{x_{n,i}\}_{i=1}^{\varphi(n)}$ such that $X = \cup_{i=1}^{\varphi(n)} B(x_{n,i}, \frac{1}{n})$. Clearly $\{x_{n,i} | n \in \mathbb{N},\ 1 \leq i \leq \varphi(n)\}$ is dense in X. □

Proposition 6.3. *Let*

> *(S1)* (X, d) *is a separable metric space.*

(X, d) *is secound countable.*

Proof. Let us fix a countable dense set $\{x_n\}_{n=1}^{\infty}$ in X. Let us arbitary open covering $\{U_\lambda\}_{\lambda\in\Lambda}$.

We set $B := \{B(x_n, \frac{1}{m} | n \in \mathbb{N}$ and $m \in \mathbb{N}$ such that there is $B(x_n, \frac{1}{m} \subset U_\lambda$ for some $\lambda \in \Lambda\}$.

There is $\varphi : B \to \Lambda$ such that

$$b \subset U_{\varphi(b)} \ (\forall b \in B) \tag{13}$$

Clearly $\{U_{\varphi(b)} | b \ni B\}$ is countable.

Let us arbitary $x \in X$. There is $\lambda \in \Lambda$ such that $x \in U_\lambda$. There is $n \in \mathbb{N}$ such that $B(x, \frac{2}{n}) \subset U_\lambda$. There is m such that $d(x, x_m) < \frac{1}{n}$. We set $b := B(x_m, \frac{1}{n})$. Clearly $x \in b \subset U_\lambda$. So $x \in b \subset U_{\varphi(b)}$. Consequently, $X = \cup_{b\in B} U_{\varphi(b)}$ □

Proposition 6.4. *Let*

> *(S1)* (X, d) *is a metric space.*

then the followings are equivalent.

> *(i)* (X, d) *is compact.*
>
> *(ii)* (X, d) *is sequentially compact.*
>
> *(iii)* (X, d) *is totally bounded and complete.*

(i) $\Longrightarrow$ *(ii).* It is easy to show. □

(ii) $\Longleftrightarrow$ *(iii).* It is easy to show. □

(iii) and (ii) $\Longrightarrow$ *(i).* We assume X is totally bounded and complete and X is not compact.

By Proposition6.3 and Proposition6.2, X is second countable.

So there is a open set covering $\{U_i\}_{i=1}^{\infty}$ such that for any finite subset $A \subset \mathbb{N}$ $X \supsetneq \cup_{i\in A} U_i$. Then $\{x_i\}_{i=1}^{\infty}$ such that $x_{n+1} \notin \cup_{i=1}^{n} U_i$. By (ii), there is a subsequence $\{x_{\varphi(i)}\}_{i=1}^{\infty}$ such that

$$\lim_{i\to\infty} x_{\varphi(i)} =: x \in X \tag{14}$$

exists.

There is n such that $x \in U_n$. There is $\epsilon > 0$ such that $B(x, \epsilon) \subset U_n$. By (14), there is $\varphi(m) > n$ such that $x_{\varphi(m)} \in B(x, \epsilon) \subset U_n$. Because $x_{\varphi(m)} \notin \cup_{i=1}^{\varphi(m)-1} \supset U_n$, $x_{\varphi(m)} \in U_n$ and $x_{\varphi(m)} \notin U_n$. It implies contradiction. □

Proposition 6.5. *Let*

> *(S1)* (X, d) *is a metric space.*
>
> *(A1)* $A \subset X$ *is dense and totally bounded.*

then (X, d) *is totally bounded.*

Proof. Let fix arbitary sequence $\{x_i\}_{i=1}^{\infty} \subset X$. By (A1), there is a sequcence $\{a_i\}_{i=1}^{\infty} \subset A$ such that $d(x_i, a_i) < \frac{1}{i}$ $(\forall i)$. By (A1) and Proposition6.1, there is a cauchy sequence $\{a_{\varphi}(i)\}_{i=1}^{\infty} \subset A$. Let fix arbitary $\epsilon > 0$. There is $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{\epsilon}{3}$ and $d(a_{\varphi(i)}, a_{\varphi(j)}) < \frac{\epsilon}{3}$ $\forall i > n_0,\ \forall j > n_0$. For any $i > n_0$ and any $j > n_0$

$$\begin{aligned} d(x_{\varphi(i)}, x_{\varphi(j)}) &\leq d(x_{\varphi(i)}, a_{\varphi(i)}) + d(a_{\varphi(i)}, a_{\varphi(j)}) + d(a_{\varphi(j)}, x_{\varphi(j)}) \\ &\leq \frac{1}{\varphi(i)} + \frac{\epsilon}{3} + \frac{1}{\varphi(j)} \\ &< \epsilon \end{aligned}$$

So $\{x_{\varphi}(i)\}_{i=1}^{\infty}$ is a cauchy sequence. Consequently X is totally bounded. □

Proposition 6.6. *Let us set* $X := [0,1]^{\mathbb{N}}$. *Let us define* $d : X \times X \to [0, \infty)$

$$d(x,y) := \Sigma_{i=1}^{\infty} \frac{|y_i - x_i|}{2^i} \tag{15}$$

then (X, d) *is a compact metric space.*

Proof. Clearly (X, d) is a metric space. By Proposition6.4, it is enough to show X is sequential compact. Let us fix arbitary $\{x_i\}_{i=1}^{\infty} \subset X$. There is a subsequence $\{x_{\varphi(1,i)}\}_{i=1}^{\infty}$ and $y_1 \in [0,1]$ such that $\lim_{i\to\infty} x_{\varphi(1,i),1} = y_1$. There is a subsequence of $\{x_{\varphi(1,i)}\}_{i=1}^{\infty}$ $\{x_{\varphi(2,i)}\}_{i=1}^{\infty}$ and $y_2 \in [0,1]$ such that $\lim_{i\to\infty} x_{\varphi(2,i),i} = y_i$ $(i = 1, 2)$. By repeating the procedure in the same way below, we get $\varphi(1,i)\}_{n,i\in\mathbb{N}}$. We set $x_{\psi(i)} := x_{\varphi(i,i)}$ (for $i \in \mathbb{N}$) and $y := (y_1, y_2, ...)$. Clearly $\{x_{\psi(i)}\}_{i=1}^{\infty}$ converges to y. □

Proposition 6.7. *Let*

(S1) (X, d) *is a separable metric space.*

there is a metric $\tilde{d}$ *such that* (X, d) *is homeomorphic to* $(X, \tilde{d})$ *and* $(X, \tilde{d})$ *is totally bounded.*

Proof. $(X, min\{d, 1\})$ is a metric space and $(X, min\{d, 1\})$ is homeomorphic to (X, d). So we can assume (X, d) satisfies $0 \leq d \leq 1$.

Let us fix $\{x_i\}_{i=1}^{\infty} \subset X$ which is dense in X. We set $i : X \ni x \mapsto (d(x, x_i))_{i=1}^{\infty}[0,1]^{\mathbb{N}}$. Clearly $i : X \to i(X)$ is homeomorphism. By Proposition6.4 and Proposition6.6, $i(X)$ is totally bounded. □

Proposition 6.8. *Let*

(S1) (X, d) *is a separable metric space.*

then there is a compact metric space $(\tilde{X}, \tilde{d})$ *and an homeomorphic mapping* $i : (X, d) \to i(X) \subset \tilde{X}$ *such that* $i(X)$ *is dense in* $\tilde{X}$

Proof. This proposition is proved by Proposition6.7 and Proposition6.5 and Proposition6.4 and Proposition3.8. □

Proposition 6.9. *Let*

(S1) (X, d) *is a metric space.*

(S2) $A \subset X$.

(S3) $r > 0$.

Then there is $f \in C_+(X)$ *such that* $0 \le f \le 1$ *on* X *and* $f|A \equiv 1$ *and* $supp(f) \subset \{x|d(x, A) \le r\}$.

Proof. We set $f : \mathbb{R} \ni x \mapsto 1 - \frac{1}{r} min(r, d(x, A)) \in [0, 1]$. f satisfies the above condition. □

By Proposition, the following holds.

