A Roundabout Proof of Schwarz's Inequality

1 Introduction

I will show a roundabout proof of Schwarz's Inequality.

Theorem 1. *H is a inner product space. Then*

$$|(u, v)| \leq ||u||||v|| \ (\forall u \in H, \forall v \in H)$$

We can gave a proof of the above theorem without the following assumption

$$u \neq 0 \ then \ ||u|| \neq 0 \ (\forall u \in H) \tag{1}$$

2 A proof

Firstly, the above theorem is obviously true when $(u, v) = 0$. So we can assume $(u, v) \neq 0$. Let $\theta \in [0, 2\pi)$ be a argument of (u, v) and let $u' := exp(i(-\theta))u$. $exp(i(-\theta))(u, v)$ is a real positive number. So, if u', v satisfies Shuwartz' inequality, u, v satisfies the one since

$$\begin{aligned} |(u, v)| &= |exp(i(-\theta))(u, v)| \\ &= exp(i(-\theta))(u, v) \\ &= (u', v) \\ &\leq ||u'||||v|| \\ &= ||u||||v|| \end{aligned} \tag{2}$$

So we can assume

$$(u, v) \in \mathbb{R} \ and \ (u, v) > 0 \tag{3}$$

Hereafter, on these assumptions, we will show

- If $||u||||v|| \neq 0$, the above theorem is true
- $||u||||v|| \neq 0$

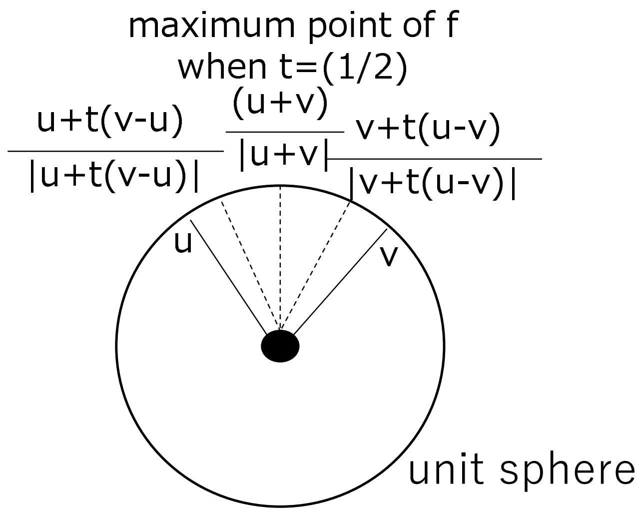

Figure 2.1: Image of poins on unit sphere

2.1 The theorem is true when u and v have not zero norm

In this case, we can assume u, v is in unit sphere. The above theorem claims that $(\frac{u}{||u||}, \frac{v}{||v||})$ reachs the maximum value 1 when $u = v$ on unit sphere $\{u \in H| \; ||u|| = 1\}$. By seeing Figure2.1 so we can guess

$$f : [0,1] \ni t \mapsto \frac{(u + t(v-u), v + t(u-v))}{(||u + t(v-u)||)(||v + t(u-v)||)} \in \mathbb{R} \tag{4}$$

reachs the maximum value 1 at $t = \frac{1}{2}$.

The following is true

$$f(0) = (u, v) \tag{5}$$

.

It is enought to show the following proposition

Proposition 1.

$$f_p : [0,1] \ni t \mapsto \frac{(u + t(v-u), v + t(u-v))}{((||u + t(v-u)|| + p)(||v + t(u-v)|| + p))} \in \mathbb{R} \tag{6}$$

reachs the maximum at $t = \frac{1}{2}$ *for all* $p > 0$.

Actually, if Proposition1 is true, then $f_p(0) \leq f_p(\frac{1}{2})$ $(\forall p > 0)$. So

$$\frac{(u,v)}{(1+p)^2} \leq \left(\frac{\frac{1}{4}||u+v||^2}{(\frac{1}{2}||u+v||+p)(\frac{1}{2}||u+v||+p)}\right) \ (\forall p > 0) \tag{7}$$

Since $(u,v) > 0$, $||u+v|| > 0$. Reaching $p \to 0$, we get Shwartz Inequality

$$(u,v) \leq 1 = ||u||||v|| \tag{8}$$

So, hereafter, we will show Proposition1.

We define

$$a : \mathbb{R} \ni t \mapsto (u + t(v-u), v + t(u-v)) \in \mathbb{R} \tag{9}$$

So, we get

$$a(t) = -||u-v||^2 t^2 + ||u-v||^2 t + (u,v) \ (\forall t \in \mathbb{R}) \tag{10}$$

So a reachs the maximum at $t = \frac{1}{2}$. We notice a reachs the maximum at $t = \frac{1}{2}$ even if u, v is not in unit sphere.

$$||u + t(v-u)||^2 = ||v + t(u-v)||^2 = ||v-u||^2 t^2 + (2(u,v) - 2)t + 1 \tag{11}$$

and

$$||v-u||^2 = -(2(u,v) - 2) \tag{12}$$

So if we define

$$b : \mathbb{R} \ni t \mapsto ((||u + t(v-u)|| + p)(||v + t(u-v)|| + p)) \in \mathbb{R} \tag{13}$$

then b reachs the minimum at $t = \frac{1}{2}$. So the theorem is true when $||u||||v|| \neq 0$.

2.2 u and v have not zero norm

We will show $||u||||v|| \neq 0$. Firstly we assume $||u|| = ||v|| = 0$. Then

$$a(\frac{1}{2}) = \frac{1}{2}(u,v) \geq a(0) = (u,v) \tag{14}$$

So $(u,v) = 0$, this contradicts with (3). Secondly we assume $||u|| = 0,\ ||v|| \neq 0$. We can assume $||v|| = 1$. We define for positive number p

$$a_p : \mathbb{R} \ni t \mapsto (pu + t(v - pu), v + t(pu - v)) \in \mathbb{R} \tag{15}$$

Similarly to the above discussion a_p reachs the maximum at $t = \frac{1}{2}$.

$$0 \leq a_{\frac{1}{(u,v)}}(\frac{1}{2}) - a(0) = 1 - 2\frac{1}{(u,v)}(u,v) = -1 \tag{16}$$

This is a contradiction. So $||u||||v|| \neq 0$.