A Proof of a determinant formula

1 Introduction

I will show the following formula.

Formula 1. *Let n be a natural number and I_n be $n \times n$ identity matrix and x, y, u, v be $n \times 1$ matrices. Then*

$$det(I_n + xy^T + uv^T) = (1 + y^T x)(1 + v^T u) - (x^T v)(y^T u)$$

2 A proof

2.1 Case1: u or v is zero

We use the following lemma.

Lemma 1. *Let n be a natural number and I_n be $n \times n$ identity matrix and x, y be $n \times 1$ matrices. Then*

$$det(I_n + xy^T) = (1 + y^T x)$$

It is easy to show the above lemma by taking advantage of orthogonal matrix $Q = \left(x, w_1, ..., w_{n-1}\right)^T$ such that $Qx = e_1$ when $|x| = 1$.

2.2 Case2: $x^T u = 0$

We will show that Formula1 is true when $x^T u = 0$. By Lemma1, we can assume $x \neq 0, u \neq 0$. Furthermore, we can assume $|x| = |u| = 1$ by taking advantage of the equation $xy^T = \frac{x}{|x|}(|x|y)^T$ and $uv^T = \frac{u}{|u|}(|u|v)^T$. By Gram-Schmit orthogonalization process theorem, we can get a orthogonal maxrix $Q = (x, u, w_1, ..., w_{n-2})^T$ s.t. $Qx = e_1, Qu = e_2$. We define $y' := Qy$ and $v' := Qv$ and $A := \begin{pmatrix} 1 + y'_1 & y'_2 \\ v'_1 & 1 + v'_2 \end{pmatrix}$. Then we get

$$\begin{aligned} det(I_n + xy^T + uv^T) &= det(Q(I_n + xy^T + uv^T)Q^T) \\ &= det(I_n + e_1 y'^T + e_2 v'^T) \\ &= det \begin{pmatrix} A & .. \\ O_{(n-2),2} & I_{n-2} \end{pmatrix} \\ &= det A \\ &= (1 + y'_1)(1 + v'_2) - y'_2 v'_1 \end{aligned}$$

Furthermore, we get the following equation by definition of y' and v'. We get Formula1 by the above equation and the following equation.

$$y'_1 = x^T y, y'_2 = u^T y, v'_1 = x^T v, v'_2 = u^T v$$

2.3 Case3: general case

We will show that Formula1 is true in general. We can get a real number a s.t. $\overline{u} := u - ax$ is orthogonal to x. We get

$$xy^T + uv^T = x(y + av)^T + \overline{u}v^T$$

So, we can use the result of case2 to get

$$\begin{aligned} det(I_n + xy^T + uv^T) &= det(I_n + x(y + av)^T + \overline{u}v^T) \\ &= (1 + (y + av)^T x)(1 + v^T \overline{u}) - x^T v(y + av)^T \overline{u} \end{aligned}$$

We can easily get

$$(1 + (y + av)^T x)(1 + v^T \overline{u}) - x^T v(y + av)^T \overline{u} = (1 + y^T x)(1 + v^T u) - (x^T v)(y^T u)$$

Consequently, we get Formula1 is true in general.