Proposition 6.10. *Let*

(S1) (X, d) *is a metric space.*

(A1) $A \subset X$ *and* $B \subset X$ *and* $d(A, B) > 0$.

then thre are $f \in C_+(X)$ *and* $g \in C_+(X)$ *such that* $0 \le f \le 1$ *on* X *and* $0 \le g \le 1$ *on* X *and* $f|A \equiv 1$ *and* $g|B \equiv 1$ *and* $d(\overline{supp(f)}, \overline{supp(g)}) > 0$.

6.2 the case of compact metric space

Proposition 6.11. *Let*

(S1) (X, d) *is a compact metric space.*

then $C(X)$ *is separable.*

Proof. By Proposition3.11, $C(X) \subset C_b(X)$. So it is enough to show $\{f \in C_+(X)|0 \le f \le 1 \text{ on } X\}$ is separable. By Proposition6.4, X is totally bounded. So for each $n \in \mathbb{N}$, there are $x_{n,1}, x_{n,2}, ..., x_{n,\varphi(n)}$ such that $X = \cup_{i=1}^{\varphi(n)} B(x_{n,i}, \frac{1}{n})$. By Proposition6.1, for each n and i and $m \in \mathbb{N}$ there is $f_{n,i,m} \in C_+(X)$ such that

$$f_{n,i,m}|B(x_{n,i}, \frac{1}{n}) \equiv 1 \tag{16}$$

and $supp(f_{n,i,m}) \subset B(x_{n,i}, \frac{1}{n} + \frac{1}{m})$ and

$$0 \le f_{n,i,m} \le 1 \tag{17}$$

on X.

We set $\Lambda := \{(n, i, m, q) \in \mathbb{N}^3 \times \mathbb{Q}|i \le \varphi(n)\}$. For each λ which is a finite subset of Λ, $g_\lambda := max\{q f_{n,i,m}|(n, i, m, q) \in \lambda\}$. Then $B := \{g_\lambda|\lambda \text{ a finite subset of } \Lambda\}$ is a countable set.

We will show $\bar{B} = \{f \in C_+(X)|0 \le f \le 1 \text{ on } X\}$. Let us fix arbitary $f \in \{f \in C_+(X)|0 \le f \le 1 \text{ on } X\}$ and $\epsilon > 0$. By Proposition3.10, there is $N \in \mathbb{N}$ such that

$$|f(x) - f(y)| < \frac{\epsilon}{2} \tag{18}$$

(for any x, y such that $d(x,y) < \frac{1}{N}$). There are $q_i \in \mathbb{Q} \cup [0,1]$ such that

$$|q_i - f(x_{2N,i})| < \frac{\epsilon}{2} \ (\forall i) \tag{19}$$

We set $g := max\{q_i f_{2N,i,2N} | i = 1, 2, ..., \varphi(2N)\}$. Clearly $g \in B$.

Let us fix arbitary $x \in X$. Because $X = \cup_{i=1}^{\varphi(2N)} B(x_{2N,i}, \frac{1}{2N})$, there is i such that $x \in B(x_{2N,i}, \frac{1}{2N})$.

By (16) and (18) and (19)

$$\begin{aligned} f(x) - \frac{\epsilon}{2} &< f(x_{2N,i} \\ &< q_i + \frac{\epsilon}{2} \\ &< q_i f_{2N,i,2N}(x) + \frac{\epsilon}{2} \\ &< g(x) + \frac{\epsilon}{2} \end{aligned} \tag{20}$$

So

$$f(x) - \epsilon < g(x) \tag{21}$$

There is j such that $g(x) = q_j f_{2N,j,2N}(x)$. By (17) and (18) and (19),

$$\begin{aligned} q_j f_{2N,j,2N}(x) &\leq q_j \\ &< f(x_{2N,j}) + \frac{\epsilon}{2} \\ &< f(x) + \epsilon \end{aligned} \tag{22}$$

So

$$|f(x) - g(x)| < \epsilon \tag{23}$$

Consequently, $\bar{B} = \{f \in C_+(X) | 0 \leq f \leq 1 \text{ on } X\}$

□

7 Finite measures on metric space

We introduce several definitions and propositions for only Section8.2.

7.1 several facts on metric space

The following three definitions are from [2].

Definition 7.1 (Elementary function family)**.** *Let*

(S1) (X, d) *is a metric space.*

$\mathscr{E} \subset Map(X, [0, \infty))$ *is called a family of elementary functions if the followings holds.*

(i) if $f, g \in \mathscr{E}$ *then* $f + g \in \mathscr{E}$*.*

(ii) if $f, g \in \mathscr{E}$ and $f \geq g$ then $f - g \in \mathscr{E}$.

(iii) if $f, g \in \mathscr{E}$ then $min\{f, g\} \in \mathscr{E}$.

Definition 7.2 (Elementary integral)**.** *Let*

(S1) (X, d) is a metric space.

(S2) $\mathscr{E} \subset Map(X, [0, \infty))$ is a elementary function family.

$l : \mathscr{E} \mapsto [0, \infty]$ is an elementary integral on $\mathscr{E}$ if the followings hold.

(i) if $f, g \in \mathscr{E}$ then $l(f + g) = l(f) + l(g)$

(ii) if $f, g \in \mathscr{E}$ and $f \leq g$ then $l(f) \leq (g)$

Definition 7.3 (Complete elementary integral)**.** *Let*

(S1) (X, d) is a metric space.

(S2) $\mathscr{E} \subset Map(X, [0, \infty))$ is a elementary function family.

(S3) $l : \mathscr{E} \mapsto [0, \infty]$ is an elementary integral.

l is a complete elementary integral if for any $\{f_n\}_{\{n=1\}}^{\infty}$ such that $\lim_{n\to\infty} f_n = f$ (pointwise convergence) and $f_n \leq f_{n+1}$ $(\forall n \in \mathbb{R})$ satisfies $\lim_{n\to\infty} l(f_n) = l(f)$

Definition 7.4 (Functional from elementary integral)**.** *Let*

(S1) (X, d) is a metric space.

(S2) $\mathscr{E} \subset Map(X, [0, \infty))$ is a elementary function family.

(S3) $l : \mathscr{E} \mapsto [0, \infty]$ is an elementary integral.

We define

$$L : \{\varphi : X \to [0, \infty)\} \ni \varphi \mapsto inf\{\Sigma_{i=1}^{\infty} l(\varphi_i) | \varphi_i \in \mathscr{E} \ (\forall i), \varphi \leq \Sigma_{i=1}^{\infty} \varphi_i\} \in [0, \infty] \tag{24}$$

Proposition 7.1. *Let*

(S1) (X, d) is a metric space.

(S2) $\mathscr{E} \subset Map(X, [0, \infty))$ is a elementary function family.

(S3) $l : \mathscr{E} \mapsto [0, \infty]$ is an elementary integral.

(A1) $[0, \infty)\mathscr{E} \subset \mathscr{E}$.

For any $\alpha > 0$ and $f \in \mathscr{E}$

$$l(\alpha f) = \alpha l(f) \tag{25}$$

Proof. Let us fix $q_1 \in (\alpha, \infty) \cap \mathbb{Q}$ and $q_2 \in (0, \alpha) \cap \mathbb{Q}$. $q_2 l(f) = l(q_2 f) \leq l(\alpha f) \leq l(q_1 f) = q_1 l(f)$. So $l(\alpha f) = \alpha l(f)$ □

Proposition 7.2 (Outer measure from elementary integral)**.** *Let*

(S1) (X, d) is a metric space.

(S2) $\mathscr{E} \subset Map(X, [0, \infty))$ *is a elementary function family.*

(S3) $l : \mathscr{E} \mapsto [0, \infty]$ *is an elementary integral.*

(S4) L *is the functional in Definition7.4.*

(S5) *We set* $\Gamma : 2^X \ni A \mapsto L(\chi_A)$.

then Γ *is outer measure on* X.

Proof. It is easy to show terms except (iii) in Definition2.8. So we will show only (iii) in Definition2.8. Let us fix $A_i{}_{i=1}^{\infty} \subset 2^X$.

Let us fix $\epsilon > 0$.

For each $i \in \mathbb{N}$, there are $\{\varphi_{i,j}\}_{j=1}^{\infty} \subset \mathscr{E}$ such that $\chi_{A_i} \le \Sigma_{j=1}^{\infty}\varphi_{i,j}$ and $\Sigma_{j=1}^{\infty} l(\varphi_{i,j}) \le \Gamma(A_i) + \frac{\epsilon}{2^i}$

So $\chi_{\cup_{i=1}^{\infty} A_i} \le \Sigma_{i=1,j=1}^{\infty}\varphi_{i,j}$.

$\Gamma(\cup_{i=1}^{\infty} A_i) \le \Sigma_{i=1,j=1}^{\infty} l(\varphi_{i,j}) \le \Sigma_{i=1}^{\infty}\Gamma(A_i) + \epsilon$

Consequently, (iii) holds.

□

Proposition 7.3. *Let*

(S1) (X, d) *is a metric space.*

(S2) $\mathscr{E} \subset Map(X, [0, \infty))$ *is a elementary function family.*

(S3) $l : \mathscr{E} \mapsto [0, \infty]$ *is an elementary integral.*

(S4) L *is the functional in Definition7.4.*

(S5) Γ *is the outer measure in Proposition7.2.*

(S6) $\mathfrak{M}_\Gamma$ *is the* σ*-algebra in Proposition3.9.*

(A1) $C_+(X) \subset \mathscr{E}$.

(A2) *If* A, B *are borel sets and* $d(A, B) > 0$ *then* $\mu(A) + \mu(B) = \mu(A \cup B)$.

then $\mathscr{B}(X) \subset \mathfrak{M}_\Gamma$.

Proof. Because $\mathfrak{M}_\Gamma$ is σ-algebra, it is enough to show that all closed sets are contained in $\mathfrak{M}_\Gamma$.

Let us fix closed set A. Let us subset B and C such that $A \subset B$ and $C \subset A^c$.

Because A is closed set, $C \subset \{x | d(x, A) > 0\}$.

For each $n \in \mathbb{N}$ we set $C_n := \{x \in C | d(x, A) > \frac{1}{n}\}$ and $D_n := \{x \in C | \frac{1}{n-1} \ge d(x, A) > \frac{1}{n}\}$.

The followings holds.

$$C = \cup_{n=1}^{\infty} D_n \tag{26}$$

$$C_N = \cup_{n=1}^{N} D_n \ (\forall N) \tag{27}$$

We assume $\Sigma_{n=1}^{\infty}\Gamma(D_n) < \infty$. Let us fix $\epsilon > 0$.

There is n_0 such that $\Sigma_{n=n_0}^{\infty}\Gamma(D_n) < \epsilon$.

Because $d(A, C_{n_0}) > 0$,

$$\begin{aligned}\Gamma(A)+\Gamma(C) &= \Gamma(A)+\Gamma(C_{n_0}\cup\cup_{n=n_0}^{\infty}D_n)\\ &\leq \Gamma(A)+\Gamma(C_{n_0})+\epsilon\\ &\leq \Gamma(A)+\Gamma(C_{n_0})+\epsilon\\ &= \Gamma(A\cup C_{n_0})+\epsilon\\ &\leq \Gamma(A\cup C)+\epsilon \end{aligned} \tag{28}$$

So if $\Sigma_{n=1}^{\infty}\Gamma(D_n) < \infty$ then $\Gamma(A) + \Gamma(C) = \Gamma(A \cup C)$.

We assume $\Sigma_{n=1}^{\infty}\Gamma(D_n) = \infty$. Then $\Sigma_{n=1}^{\infty}\Gamma(D_{2n}) = \infty$ or $\Sigma_{n=1}^{\infty}\Gamma(D_{2n-1}) = \infty$. We assume $\Sigma_{n=1}^{\infty}\Gamma(D_{2n}) = \infty$.

If $n_1 \neq n_2$ then $d(D_{n_1}, D_{n_2}) > 0$. So $\Gamma(C) \geq \Gamma(\cup_{n=1}^{\infty}D_{2n}) \geq \Sigma_{n=1}^{\infty}\Gamma(D_{2n}) = \infty$. So if $\Sigma_{n=1}^{\infty}\Gamma(D_{2n}) = \infty$ then $\Gamma(B) + \Gamma(C) = \Gamma(A \cup C) = \infty$.

Similary, if $\Sigma_{n=1}^{\infty}\Gamma(D_{2n-1}) = \infty$ then $\Gamma(B) + \Gamma(C) = \Gamma(A \cup C) = \infty$. □

Proposition 7.4. *Let*

- *(S1)* (X, d) *is a metric space.*
- *(S2)* $\mathscr{E} \subset Map(X, [0, \infty))$ *is a elementary function family.*
- *(S3)* $l : \mathscr{E} \mapsto [0, \infty]$ *is an elementary integral.*
- *(S4)* $\{f_n\}_{n=1}^{\infty} \subset \mathscr{E}$ *and* $f_n \geq f_{n+1}$ *on* X $(\forall n)$*.*
- *(A1) There is* $f \in \mathscr{E}$ *such that* $\lim_{n\to\infty} ||f_n - f||_{\infty} = 0$
- *(A2)* $\mathbb{R}\mathscr{E} \subset \mathscr{E}$

then

$$\lim_{n\to\infty} l(f_n) = l(f) \tag{29}$$

Proof. $|l(f) - l(f_n)| = l(f - f_n) \leq ||f - f_n||_{\infty} l(1) \to 0$ $(n \to \infty)$ □

Proposition 7.5. *Let*

- *(S1)* (X, d) *is a metric space.*
- *(S2)* $l : \mathscr{E} \mapsto [0, \infty]$ *is an elementary integral on* $\mathscr{E} := \{f|f$ *is non-negative borel measurable on* X $\}$*.*
- *(S3)* L *is the functional in Definition7.4.*
- *(S4)* $h_1, h_2 \in \mathscr{E}$*.*
- *(A1)* $d(supp(h_1), supp(h_2)) > 0$*.*

then $L(h_1 + h_2) = L(h_1) + L(h_2)$

Proof. Let us fix arbitary $\epsilon > 0$. Let us fix f and g in Proposition6.10.

Let us fix $\{\varphi_i\} \subset \mathscr{E}$ such that $h_1 + h_2 \leq \Sigma_{i=1}^{\infty}\varphi_i$ and $\Sigma_{i=1}^{\infty}l(\varphi_i) \leq L(h_1 + h_2) + \epsilon$.

By definition of f and g,

$$h_1 + h_2 \leq (f + g)\Sigma_{i=1}^{\infty}\varphi_i \tag{30}$$

and

$$h_1 \leq f\Sigma_{i=1}^{\infty}\varphi_i \tag{31}$$

and

$$h_2 \leq g\Sigma_{i=1}^{\infty}\varphi_i \tag{32}$$

So

$$\begin{aligned} L(h_1+h_2)+\epsilon &\geq \Sigma_{i=1}^{\infty} l(\varphi_i) \\ &\geq \Sigma_{i=1}^{\infty}(l(f\varphi_i)+\Sigma_{i=1}^{\infty} l(g\varphi_i)) \\ &\geq L(h_1)+L(h_2) \end{aligned} \tag{33}$$

Consequently

$$L(h_1)+L(h_2) \leq L(h_1+h_2) \tag{34}$$

□

Proposition 7.6. *Let*

- *(S1)* (X, d) *is a metric space.*
- *(S2)* $l : \mathscr{E} \mapsto [0, \infty]$ *is an elementary integral on* $C_+(X)$*.*
- *(S3)* L *is the functional in Definition7.4.*
- *(S4)* Γ *is the outer measure in Proposition7.2.*
- *(S5)* $\mathfrak{M}_\Gamma$ *is the* σ*-algebra in Proposition3.9.*

then $\mathscr{B}(X) \subset \mathfrak{M}_\Gamma$*.*

Proof. Let us fix arbitary borel sets A, B such that $d(A, B) > 0$.

By Proposition7.5, $\Gamma(A \cup B) = L(\chi_{A\cup B}) = L(\chi_A + \chi_B) = L(\chi_A) + L(\chi_B) = \Gamma(A) + \Gamma(B)$.

By Proposition7.3, $\mathscr{B}(X) \subset \mathfrak{M}_\Gamma$. □

7.2 several facts on compact metric spaces

Proposition 7.7. *Let*

- *(S1)* (X, d) *is a compact metric space.*
- *(S2)* l *is an elementary integral on* $C_+(X)$*.* $C_+(X) := \{f \in C(X) | f \geq 0\}$

then there is an unique measure μ *on* $(X, \mathscr{B}(X))$ *such that for any* $f \in C_+(X)$

$$l(f) = \int_X f\mu \tag{35}$$

Existence. Let us fix $f \in C_+(X)$.

By replacing f by $||f||_\infty - f$, it is enough to show

$$\int_X f d\mu l(f) \leq l(f) \tag{36}$$

By an argument similar to one in the proof of Proposition8.4, there are $a_{m,i_{1\leq m\leq\infty,1\leq i\leq\varphi(m)}} \subset \mathbb{R}$ such that

$$0 = a_{m,1} \leq a_{m,2} \leq ... \leq a_{m,\varphi(m)} > ||f||_\infty \ (\forall m \in \mathbb{N}) \tag{37}$$

$$|a_{m,i} - a_{m,i+1}| \leq \frac{1}{2^m} \ (\forall m, \forall i) \tag{38}$$

$$\mu(\{f = a_{m,i}\}) = 0 \ (\forall m, \forall i) \tag{39}$$

We set

$$h_m := \Sigma_{i=1}^{\varphi(m)} a_{m,i}\chi_{[a_{m,i},a_{m,i+1})} \ (m \in \mathbb{N}) \tag{40}$$

and

$$h_{m,n} := \Sigma_{i=1}^{\varphi(m)} a_{m,i}\chi_{(a_{m,i}+\frac{1}{n},a_{m,i+1}-\frac{1}{n})} \ (m \in \mathbb{N},\ 1 \leq i \leq \varphi(m)) \tag{41}$$

Let us fix $\epsilon > 0$.

By Proposition3.10, $f \in C_u(X)$.

By (39), there is m, n such that

$$|\int_X f d\mu - \int_X h_{m,n} d\mu| < \epsilon \tag{42}$$

Because $f \in C_u(X)$, if $i \neq j$ then $d(f^{-1}((a_{m,i} + \frac{1}{n}, a_{m,i+1} - \frac{1}{n})), f^{-1}((a_{m,j} + \frac{1}{n}, a_{m,j+1} - \frac{1}{n}))) > 0$.

So

$$l(f) \geq L(h_{m,n} \geq \int_X h_{m,n} d\mu \tag{43}$$

Therefore,

$$\int_X f d\mu - \epsilon \leq l(f) \tag{44}$$

Consequently,

$$\int_X f d\mu \leq l(f) \tag{45}$$

□

Uniqueness. Let us fix arbitary $\mu_1 \in \mathscr{P}(X)$ and arbitary $\mu_2 \in \mathscr{P}(X)$ such that

$$\int_X f d\mu_1 = \int_X f d\mu_2 \ (\forall f \in C_+(X)) \tag{46}$$

We set $\mathscr{B} := A \in \mathscr{B}(X) | \mu_1(A) = \mu_2(A)$. Clearly $\mathscr{B}$ is σ-algebra.
Let us fix closed set A.
By Proposition6.1, there are $\{f_m\}_{m=1}^{\infty} \subset C_+(X)$ such that

$$||f_m||_\infty \le 1 \ (\forall m) \tag{47}$$

and

$$lim_{m\to\infty} f_m = \chi_A \ (pointwize\ convergence) \tag{48}$$

By Lebesugue's convergence theorem, $\mu_1(A) = \mu_2(A)$.
So $A \in \mathscr{B}$.
Consequently $\mathscr{B} \subset \mathscr{B}(X)$.

□

8 Weak convergence of probability distributions

8.1 the case of single variate

Proposition 8.1 (Helly's selection theorem)**.** *Let*

(S1) $(\Omega, \mathscr{F}, P)$ is a probability space.

(S2) Let $\{\mu_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathbb{R})$ and denote F_{μ_n} by F_n $(n = 1, 2, 3, ...)$.

Then there is a subsequence $\{F_{\alpha(n)}\}_{n=1}^{\infty}$ and $F : \mathbb{R} \to [0, \infty)$ such that F is monotone increasing and right continuous, and $F_{\alpha(n)}(x) \to F(x)$ for any point x at which F is continuous.

Proof. There is $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ such that $\overline{\{x_n\}_{n=1}^{\infty}} = \mathbb{R}$. Let fix such $\{x_n\}_{n=1}^{\infty}$. Because $0 \le F_n(x_m) \le 1$ (for any m, n in $\mathbb{N}$), there is a subsequence $\{\alpha(n)\}_{n=1}^{\infty} \subset \mathbb{N}$ and $\{F(x_n)\}_{n=1}^{\infty} \subset [0, 1]$ such that $F_{\alpha(m)}(x_n) \to F(x_n)$ $(m \to \infty)$. We fix such $\{\alpha(n)\}_{n=1}^{\infty}$ and $F(x_n)_{n=}^{\infty}$ We define $F(x) := inf_{m \in \{k | x \le x_k\}} F(x_m)$. By the definition of F, F is right continuous and monotone increasing. Arbitrarily take $x \in \mathbb{R}$ at which F is continuous and fix it. Arbitrarily take $\epsilon > 0$ and fix it. Let pick $x_{\alpha(m1)}$ and $x_{\alpha(m2)}$ such that $x_{\alpha(m1)} \le x \le x_{\alpha(m2)}$ and $(F(x_{\alpha(m2)}) - F(x_{\alpha(m1))}) \le \frac{\epsilon}{8}$. There is a $n_0 \in \mathbb{N}$ such that $|F_n(x_{\alpha(m1)}) - F(x_{\alpha(m1)})| \le \frac{\epsilon}{8}$ and $|F_n(x_{\alpha(m2)}) - F(x_{\alpha(m2)})| \le \frac{\epsilon}{8}$ for any $n \ge n_0$. Let fix such n_0 and $m1$ and $m2$. For any $n \ge n_0$

$$\begin{aligned} |F_n(x_{\alpha(m1)}) - F(x)| &\le |F_n(x_{\alpha(m1)}) - F(x_{\alpha(m1)})| + |F(x_{\alpha(m1)}) - F(x)| \\ &\le \frac{\epsilon}{4} \end{aligned} \tag{49}$$

and

$$
\begin{aligned}
|F_n(x_{\alpha(m2)}) - F(x)| &\leq |F_n(x_{\alpha(m1)}) - F(x_{\alpha(m1)})| + |F(x_{\alpha(m1)}) - F(x)| \\
&\leq \frac{\epsilon}{4} \qquad (50)
\end{aligned}
$$

So for any $n \geq n_0$

$$|F_n(x_{\alpha(m1)}) - F_n(x_{\alpha(m2)})| \leq \frac{\epsilon}{2} \qquad (51)$$

Arbitrarily take $n \geq n_0$ and fix it. Because $F_n(x_{m1}) \leq F_n(x) \leq F_n(x_{m2})$,

$$max\{|F_n(x_{\alpha(m1)}) - F_n(x)|, |F_n(x_{\alpha(m2)}) - F_n(x)|\} \leq \frac{\epsilon}{2} \qquad (52)$$

By (49) and (50) and (52),

$$|F_n(x) - F(x)| \leq \epsilon \qquad (53)$$

□

Proposition 8.2. *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S2) Let* $\{\mu_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathbb{R})$.

If $\{\mu_n\}_{n=1}^{\infty}$ *is tight then* $\{\mu_n\}_{n=1}^{\infty}$ *is weakly compact.*

Proof. By Proposition8.1, there is $F : \mathbb{R} \to [0, \infty)$ such that F is monotone increasing and right continuous, and for any point x at which F is continuous

$$F_{\alpha(n)}(x) \to F(x) \ (n \to \infty) \qquad (54)$$

Here we denote F_{μ_n} by F_n. Because of tightness of $\{\mu_n\}_{n=1}^{\infty}$, $limit_{x\to\infty}(F(x) - F(-x)) = 1$. So there is a probability measure μ such that F is a distribution function of μ. By (54), $\mu_n \Longrightarrow \mu \ (n \to \infty)$. □

Proposition 8.3. *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S2) Let* $\{\mu_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathbb{R})$. *and* $\mu \in \mathcal{P}(\mathbb{R})$
- *(A1)* $\mu_n \Longrightarrow \mu \ (n \to \infty)$.
- *(A2) Let* f *be an arbitary bouded continuous function on* $\mathbb{R}$.

then

$$\lim_{n\to\infty} \int f d\mu_n(x) = \int f d\mu(x) \qquad (55)$$

Proof. Let us fix arbitary $f \in C_b(\mathbb{R})$ and $\epsilon > 0$.

Because $\mu(\mathbb{R}) = 1$ and $\mathbb{R} = \cup_{a \in \mathbb{R}} a$, for each $n \in \mathbb{N}$ $\{a \in \mathbb{R} | \mu(a) > \frac{1}{n}\}$ is finite. So $\{a \in \mathbb{R} | \mu(a) > 0\}$ is at most coutable.

So there is $r_1 > 0$ and $r_2 > 0$ such that

$$1 - \mu((-r_1, r_2)) < \frac{\epsilon}{3(||f||_\infty + 1)} \tag{56}$$

and $\mu(-r_1) = 0$ and $\mu(-r_2) = 0$.

Because f is uniformly continuous on X,

So there are $a_{m,i_{1 \leq m \leq \infty, 1 \leq i \leq \varphi(m)}} \subset \mathbb{R}$ such that

$$-r_1 = a_{m,1} \leq a_{m,2} \leq ... \leq a_{m,\varphi(m)} = r_2 \ (\forall m \in \mathbb{N}) \tag{57}$$

and

$$|a_{m,i} - a_{m,i+1}| \leq \frac{1}{2^m} \ (\forall m, \forall i) \tag{58}$$

and

$$\mu(\{a_{m,i}\}) = 0 \ (\forall m, \forall i) \tag{59}$$

For each $m \in \mathbb{N}$, set $f_m := \Sigma_{i=1}^{\varphi(m)} f(a_i) \chi_{[a_i, a_{i+1})}$.

Because $\lim_{m \to \infty} f_m = f$ (pointwize convergence), by Lebesugue's convergence theorem there is $m \in \mathbb{N}$ such that

$$|\int_{-r_1}^{r_2} f_m \mu - \int_{-r_1}^{r_2} f \mu| < \frac{\epsilon}{3} \tag{60}$$

Because

$$\int_{-r_1}^{r_2} f_m \mu = \Sigma_{i=1}^{\varphi(m)} f(a_i) \mu([a_i, a_{i+1})) \tag{61}$$

and

$$\int_{-r_1}^{r_2} f_m \mu_n = \Sigma_{i=1}^{\varphi(m)} f(a_i) \mu_n([a_i, a_{i+1})) \ (\forall n) \tag{62}$$

So there is n_0 such that

$$|\int_{-r_1}^{r_2} f_m \mu_n - \int_{-r_1}^{r_2} f_m \mu| < \frac{\epsilon}{3} \ (\forall n \geq n_0) \tag{63}$$

By (56) and (60) and (63),

$$|\int_{\mathbb{R}} f \mu_n - \int_{\mathbb{R}} f \mu| < \epsilon \ (\forall n \geq n_0) \tag{64}$$

□

8.2 the case of multi variates

Definition 8.1 (Weak convergence)**.** *Let*

- *(S1)* (X, d) *is a metric space.*
- *(S2)* $\{\mu_n\}_{n=1}^{\infty} \subset \mathscr{P}(X)$.
- *(S3)* $\mu \in \mathscr{P}(X)$.

We say $\{\mu_n\}_{n=1}^{\infty}$ *weakly converges to* μ *if for any borel set* A *such that* $\mu(\partial(A)) = 0$ $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ *Denote* $\mu_n \Longrightarrow \mu$ *by weak convergence.*

The following proposition gives the equivalent definition of weak convergence.

Proposition 8.4. *Let*

- *(S1)* (X, d) *is a metric space.*
- *(S2)* $\{\mu_n\}_{n=1}^{\infty} \subset \mathscr{P}(X)$.
- *(S3)* $\mu \in \mathscr{P}(X)$.

then the followings are equivalent.

- *(i)* $\mu_n \Longrightarrow \mu$.
- *(ii) Set* $C_b(X) := \{f \in C(X) | ||f||_\infty < \infty\}$. *For any* $f \in C_b(X)$

$$\lim_{n\to\infty} \int f d\mu_n = \int f d\mu \tag{65}$$

- *(iii) Set* $C_u(X) := \{f \in C(X) |\ f$ *is uniformly continuous on* $X\}$. *For any* $f \in C_b(X) \cap C_u(X)$

$$\lim_{n\to\infty} \int f d\mu_n = \int f d\mu \tag{66}$$

- *(iv) For any closed set* A

$$\overline{\lim_{n\to\infty}} \mu_n(A) \leq \mu(A) \tag{67}$$

- *(v) For any closed set* U

$$\underline{\lim_{n\to\infty}} \mu_n(U) \geq \mu(U) \tag{68}$$

(i) $\Longrightarrow$ *(ii):* Let fix arbitary $f \in C_b(X)$. Because $\cup_{a\in\mathbb{R}}\{f = a\} = X$ and $\mu(X) = 1$, for any $n \in \mathbb{N}$ $\{a \in \mathbb{R}|\mu(\{f = a\}) > \frac{1}{n}\}$ is a finite set. So $\{a \in \mathbb{R}|\mu(\{f = a\}) > 0\} = \cup_{n=1}^{\infty}\{a \in \mathbb{R}|\mu(\{f = a\}) > \frac{1}{n}\}$ is at most countable.

So there are $a_{m,i_{1\leq m\leq\infty,1\leq i\leq\varphi(m)}} \subset \mathbb{R}$ such that

$$-||f||_\infty > a_{m,1} \leq a_{m,2} \leq ... \leq a_{m,\varphi(m)} > ||f||_\infty \ (\forall m \in \mathbb{N}) \tag{69}$$

$$|a_{m,i} - a_{m,i+1}| \leq \frac{1}{2^m} \ (\forall m, \forall i) \tag{70}$$

$$\mu(\{f = a_{m,i}\}) = 0 \ (\forall m, \forall i) \tag{71}$$

For $m \in \mathbb{N}$ set

$$g_m := \Sigma_{i=1}^{\varphi(m)} a_{m,i+1} \chi_{\{a_{m,i} \leq f \leq a_{m,i+1}\}} \tag{72}$$

and

$$h_m := \Sigma_{i=1}^{\varphi(m)} a_{m,i} \chi_{\{a_{m,i} \leq f \leq a_{m,i+1}\}} \tag{73}$$

Because for any m and i $\partial\{a_{m,i} \leq f \leq a_{m,i+1}\} \subset \{f = a_{m,i}\} \cup \{f = a_{m,i+1}\}$, for any m and i

$$\mu(\partial\{a_{m,i} \leq f \leq a_{m,i+1}\}) = 0 \tag{74}$$

Let fix arbitary $\epsilon > 0$.

By Lebesugue's convergence theorem, there is $m \in \mathbb{N}$ such that $\int g_m d\mu - \int h_m d\mu \leq \epsilon$.

By (i),

$$\begin{aligned} \int f d\mu - \epsilon &\leq \int h_m d\mu \\ &= \lim_{n\to\infty} \int h_m d\mu_n \\ &\leq \underline{\lim}_{n\to\infty} \int f d\mu_n \end{aligned} \tag{75}$$

and

$$\begin{aligned} \int f d\mu + \epsilon &\geq \int g_m d\mu \\ &= \lim_{n\to\infty} \int g_m d\mu_n \\ &\geq \overline{\lim}_{n\to\infty} \int f d\mu_n \end{aligned} \tag{76}$$

Consequently, $\int f d\mu = \lim_{n\to\infty} \int f d\mu_n$.

□

(ii) $\Longrightarrow$ *(iii):* It's trivial. □

(iii) $\Longrightarrow$ *(iv):* Let fix arbitary closed set A. We set

$$f_n(x) := |1 - min(1, d(x, A))|^n \ (n \in \mathbb{N}, x \in x) \tag{77}$$

$f_n \in C_b(X) \cap C_u(X)$ $(\forall n)$ and $\lim_{n\to\infty} f_n \to \chi_A$ $(pointwiseconvergence)$ and

$$\int f_m d\mu_n \geq \mu_n(A) \tag{78}$$

By Lebesugue's convergence theorem,

$$\mu(A) \geq \overline{\lim_{n\to\infty}} \mu_n(A) \tag{79}$$

$\square$

(iv) $\iff$ *(v):* It's trivial. $\square$

(iv) and (v) $\implies$ *(i):* Let $A \in \mathscr{B}(X)$ and $\mu(\partial A) = 0$. By (iv),

$$\begin{aligned} \overline{\lim_{n\to\infty}} \mu_n(A) &\leq \overline{\lim_{n\to\infty}} \mu_n(\overline{A}) \\ &\leq \mu(\overline{A}) \\ &= \mu(\overline{A} \setminus A) + \mu(A) \\ &\leq \mu(\partial) + \mu(A) \\ &= \mu(A) \end{aligned} \tag{80}$$

In the same way as above we obtain

$$\underline{\lim_{n\to\infty}} \mu_n(A) \quad \geq \quad \mu(A) \tag{81}$$

Consequently

$$\lim_{n\to\infty} \mu_n(A) = \mu(A) \tag{82}$$

$\square$

The following is the definition of a metric of $\mathscr{P}(\mathbb{R})$.

Proposition 8.5. *Let*

- *(S1)* (X, d) *is a compact metric space.*
- *(S2)* $\{f_n\}_{n=1}^{\infty}$ *is a dense subset of* (X, d)*. By Proposition6.11, such subsets always exist.*
- *(S3)* $\tau(\mu_1, \mu_2) := \Sigma_{n=1}^{\infty} |\int f_n d\mu_1 - \int f_n d\mu_2|$ $(\mu_1, \mu_2 \in \mathscr{P}(\mathbb{R}))$.

then the followings hold.

- *(i)* τ *is a metric on* $\mathscr{P}(\mathbb{R})$.
- *(ii) for any* $\{\mu_n\}_{n=1}^{\infty} \subset \mathscr{P}(\mathbb{R})$ *and* $\mu \in \mathscr{P}(\mathbb{R})$, $\mu_n \Longrightarrow \mu$ $(n \to \infty)$ *is equivalent to* $\tau(\mu_n, \mu) \to 0$ $(n \to \infty)$.

(i): Let fix $\mu_1 \in \mathscr{P}(X)$ and $\mu_2 \in \mathscr{P}(X)$ such that $\tau(\mu_1, \mu_2) = 0$. It is enough to show $\mu_1 = \mu_2$ for showing (i). By (S2), for any $f \in C_+(X)$ $\int f d\mu_1 = \int f d\mu_2$. By uniqueness in Proposition7.7, $\mu_1 = \mu_2$. $\square$

(ii): Let us assume $\tau(\mu_n, \mu) \to 0$ $(n \to \infty)$. Let us fix arbitary $\epsilon > 0$. There is $m \in \mathbb{N}$ such that $||f - f_m||_\infty < \frac{\epsilon}{3}$. There is $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$

$$|\int_X f_m d\mu_n - \int_X f_m d\mu| < \frac{\epsilon}{3}. \tag{83}$$

For any $n \geq n_0$

$$\begin{aligned} |\int_X f d\mu_n - \int_X f d\mu| &< |\int_X f d\mu_n - \int_X f_m d\mu_n| \\ &\quad +|\int_X f_m d\mu_n - \int_X f_m d\mu| + |\int_X f_m d\mu - \int_X f_m d\mu| \\ &< \epsilon \end{aligned} \tag{84}$$

Consequently, $\mu_n \Longrightarrow \mu$ $(n \to \infty)$.

The inverse is clear. □

Proposition 8.6. *$(\mathscr{P}(X), \tau)$ is a compact metric space.*

Proof. By Proposition6.4, it is enough to show $(\mathscr{P}(X), \tau)$ is sequencially compact.

Let us fix arbitary $\mu_n{}_{n=1}^\infty \subset \mathscr{P}(X)$.

For any $m \in \mathbb{N}$, $\{\int f_m \mu_n\}_{n=1}^\infty$ is bounded.

For each $m \in \mathbb{N}$, there is $\{\varphi(m,n)\}_{n=1}^\infty$ such that $l(f_m) := \lim_{n\to\infty} \int f_m d\mu_{\varphi(m,n)}$ exists and $|l(f_m) - \int f_m d\mu_{\varphi(m,n)}| < \frac{1}{m}$ $(\forall n \geq m)$.

We set $\psi(m) := \varphi(m,m)$ $(m \in \mathbb{N})$.

By the definition of ψ, for any $m \in \mathbb{N}$ $l(f_m) = \lim_{n\to\infty} \int f_m d\mu_{\psi(n)}$.

Let us fix arbitary $f \in C_b(X)$ and $\epsilon > 0$. There is $k \in \mathbb{N}$ such that $||f - f_k|| < \frac{\epsilon}{3}$.

There is $n_0 \in \mathbb{N}$ such that for any $m \geq n_0$ and any $n \geq n_0$ $|\int f_k d\mu_{\psi(m)} - \int f_k d\mu_{\psi(m)}| < \frac{\epsilon}{3}$

So for any $m \geq n_0$ and any $n \geq n_0$ $|\int f d\mu_{\psi(m)} - \int f d\mu_{\psi(m)}| < \epsilon$.

So $l(f) := \lim_{m\to\infty} \int f d\mu_{\psi(m)}$ exists.

Clearly l is an elementary integral on $C_+(X)$.

So by Proposition7.7, there is $\mu \in \mathscr{P}(X)$ such that

$$l(f) = \int_X f d\mu \ (\forall f \in C_+(X)) \tag{85}$$

Clearly $\mu_{\psi(n)} \Longrightarrow \mu$ $(n \to \infty)$.

□

Proposition 8.7. *Let*

(S1) *(X, d) is a separable metric space.*

(A1) *$\{\mu_n\}_{n=1}^\infty \subset \mathscr{P}(X)$ is tight.*

There is a subsequence $\mu_\varphi(n)_{\{n=1\}}^\infty$ and $\mu \in \mathscr{P}(X)$ such that $\mu_\varphi(n) \Longrightarrow \mu$ $(n \to \infty)$.

Proof. Let $(\tilde{X}, \tilde{d})$ be a compact metric space in Proposition6.8 and $i : X \to \tilde{X}$ in Proposition6.8. By Proposition7.7, for each $n \in \mathbb{N}$ there is a measure $\tilde{\mu}_n$ such that for any $g \in C_+(\tilde{X})$ and $n \in \mathbb{N}$

$$\int_X g \circ i d\mu_n = \int_{\tilde{X}} g d\tilde{\mu}_n \tag{86}$$

There is an increasing sequence of compact sets $\{K_n\}_{n=1}^{\infty}$ such that

$$\mu_m(K_n) > 1 - \frac{1}{n} \tag{87}$$

$(\forall m \in \mathbb{N}, \forall n \in \mathbb{N})$

Let $K := \cup_{n=1}^{\infty} K_n$. By (87), for any $m \in \mathbb{N}$

$$\mu_m(K) = \tilde{\mu}_m(i(K)) = 1 \tag{88}$$

For $n \in \mathbb{N}$ and $x \in \tilde{X}$, $g_{m,n}(x) := (1 - min1, d(x, K_m))^n$. $\int_{\tilde{X}} g_{m,n} d\tilde{\mu}_l \geq \tilde{\mu}_m(K_m) \geq 1 - \frac{1}{m}$. By reaching $n \to \infty$, $\mu_m(K_m) = \tilde{\mu}(i(K_m)) \geq 1 - \frac{1}{m}$. By reaching $m \to \infty$,

$$\tilde{\mu}(i(K)) = 1 \tag{89}$$

By Proposition, there is a subsequence $\{\tilde{\mu}_{\varphi(n)}\}_{n=1}^{\infty}$ and $\tilde{\mu} \in \mathscr{P}(\tilde{X})$ such that $\tilde{\mu}_n \Longrightarrow \tilde{\mu}$ $(n \to \infty)$.

Because for any $n \in \mathbb{N}$ $i(K_n)$ is compact, $i(K_n) \in \mathscr{B}(\tilde{X})$. So $i(K) \in \mathscr{B}(\tilde{X})$.

We will show

$$\mathscr{B}(X) \subset \mathscr{B} := \{A \subset X | i(A \cap K) \mathscr{B}(\tilde{X})\} \tag{90}$$

Because i is injective, if $\{A_n\}_{n=1}^{\infty} \subset \mathscr{B}$ then $\cup_{n=1}^{\infty} A_n \in \mathscr{B}$. And if $A\mathscr{B}$ then $i(A^c \cap K) = i(K) \cap i(A \cap K)^c \in \mathscr{B}$ So $\mathscr{B}$ is a σ-algebra. For any closed set A, $A \in \mathscr{B}$. So (90) holds.

For $A \in \mathscr{B}(X)$, we define

$$\mu(A) := \tilde{\mu}(i(A \cup K)) \tag{91}$$

By (89),

$$\mu(K) = 1 \tag{92}$$

Let me fix arbitary $f \in C_b(X) \cap C_u(X)$. Because $f \in C_u(X)$ and $i(X) is dense in \tilde{X}$, there is $\tilde{f} \in C_b(\tilde{X}) \cap C_u(\tilde{X})$ such that $\tilde{f}|i(X) = f \circ i^{-1}$.

By the definition of $\{\mu_n\}_{n=1}^{\infty}$ and μ,

$$\begin{aligned}\lim_{n\to\infty}\int_X f d\mu_n &= \lim_{n\to\infty}\int_X \tilde{f}\circ i d\mu_n \\ &= \lim_{n\to\infty}\int_{\tilde{X}} \tilde{f} d\tilde{\mu}_n \\ &= \int_{\tilde{X}} \tilde{f} d\tilde{\mu} \\ &= \int_{i(K)} \tilde{f} d\tilde{\mu} \\ &= \int_{i(K)} f\circ i^{-1} d\tilde{\mu} \\ &= \int_K f d\mu \\ &= \int_X f d\mu \qquad (93)\end{aligned}$$

□

9 Characteristic functions of probability distribution

9.1 the case of single variate

By Fubini's theorem, the following holds.

Proposition 9.1. *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S2) Let* $\mu \in \mathcal{P}(\mathbb{R})$.
- *(S3) Let* $f \in L^1(\mathbb{R})$.

then

$$\int_{\mathbb{R}} f(t)\varphi_\mu(t)dt = \int_{\mathbb{R}} \mathscr{F}^{-1}(f)(x)d\mu(x) \qquad (94)$$

Proposition 9.2 (Uniqueness of characteristic function)**.** *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S2) Let* $\mu \in \mathcal{P}(\mathbb{R})$ *and* $\mu' \in \mathcal{P}(\mathbb{R})$.

If $\varphi_\mu = \varphi_{\mu'}$ *then* $\mu = \mu'$.

Proof. Let us arbitary $f \in C_c^\infty(\mathbb{R}^n)$. By Proposition5.3, $\mathscr{F}(f) \in L^1(\mathbb{R}^n)$. By Proposition5.2, $\int_{\mathbb{R}} f(x)d\mu(x) = \int_{\mathbb{R}} f(x)d\mu'(x)$. By Proposition4.3, $\mu = \mu'$. □

This proposition states that convergence of distributions in law is derived from each point convergence of the characteristic function.

Proposition 9.3 (Levy's continuity theorem(single variate case))**.** *Let*

- *(S1)* $\{\mu_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathbb{R})$
- *(S2)* φ_n *is the characteristic function of* μ_n $(n = 1, 2, ...)$
- *(A1)* $\{\mu_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathbb{R})$ *then the followings are equivalent.*
  - *(i) There is a* φ *s.t* φ *is a measurable function on* $\mathbb{R}$ *and* φ *is continuous at* 0 *and* $\varphi(0) = 1$ *and* $\varphi_n \xrightarrow[n\to\infty]{} \varphi$ *(converge pointwise). Below, we fix such* φ.
  - *(ii) Then there is a* $\mu \in \mathcal{P}(\mathbb{R})$ *such that* φ *is the characteristic function of* μ *and* $\mu_n \Longrightarrow \mu$ $(n \to \infty)$.

(i) $\Longrightarrow$ *(ii).* The followings are strategy of the proof.
–Memo

(STEP1) Showing $\{\mu_n\}_{n=1}^{\infty}$ is tight.
(STEP2) Getting μ of the subject.

–
(STEP1)
For each $m \in \mathbb{N}$, there is a measurable function f_m such that f_m continuous at 0 and $f_m(0) = 1$ and $supp(f) \subset [\frac{-1}{m}, \frac{-1}{m}]$ is compact and $f_m \leq 1$ in $\mathbb{R}$ and $\mathcal{F}^{-1} f_m \leq 1$ in $\mathbb{R}$. $\{\chi_{[-\frac{1}{m},\frac{1}{m}]}\}_{m=1}^{\infty}$ sutisfies the above conditions. Fix such $\{f_m\}_{m=1}^{\infty}$.

We get

$$\int_{\mathbb{R}} f_m(x)\varphi_n(x)dx = \int_{\mathbb{R}} \mathcal{F}^{-1} f_m(x) d\mu_n(x) \tag{95}$$

So

$$1 - \frac{m}{2}\int_{\mathbb{R}} f_m(x)\varphi_n(x)dx = 1 - \frac{m}{2}\int_{\mathbb{R}} \mathcal{F}^{-1} f_m(x) d\mu_n(x) \tag{96}$$

Call the left side of the above (96) $I_{m,n}$ and call the right side of the above (96) $J_{m,n}$. Fix any $\varepsilon > 0$.

(STEP1-1)
–Memo
We will show that $I_{m,n} < \varepsilon$ for sufficient large m, n. We will show this statement using the dominated convergence theorem and continuity of φ at 0
–
(STEP1-2)
–Memo
We will show that $J_{m,n} > \mu_n(\{x \in \mathbb{R} |\ |x| \geq m\})$ for sufficient large m, n. We will show this statement using the dominated convergence theorem and continuity of φ at 0

–

The following holds.

$$\mathcal{F}^{-1}f_m(x) = \frac{1}{m}\mathcal{F}^{-1}f_m(\frac{x}{m}) \tag{97}$$

So

$$\begin{aligned} J_{m,n} &= 1 - \frac{1}{2}\int_{\mathbb{R}} \mathcal{F}^{-1}f_m(\frac{x}{m})d\mu_n(x) \\ &= \int_{\mathbb{R}} 1 - \frac{1}{2}\mathcal{F}^{-1}f_m(\frac{x}{m})d\mu_n(x) \\ &= \int_{\{x\in\mathbb{R}|\ |x|\geq m\}} 1 - \frac{1}{2}\mathcal{F}^{-1}f_m(\frac{x}{m})d\mu_n(x) \end{aligned} \tag{98}$$

In (98), we use statement $\mathcal{F}^{-1}f_m \leq 1$ in $\mathbb{R}$ ($\forall m \in \mathbb{N}$).

$$\begin{aligned} 1 - \frac{1}{2}\mathcal{F}^{-1}f_m(\frac{x}{m}) &\geq 1 - \frac{1}{2}max_{y\in supp(|f_m|)}|f_m(y)|\frac{m}{|x|} \\ &\geq \frac{1}{2} \end{aligned} \tag{99}$$

So

$$J_{m,n} \geq \frac{1}{2}\mu_n(\{x \in \mathbb{R}|\ |x| \geq m\}) \tag{100}$$

By (STEP1-1) and (100) for sufficient large m and n we get

$$2\epsilon \geq \mu_n(\{x \in \mathbb{R}|\ |x| \geq m\}) \tag{101}$$

So We have shown $\{\mu_n\}_{n=1}^{\infty}$ is tight.

(STEP2)

By (STEP1), there is a subsequence $\{\mu_{\psi(n)}\}_{n=1}^{\infty}$ which converges to a μ in law. It is enough to show for any subsequence of $\{\mu_n\}_{n=1}^{\infty}$ the subsequence has some subsequnece of the subsequence which converges to μ in law. Let fix any subsequence $\{\mu_{\omega(n)}\}_{n=1}^{\infty}$. There is a subsequence $\{\mu_{\omega(\alpha(n))}\}_{n=1}^{\infty}$ which converges to μ'. By increasing n to ∞ in (96) and Proposition8.3, $\phi_\mu = \phi$ and $\phi_{\mu'} = \phi$. By uniqueness of characteristic function, $\mu = \mu'$.

□

(ii) $\Longrightarrow$ *(i)*. $\varphi_\mu : \mathbb{R} \ni t \mapsto \int_\Omega exp(itx)d\mu$. It is easy to show φ_μ is continuous at 0.

By Proposition8.3,

$$\int_{\mathbb{R}} exp(itx)d\mu(x) = \lim_{n\to\infty}\int_{\mathbb{R}} exp(itx)d\mu_n\ (\forall t) \tag{102}$$

□

9.2 the case of multi variates

Proposition 9.4 (Levy's continuity theorem(multi variate case)). *Let*

- *(S1)* $\{\mu_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathbb{R}^N)$
- *(S2)* φ_n *is the characteristic function of* μ_n $(n = 1, 2, ...)$
- *(A1)* $\{\mu_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathbb{R}^N)$
- *(A1) There is a* φ *s.t* φ *is a measurable function on* $\mathbb{R}^N$ *and* φ *is continuous at* 0 *and* $\varphi(0) = 1$ *and* $\varphi_n \xrightarrow[n\to\infty]{} \varphi$ *(converge pointwise). Below, we fix such* φ.

Then there is a $\mu \in \mathcal{P}(\mathbb{R}^N)$ *such that* φ *is the characteristic function of* μ *and* $\mu_n \Longrightarrow \mu$ $(n \to \infty)$.

Proof. By an argument which is similar to the proof of Proposition9.3, we can show that $\{\mu_n\}_{n=1}^{\infty}$ is tight.

By Proposition8.7 and uniqueness of fourier transformation in $\mathbb{R}^N$ and Proposition8.4, there is $\mu \in \mathscr{P}(R)^N$ such that $\mu_n \Longrightarrow \mu$ $(n \to \infty)$ and $\varphi_\mu = \varphi$. □

10 A proof of the central limit theorem

10.1 the case of single variate

Theorem 10.1 (Central limit theorem). *Let*

- *(S1)* $(\Omega, \mathscr{F}, P)$ *is a probability space.*
- *(S2)* $\{X_i\}_{i=1}^{\infty}$ *is a sequence of random variables on* $(\Omega, \mathscr{F}, P)$.
- *(A1)* $\exists \mu$ *such that* $X_i \sim \mu$ $(\forall i)$. *Bellow, we fix such* μ.
- *(A2)* $\{X_i\}_{i=1}^{N}$ *are independent for any* $N \in \mathbb{N}$.
- *(A3)* $E[\mu] = \nu$ *and* $V[\mu] = \sigma^2$ *and* $\sigma > 0$.

then $P_{\sqrt{n}(\bar{X}-\nu)}$ *weakly converges to* $N(0, \sigma)$.

Proof. We can assume $\nu = 0$ and $\sigma = 1$. Bellow, we assume that.
Let $Y_{i,n} := \frac{X_i}{\sqrt{n}}$ $(i = 1, 2, ..., n)$ and $Y_n := \sum_{i=1}^{n} Y_{i,n}$ $(n = 1, 2, ...)$. By $(A1)$, $\varphi_{Y_{i,n}} = \varphi_{Y_{1,n}}$ $(\forall i, \forall n)$. Let $\varphi_n := \varphi_{Y_n}$ and $\psi_n := \varphi_{Y_{1,n}}$ $(n = 1, 2, ...)$. And let $\psi_\mu : \mathbb{R} \ni s \mapsto \int_{\mathbb{R}} exp(isx) d\mu(x)$. Then $\varphi_n = (\psi_n)^n$ and $\psi_n(t) = \psi_\mu(\frac{t}{\sqrt{n}})$ and $(\forall t \in \mathbb{R})$. We will show the following equation. By Proposition3.7,

$$\varphi_{Y_{1,n}}(t) = 1 - \frac{t^2}{2n} + o(\frac{1}{n})(n \to \infty) \tag{103}$$

By the above equation and Proposition3.4,

$$\varphi_n(t) = (1 - \frac{t^2}{2n} + o(\frac{1}{n}))^n \to exp(-\frac{t^2}{2})\ (n \to \infty) \tag{104}$$

By Proposition9.3, there is a $\mu_0 \in \mathscr{P}(\mathbb{R})$ such that $P_{\sqrt{n}\bar{X}}$ converges to μ_0 in law and $\varphi_{\mu_0} = exp(-\frac{(\cdot)^2}{2})$. Because $\varphi_{N(0,1)} = exp(-\frac{(\cdot)^2}{2})$ and uniqueness of characteristic function, $P_{\sqrt{n}\bar{X}}$ converges to $N(0,1)$ □

10.2 the case of multi variates

Theorem 10.2 (Central limit theorem(multi variate case))**.** *Let*

(S1) $(\Omega, \mathscr{F}, P)$ *is a probability space.*

(S2) $\{X_i\}_{i=1}^{\infty}$ *is a sequence of* N*-dimensional vectors of random variables on* $(\Omega, \mathscr{F}, P)$*.*

(A1) $\exists \mu$ *such that* $X_i \sim \mu$ $(\forall i)$*. Bellow, we fix such* μ*.*

(A2) $\{X_i\}_{i=1}^{n}$ *are independent for any* $n \in \mathbb{N}$*.*

(A3) $E[\mu] = \nu$ *and* $cov[\mu] = \sigma^2$ *and* σ *is* N*-by-*N *positive definite symmetric matrix.*

then $P_{\sqrt{n}(\bar{X}-\nu)}$ *weakly converges to* $N(0, \Sigma)$*.*

Proof. Let us fix arbitary $\boldsymbol{t} \in \mathbb{R}^N$ and $s \in \mathbb{R}$. Let us set $Y_n := s\boldsymbol{t}^T(X_n - \nu)$.

The following holds.

$$\varphi_{\sqrt{n}(\bar{X}-\nu)}(s\boldsymbol{t}) = E(exp(\sqrt{n}is\boldsymbol{t}^T(\bar{X}-\nu))) = \varphi_{\sqrt{n}(\bar{Y}-\nu)}(s) \tag{105}$$

By Theorem10.1 and Proposition9.3 and Proposition3.3,

$$\lim_{n\to\infty} \varphi_{\sqrt{n}(\bar{Y}-\nu)}(s) = exp(-\frac{s^2\boldsymbol{t}^T\Sigma^2\boldsymbol{t}}{2}) \tag{106}$$

By setting $s = 1$,

$$\lim_{n\to\infty} \varphi_{\sqrt{n}(\bar{X}-\nu)}(s\boldsymbol{t}) = exp(-\frac{\boldsymbol{t}^T\Sigma^2\boldsymbol{t}}{2}) \tag{107}$$

By Proposition9.4 and Proposition3.3, $P_{\sqrt{n}(\bar{X}-\nu)}$ weakly converges to $N(0, \Sigma)$. □

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